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ON WEAK RECURRENT POINTS OF ULTIMATELY NONEXPANSIVE MAPPINGS

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ABSTRACT. Conditions are investigated which imply that every ultimately nonexpansive mapping has a weak recurrent point. Examples are presented which show that better results can not be obtained.

1. Introduction. Throughout this paper X will represent a real Banach space, H a real Hilbert space, and K a weakly compact (usually convex) subset of either X or H. The functions f and F will be ultimately nonexpansive self mappings (see below) of K. In §2 f is arbitrary, while f and F are specific functions in §3 and §4.

DEFINITIONS. A fixed point of a self mapping f of a set K is a point x such that $f(x) = x$.

A function $f : K \to K \subseteq X$ is said to be nonexpansive if $||f(x)-f(y)|| \leq ||x-y||$ for all x and y in K, and is said to be ultimately nonexpansive if it is continuous and $\limsup_n ||f^n(x)-f^n(y)|| \leq ||x-y||$. If, for each x and y, there is an N such that $||f^n(x) - f^n(y)|| \leq ||x-y||$ for $n > N$, then f is said to be asymptotically nonexpansive.

Note that nonexpansive $\Rightarrow$ asymptotically nonexpansive $\Rightarrow$ ultimately nonexpansive.

There is a considerable amount of literature on nonexpansive mappings, especially with regards to the existence of fixed points. We refer the reader to Kirk [8] for a survey of these results. In general, in "nice" spaces (notably uniformly convex and those with normal structure) every nonexpansive self mapping of a weakly compact convex set has a fixed point.

Ultimately and asymptotically nonexpansive functions have not proven as productive with regards to fixed points. Some of the earlier theorems on nonexpansive mappings have been generalized to asymptotically (cf.[7]) and ultimately (cf.[4]) nonexpansive mappings. These

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either assume that K is compact or that there is a recurrent point (see below).

In [**10**], we gave an example of an asymptotically nonexpansive self mapping of a weakly compact convex subset of a Hilbert space which has no fixed point. The question then arises as to what type of properties do such mappings have.

DEFINITIONS. Let $f : K \to K \subseteq X$. Then a recurrent (weak recurrent) point of f is a point x such that $f^{n_i}(x) \to x$ for some subsequence $\{f^{n_i}(x)\}$ of $\{f^n(x)\}$.

If $\lim f^n(x) = x$, and f is continuous it is easy to see that x is a fixed point of f. Thus we say that x is a weak fixed point of f if $\{f^n(x)\}$ converges weakly to X, written $w - \lim f^n(x) = x$. The example given in [**10**] has a weak fixed point. Our motivation for the work of this paper was initiated by this observation and our wondering if perhaps all ultimately (or asymptotically) nonexpansive maps had such a point. The example given in §3 shows that they need not.

This leads us to the investigation of recurrent points of ultimately nonexpansive maps. Recurrent points of nonexpansive maps, both weak and strong, as well as the set of cluster and weak cluster points of the sequence of iterates of a point have received attention in recent years (cf. [**2**], [**9**]). As mentioned, Edelstein [**4**] showed that if $f : K \to K \subseteq X$ is ultimately nonexpansive (K convex and weakly compact) and has a recurrent point, then f has a fixed point. In view of this, it follows that the example in [**10**] has no recurrent point. Weak recurrent points proved to be more fruitful. We show in §2 that every ultimately nonexpansive self map of a weakly compact convex set K which has the fixed point property for nonexpansive maps (that is if $g : K \to K$ is nonexpansive, then g has a fixed point) has a weak recurrent point.

Most of the examples in this paper, as well as that in [**10**], satisfy a condition much stronger than asymptotically nonexpansive. Indeed, for each x and y in K,

$$(*) \qquad \lim_n ||f^n(x) - f^n(y)|| = 0.$$

This is also stronger than asymptotically regular (cf.[**1**]). In passing we mention that (*) implies that the set of weak cluster points of $\{f^n(x)\}$ is the same for every x. Hence if, for some x and y, y is a weak cluster point of $\{f^n(x)\}$, then y is a weak recurrent point. Thus if f satisfies

(*) the above mentioned result is easy.

