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A NOTE ON βD–D

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ABSTRACT. Let $U(\kappa)$ be the space of ultrafilters on the infinite regular cardinal κ and let SL_κ be the generalization of "Solovay's Lemma" SL_ω to κ. Our main result is to show that, assuming SL_ω, every cellular family of fewer than 2^κ open subsets of $U(\kappa)$ has a C^*–embedded selection. Further results are provided which are intended to exhibit the need for assuming SL_κ in the above result.

1. Introduction. In this paper we present some results concerning the C^*–embedding of subsets of βD–D. βD–D (also denoted D^*) denotes the remainder of the Cech–Stone compactification of the discrete space D, and a subset is said to be C^*–embedded if every bounded continuous real–valued function on the subspace can be extended to one on the entire space. For backgound on C^*–embedding and Cech–Stone compactifications the reader is referred to [**5**,**9**].

There are many known related results. Before stating some of the known results, we introduce some notation.

DEFINITION.

(i) A cellular family of subsets of a space is a family, any two members of which are pairwise disjoint.

(ii) A clopen subset is one that is both closed and open.

(iii) If κ is an infinite cardinal, then a uniform ultrafilter on a set of size κ is an ultrafilter all of whose members has cardinality κ. Given a discrete set D of size κ, the subspace of βD–D consisting of the uniform ultrafilters is denoted $U(\kappa)$. Thus, $U(\omega) = \beta N$–N.

(iv) If κ is an infinite cardinal, then Solovay's Lemma for cardinal κ(denoted SL_κ) is the following statement: Suppose λ is an infinite cardinal for which $\lambda < 2^\kappa$. Let $\{F_i\}_{i<\lambda}$ and $\{G_j\}_{j<\lambda}$ be collections of subsets of κ such that if $j < \kappa$ and S is a subset of κ for which $|S| < \kappa$, then $|G_j - \bigcup_{i\in S} F_i| = \kappa$. Then there is a set $B \subset \kappa$ such that, for any $i < \kappa$, $|B \cap F_i| < \kappa$ and $|B \cap G_i| = \kappa$.

Spaces of the form $U(\kappa)$ have been studied in detail in [**3**]. Solovay's

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Lemma (SL_ω–which is a consequence of Martin's Axiom –MA) is discussed along with Martin's Axiom in [**6,7,10**].

We now list several known results concerning the C^*–embedding in $U(\kappa)$ of unions of clopen subsets:

(1) If κ is a regular cardinal, then any union of κ clopen subsets of $U(\kappa)$ is C^*–embedded in $U(\kappa)$ [**3**];

(2) If κ is a regular cardinal, then it is consistent that $U(\kappa)$ has a pairwise disjoint family of κ^+ non–empty clopen subsets whose union is C^*–embedded. However, if MA is true, then $U(\omega)$ has no such family that is uncountable [**8**];

(3) In $U(\omega)$ there is (absolutely) a family of ω_1 pairwise disjoint non–empty clopen subsets whose union is not C^*–embedded - this can be found as an exercise in [**6**]; and

(4) If κ is an infinite cardinal with cofinality λ, then the union of λ pairwise disjoint clopen subsets of $U(\kappa)$ is C^*–embedded – the proof is similar to that of (1) above.

Known results concerning the C^*–embedding in $U(\kappa)$ of discrete subsets are:

(1) $2^\kappa < 2^{\kappa^+}$ implies that no discrete subset of size κ^+ is C^*–embedded in $U(\kappa)$ – this follows from a simple cardinality argument concerning numbers of continuous real–valued functions on $U(\kappa)$:

(2) If κ is any infinite cardinal, then there is a subset $D \subset U(\kappa)$ such that D is discrete, $|D| = \omega_1$ and D is not C^*–embedded in $U(\kappa)$ [**2**];

(3) If $\kappa \leq \lambda$ and $2^\kappa = 2^\lambda$ then $\beta\kappa$ can be mapped continuously onto $\{0,1\}^{2^\lambda}$. Since $\{0,1\}^{2^\lambda}$ contains $\beta\lambda$, and hence a C^*–embedded discrete subset of cardinality λ, it follows that $\beta\kappa$ does also. Finally, since $\beta\kappa$ can be embedded in $U(\kappa)$, it follows that $U(\kappa)$ contains a C^*–embedded discrete subset of cardinality λ. Thus, $2^\kappa = 2^\lambda$ if and only if $U(\kappa)$ contains a discrete C^*-embedded subset of cardinality λ.