2. In this section we show that every ultimately nonexpansive self mapping of a weakly compact convex subset K of a Banach space X, such that K has the fixed point property for nonexpansive maps, has a weak recurrent point. The proof uses universal nets. We refer the reader to a book on general topology, such as [**6**], for details.

The following lemma is a simple application of "Whitleys Construction" (cf. [**5** , p.147]). It undoubtedly appears in the literature.

LEMMA 2.1. *Let* $\{x_n\}$ *be a sequence in a weakly compact set* K. *Then if* x *is a weak cluster point of* $\{x_n\}$ *there is a subsequence* $\{x_{n_i}\}$ *of* $\{x_n\}$ *with* $w - \lim x_{n_i} = x$.

PROOF. By Whitleys Construction (cf. [**5**, p.147]) there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which has the unique weak cluster point x. Since K is weakly sequentially compact (the Eberlein - Smulian Theorem) this immediately shows that $w - \lim x_{n_i} = x$.

LEMMA 2.2. *Let* $\{f_\alpha : \alpha \in D\}$ *be a subnet of the sequence of iterates of an ultimately nonexpansive function* $f : K \to K$. *Then*

(a) $\limsup_\alpha ||f_\alpha(x) - f_\alpha(y)|| \leq ||x - y||$, *for* $x, y \in K$,

(b) *If* $w - \lim_\alpha f_\alpha(x) = x'$, $w - \lim_\alpha f_\alpha(y) = y'$ *then* $||x' - y'|| \leq ||x - y||$.

PROOF. Let $x, y \in K$ and $\varepsilon > 0$ be given. Then there is an integer M such that for $n > M, ||f^n(x) - f^n(y)|| \leq (1 + \varepsilon)||x - y||$. Pick $\beta \varepsilon D$ such that $N(\alpha) \geq M$ for $\alpha \geq \beta$. Then, for $\alpha \geq \beta$,

$$||f_\alpha(x) - f_\alpha(y)|| = ||f^{N(\alpha)}(x) - f^{N(\alpha)}(y)|| \leq (1 + \varepsilon)||x - y||.$$

This proves (a).

As is well known [**3**, p.29] that the norm is weakly lower semicontinuous, thus $||x' - y'|| \leq \limsup_\alpha ||f_\alpha(x) - f_\alpha(y)|| \leq ||x - y||$.

COROLLARY 2.3. *If $w - \lim_{\alpha\in D} f_\alpha(x)$ exists for every $x \in K$, then $x \to w - \lim_\alpha f_\alpha(x)$ is a nonexpansive self mapping of K.*

The following is the main result of this paper.

THEOREM 2.4. *Let X be a Banach space, and $K \subseteq X$ be weakly compact and have the fixed point property for nonexpansive maps. Then if $f : K \to K$ is ultimately nonexpansive, f has a weak recurrent point.!*

PROOF. Let $\{n_\alpha : \alpha \in D\}$ be a universal subnet of the sequence of integers. Then $\{f^{n_\alpha}(x) : \alpha \in D\}$ is a universal subnet in the weakly compact set K, hence $w-\lim_{\alpha\in D} f^{n_\alpha}(x)$ exists for each $x \in K$. Lemma 2.2 shows that, for x in K, $x \to w-\lim_{\alpha\in D} f^{n_\alpha}(x)$ is nonexpansive and hence possesses a fixed point x_0. Thus $x_0 = w-\lim_{\alpha\in D} f^{n_\alpha}(x_0)$, hence x_0 is a weak cluster point of $\{f^n(x_0)\}$. Lemma 2.1 shows that there is a subsequence $\{f^{n_i}(x_0)\}$ of $\{f^n(x_0)\}$ such that $x_0 = w - \lim f^{n_i}(x_0)$.

Before proceeding to §3, where we present some examples related to Theorem 2.4, we prove a proposition which extends Corollary 2.3.

DEFINITIONS. If $\{x_\alpha : \alpha \in D\}$ is a net in X, then $W\{x_\alpha : \alpha \in D\}$ will denote the set of weak cluster points of $\{x_\alpha : \alpha \in D\}$. For a set K in $X, WC(K)$ will denote the class of all relatively weakly compact subsets of K. If U, V are sets in X, then $H(U,V)$ will denote the Hausdorf distance between U and V.

PROPOSITION 2.5. Let $K \subseteq X$ be weakly compact, and let $\{f_\alpha : \alpha \in D\}$ be a subnet of the sequence of iterates of an ultimately nonexpansive mapping $f : K \to K$. Then the function $G : K \to WC(K)$, defined by $G(x) = W\{f_\alpha(x) : \alpha \in D\}$, is norm to Hausdorf metric nonexpansive.

PROOF. Let $x, y \in K$ be given, $S = W\{f_\alpha(x) : \alpha \in D\}, T = W\{f_\alpha(y) : \alpha \in D\}$ and $x' \in S$. Then there is a subnet $\{f_\alpha : \alpha \in D'\}$ of $\{f_\alpha : \alpha \in D\}$ such that $x' = w - \lim_{\alpha\in D'} f_\alpha(x)$, and a subnet $\{f_\alpha : \alpha \in D''\}$ of $\{f_\alpha : \alpha \in D'\}$ such that $\{f_\alpha(y) : \alpha \in D''\}$

weakly converges to, say, y'. Then $x' = w - \lim_{\alpha \in D''} f_\alpha(x)$ and $y' = w - \lim_{\alpha \in D''} f_\alpha(y)$ so Lemma 2.2 shows that $||x' - y'|| \leq ||x - y||$. Thus

$$S \subseteq \bigcup_{z \in T} \overline{B}(z, ||x - y||).$$

Similarly,

$$T \subseteq \bigcup_{z \in S} \overline{B}(z, ||x - y||)$$

showing that $H(S,T) \leq ||x - y||$.

REMARK. Proposition 2.5 shows that $x \to W\{f^n(x) : n = 1, 2, \dots\}$ is norm to Hausdorff metric nonexpansive. If X has the fixed point property for nonexpansive maps, it is an open question as to whether or not every nonexpansive mapping $F : K \to WC(K)$ has a fixed point. A positive answer to this would provide another proof to Theorem 2.3.

3. In this section we present an example of an asymptotically nonexpansive function F, acting on a weakly compact subset C of a Hilbert space such that $W\{F^n(x) : n = 1, 2, \dots\}$ is not a singleton for any $x \in C$. Hence F has no weak fixed points. (That is, no point x such that $x = w - \lim F^n(x)$). To do this, the function f given in [**10**] is used. We now define f, and list some of its properties, but refer the reader to [**10**] for details.

For a sequence $\{x_i\}$ in a Hilbert space H, the closed span of $\{x_i\}$, denoted $\overline{\text{span}}\,\{x_i\}$, is the closure of $\{\sum_{i=1}^n \lambda_i x_i : \lambda_i \text{ real}, n = 1, 2, \dots\}$. If δ_{ij} is the Kronecker δ, that is

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i \neq j \end{cases},$$

and $\{x_i\}$, $\{x_i^*\}$ are sequences in H such that $x_i^*(x_j) \overset{(\text{def})}{=} < x_i^*, x_j >= \delta_{ij}$, then $\{x_i, x_i^*\}$ is called a biorthogonal system for H. The cone associated with a biorthogonal system, denoted $\kappa\{x_i, x_i^*\}$. is defined by

$$\kappa\{x_i, x_i^*\} = \{x \in X : x_i^*(x) \geq 0, i = 1, 2, \dots\}.$$

Let $\{x_i, x_i^*\}$ be a biorthogonal system in a Hilbert space H with $||x_i|| = 1, \langle x_i, x_j \rangle \geq 0, H = \overline{\text{span}}\{x_i^*\}$ and $\lim_n \langle x_n, x_{n+1} \rangle = 1$, for all i. (An example of such a system is given in [**10**]). As noted in [**10**], for each $x \in \kappa\{x_i, x_i^*\}$.