For the final result that we state, we require the following definition. Given the pairwise disjoint family of clopen subsets of $U(\kappa)$, $\{A_\alpha : \alpha < \lambda\}$, the set $D = \{d_\alpha : \alpha < \lambda\}$ is called a selection for $\{A_\alpha : \alpha < \lambda\}$ if, for each $\alpha < \lambda, |D \cap A_\alpha| = 1$.

(4) From (1) it follows that if $2^\kappa < 2^{\kappa^+}$, then $U(\kappa)$ has no C^*–embedded selections of cardinality κ^+. However, in $U(\kappa)$, every selection of size $\mathrm{cf}(\kappa)$ is C^*–embedded [**3**] (in fact, in $U(\omega)$ every countable subset is C^*-embedded [**9**]).

The main results of this paper concern the relationship between Solovay's Lemma and the C^*–embedding of selections from cellular families of clopen sets in $U(\kappa)$.

2. The main results. We begin this section with a result linking Solovay's Lemma and C^*–embedding of selections from cellular families in $U(\kappa)$.

THEOREM 2.1. *Suppose that κ is an infinite regular cardinal number, and assume that SL_κ is true. If λ is a cardinal, $\kappa \leq \lambda$ and λ less than 2^κ, and if $\{A_i : i < \lambda\}$ is a cellular family of non–empty clopen subsets of $U(\kappa)$, then there is a selection D for $\{A_i : i < \lambda\}$ such that D is C^*–embedded in $U(\kappa)$.*

PROOF. Let $\{F_i : i < \lambda\}$ be the family of subsets of the discrete space of size κ such that, for each $i < \lambda, cl_{\beta\kappa}(F_i) \cap U(\kappa) = A_i$. Since the A_i's are pairwise disjoint, it follows that any pair of F_i's intersect in a set of size less than κ.

Let $\{K_t\}_{t<2^\lambda}$ be an enumeration of all subsets of λ. It was shown in [**7**], that under the assumed conditions, we have $2^\lambda = 2^\kappa$. We will define by induction on 2^λ, for each $t < 2^\lambda$, a sequence $\{\mathcal{B}_i^t\}_{i<\lambda}$ such that:

(i) For each $t < 2^\lambda$ and $i < \lambda$, $\mathcal{B}_i^t$ is a family of subsets of κ closed under finite intersections (i.e., if $S \subset \mathcal{B}_i^t$ and S is finite, then $\cap S \in \mathcal{B}_i^t$) and every member of $\mathcal{B}_i^t$ has cardinality κ;

(ii) If $i \neq j$ and $C \in \mathcal{B}_i^t, D \in \mathcal{B}_j^t$, then $|C \cap D| < \kappa$;

(iii) If $i < \lambda$ and $t < u < 2^\lambda$, then $\mathcal{B}_i^t \subset \mathcal{B}_i^u$;

(iv) For each $i < \lambda$ and $t < 2^\lambda, |\mathcal{B}_i^t| < 2^\lambda$.