$$x = \sum x_i^*(x) x_i$$

(hence we say that $\{x_i\}$ is a basis for $\kappa\{x_i, x_i^*\}$) and

$$\sum x_i^*(x)^2 \leq ||x||^2.$$

Define the set K by

$$K = \kappa\{x_i, x_i^*\} \bigcap B,$$

where B is the closed unit ball of H. For $x \in K$, define

$$f_1(x) = (1 - ||x||)x_1 + \sum_{i=1}^{\infty} x_i^*(x)^2 x_{i+1}$$

and

$$f(x) = \frac{f_1(x)}{||f_1(x)||}.$$

It is shown in [**10**] that f is an asymptotically nonexpansive self mapping of K having no fixed points, and satisfies (*). Since $H = \overline{\text{span}}\{x_i^*\}$ it is easy to verify that $w - \lim f^n(x) = 0$. (Note that $||f(x)|| = 1$, so, for $x \in f(K), f(x)$ is, loosely speaking, a shift on the basis $\{x_i\}$.) In particular, $w - \lim f^n(0) = 0$, so 0 is a weak fixed point for f.

Before defining F, we present an intuitive discussion as to the nature of the orbit of a point x under F, i.e., the set $\{F^n(x) : n = 1, 2, \dots\}$. The functon F is defined on the set $C = [0,1] \times K$, where K is as above, and has the form

$$F(r, x) = (g(r, x), f(x)).$$

where $r \in [0,1], x \in K$, and $g : C \to [0,1]$. Let $(r, x) \in C$ be arbitrary, but fixed. Then there is a sequence $\{r_n\} \subset [0,1]$ such that $F^n(r, x) = (r_n, f^n(x))$. Since $w - \lim f^n(x) = 0$, for every $x \in K$, it follows that

$$W\{F^n(r, x) : n = 1, 2, \dots\} \subset [0,1] \times \{0\}.$$

In order for F to have no weak fixed point, the sequence $\{r_n\}$ must not converge to r, if $x = 0$. (In particular, this shows that the function g

cannot be independent of x, for otherwise $g: [0,1] \to [0,1]$ would have a fixed point, call it r' and $(r', 0)$ would be a weak fixed point of F.)

Now we also wish F to satisfy (*). Thus if $(s, y) \in C$ is also arbitrary, but fixed, and $F^n(s,y) = (s_n, f^n(y))$, then, since f satisfies (*), $|s_n - r_n|$ must tend to zero.

To obtain the first objective, we construct g so that the sequence $\{r_n\}$, eventually, moves back and forth across $[0,1]$ in progressively smaller steps. More precisely, there is an M such that $\{r_n : n > M\}$ will increase until it becomes one, remain at one for some terms, then decrease until 0, etc. Furthermore, the difference between successive terms of $\{r_n\}$ tends to zero, ensuring that $\{r_n\}$ is dense in $[0,1]$, so does not converge. This, along with the fact that $w - \lim f^n(x) = 0$ shows that $W\{F^n(r,x) : n = 1, 2, \dots\} = [0,1] \times \{0\}$.

In order to ensure that $|s_n - r_n| \to 0$, g is constructed so that the "waiting time" of $\{r_n\}$ at one and zero becomes progressively longer. This allows $\{s_n\}$ to eventually "catch" $\{r_n\}$. The continuity of g ensures that once $\{s_n\}$ has caught $\{r_n\}$, and $||f^n(x) - f^n(y)||$ has become sufficiently small (recall f satisfies (*)), the values of $\{s_n\}$ and $\{r_n\}$ cannot diverge rapidly. In fact, before they diverge too far, the other end of $[0,1]$ is encountered where they catch one another once more.

To define F, some technical definitions are needed. For an integer k let P_k (also denoted $P(k)$) be the integer satisfying $2^{P_k} < k \le 2^{P_{k+1}}$. Define a functional h_1 on ℓ^1 (hence h_1 is in ℓ^∞) by

$$h_1 = 2 \sum_{k=1}^{\infty} \frac{(-1)^{P_k}}{2^{P_k}} e_k,$$

where $\{e_k\}$ is the standard basis for ℓ^∞. For $x \in K$, as in **[10]**, Lemma 4, $f^n(x) = \sum_{i=0}^{\infty} \lambda_i^{(n)} x_{i+n}$ for some sequence $\{\lambda_i^{(n)}\}$. Note that $\lambda_i^{(n)} \ge 0$, since $f^n(x) \in K$, and, for $n \ge 1, \sum_{i=0}^{\infty} \lambda_i^{(n)} < \infty (**)$, so $(\lambda_i^{(n)}) \in \ell^1$. Thus define

$$h(x) = h_1((\lambda_i^{(1)})).$$

For $r \in R$, define

$$H(r) = \begin{cases} 1, & r \ge 1 \\ r, & 0 \le r \le 1 \\ 0, & r \le 0 \end{cases}$$

and, for $r \in [0,1]$,

$$g(r,x) = H(r + h(x)).$$

Define C to be the set $[0,1] \times K$ of the Hilbert space $R \times H$ with the usual norm.

Finally, for $(r,x) \in C$, define

$$F(r,x) = (g(r,x), f(x)).$$

We will not subject the reader to a proof that F is ultimately nonexpansive and that $W\{F^n(x) : n = 1,2,\dots\}$ is not a singleton. Our proof is long and tedious, yet proceeds in a straightforward fashion. The following remarks might help anyone interested in working through the proof.

REMARKS. (1)With $\lambda_i^{(n)}$ as above and $n \geq 1$,

$$h(f^{n-1}(x)) = 2\sum_{k=n}^{\infty} \frac{(-1)^{P_k}}{2^{P_k}} \lambda_{r-n}^{(n)}.$$

(2) Much of the proof of this example is based on the observations that if $h(f^n(x)) \geq 0$, (or ≤ 0) for all $0 \leq n < m$, then

$$F^m(r,x) = (H(r + \sum_{n=0}^{m-1} h(f^n(x)), f^m(x)),$$

and if, in addition

$$a \leq \sum_{n=0}^{m-1} h(f^n(x)) \leq b$$

and r_m is defined by

$$F^m(r,x) = (r_m, f^m(x)),$$

then

$$H(r+a) \leq r_m \leq H(r+b).$$

(3) Since f, h, and H are continuous, to show that F is continuous it need only be shown that $x \rightarrow (\lambda_i^{(1)})_{i=0}^{\infty}$ is continuous from K to ℓ^1.

4. Further examples. In this section we outline two further examples built on the function F.

EXAMPLE 4.1. There may be the objection that both F and f satisfy (*), a condition much stronger than ultimately nonexpansive. The following function F_1 seems to be "truly" asymptotically nonexpansive.

Let F, C, K, and H be as in the previous section. We define F_1 on a subset C_1 of $R^2 \times H$ with the usual norm. Let C_1 be $[-1,1] \times C$ and let $F_1 : C_1 \to C_1$ be given by $F_1(r,x) = (-r, F(x))$ for $x \in C, r \in [-1,1]$. Since $||F^n(x) - F^n(y)|| \to 0$, for $x, y \in C$, it is straightforward to show that F_1 is asymptotically nonexpansive. Also, for $r, s \in [-1,1]$ and $x \in C, ||F_1^n(r,x) - F_1^n(s,x)|| = ||(r,x) - (s,x)||$ for all n, so, in some sense F_1 is truly asymptotically nonexpansive. Note that $W\{F_1^n(r,x) : n = 1,2,\dots\} = \{r, -r\} \times W\{F^n(x) : n = 1,2,\dots\}$, so is not a singleton for any $(r,x) \in C$, and is not convex or connected for $r \neq 0$.

EXAMPLE 4.2. We feel one more example is worth mentioning. This example, F_2, satisfies $(*)$, and for any x, $W\{f^n(x) : n = 1,2,\dots\}$is connected, but not convex.

Let C, K, H, f, g and F be as in the previous section. Let C_2 be the subset $[0,1] \times [0,1] \times K$ of $\mathbf{R}^2 \times H$ (with the usual norm), and let $\gamma : [0,1] \to [0,1]$ be an arbitrary continuous function.

Define $F_2 : C_2 \to C_2$ by

$$\begin{aligned} F_2(r', r, x) &= (\gamma(g(r,x)), g(r,x), f(x)) \\ &= (\gamma(g(r,x), F(r,x)). \end{aligned}$$

Since $W\{F^n(r,x) : n = 1,2,\dots\} = [0,1] \times \{0\}$, $W\{F_2^n(r,x) : n = 1,2,\dots\} = \{(\gamma(r), r) \times \{0\} : rE[0,1]\}$, which, for suitable γ, need not be convex. Finally, since F satisfies $(*)$ and γ is uniformly continuous, F_2 must satisfy $(*)$.

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