For each $i < \lambda$, let $\mathcal{B}_i^0 = \{F_i\}$. Suppose that $t < 2^\lambda$ and $\{\mathcal{B}_i^s\}_{i<\lambda,s<t}$ has been defined satisfying conditions (i)-(iv). We define sets $\mathcal{U}_t$ and $\mathcal{V}_t$ as follows: $\mathcal{U}_t = \cup\{\mathcal{B}_i^s : i \in K_t, s < t\}$ and $\mathcal{V}_t = \cup\{\mathcal{B}_i^s : i \notin K_t, s < t\}$. If $C \in \mathcal{U}_t$ and $D \in \mathcal{V}_t$, then $|C \cap D| < \kappa$. Since $2^\kappa = 2^\lambda$ is a regular cardinal (this is a consequence of SL_κ and Konig's Theorem), since $|\lambda - K_t| < 2^\kappa, |K_t| < 2^\kappa, |t| < 2^\kappa$, and for each $i < \lambda, s < t, |\mathcal{B}_i^s| < 2^\kappa$, it follows that $|\mathcal{U}_t| < 2^\kappa$ and $|\mathcal{V}_t| < 2^\kappa$. Thus, $\mathcal{U}_t$ and $\mathcal{V}_t$ satisfy the requirements of SL_κ, and it follows that there is a set $X_t \subset \kappa$ such that for each $C \in \mathcal{U}_t, |X_t \cap C| = \kappa$, and for each $D \in \mathcal{V}_t, |X_t \cap D| < \kappa$. If $i \notin K_t$, we define $\mathcal{B}_i^t$ to be $\mathcal{B}_i^t = \cup_{s<t}\mathcal{B}_i^t$. If $i \in K_t$, we define $\mathcal{B}_i^t$ to be $\{X_t \cap C : C \in \cup_{s<t}\mathcal{B}_i^s\} \cup \cup_{s<t}\mathcal{B}_i^s$. The family $\{\mathcal{B}_i^t\}_{i<\kappa}$ satisfies (i)-(iv). This completes the induction, and provides the desired sequence of families of subsets of κ, $\{\mathcal{B}_i^t\}_{i<\lambda,t<2^\kappa}$.

Since, for each $i < \lambda, \cup_{t<2^\kappa}\mathcal{B}_i^t$ is a family of subsets of κ such that any finite subfamily intersects in a set of size κ, it follows that $\cap\{\mathrm{cl}_{\beta\kappa}(C) : C \in \cup_{t<2^\kappa}\mathcal{B}_i^t\} \cap U(\kappa) = G_i$ is a non–empty, compact subset of $U(\kappa)$ contained in A_i. We show that any selection for the family $\{G_i : i < \lambda\}$ is C^*–embedded in $U(\kappa)$.

Suppose that $D = \{d_i : i < \lambda\}$ is a selection for the G_i's. Then D is a discrete subset of $U(\kappa)$. In order to show that D is C^*–embedded in $U(\kappa)$, since D is discrete, then Urysohn Extension Theorem for compact zero–dimensional spaces shows that it is sufficient to verify that, for any subset $Z \subset D$, there is a clopen set $M = U(\kappa) \cap \mathrm{cl}_{\beta\kappa}(M_0)$ in $U(\kappa)$ such that $Z \subset M$ and $(D - Z) \cap M = \phi$. But since $Z \subset D$, there is a $t < 2^\lambda = 2^\kappa$ such that $Z = \{d_i : i \in K_t\}$, and hence $D - Z = \{d_i : i \notin K_t\}$. It follows from the construction of the family $\{\mathcal{B}_i^t\}$ that if $i \in K_t$, then $G_i \subset U(\kappa) \cap \mathrm{cl}_{\beta\kappa}(X_t)$, and if $i \notin K_t$, then $G_i \cap U(\kappa) \cap \mathrm{cl}_{\beta\kappa}(X_t) = \phi$. Therefore, with $M_0 = X_t, Z \subset M$ and $(D - Z) \cap M = \phi$, showing that D is C^*–embedded in $U(\kappa)$. This completes the proof of the theorem.

There are two comments to make on Theorem 2.1. The union of the C^*–embedded selections for the family $\{A_i\}_{i<\lambda}$ is dense in $\cup_{i<\lambda}A_i$. Also, note that when Solovay's Lemma was applied at stage $t(< 2^\lambda)$ to the families $\mathcal{U}_t$ and $\mathcal{V}_t$, we had that if $C \in \mathcal{U}_t$ and $D \in \mathcal{V}_t$ then $|C \cap D| < \kappa$, whereas Solovay's Lemma only requires that $|C - \cup S| = \kappa$ for $C \in \mathcal{U}_t$ and $S \subset \mathcal{V}_t$ with $|S| < \kappa$.

In contrast to Theorem 2.1, it can be shown that it is consistent with the negation of SL_κ for κ regular that every cellular family of less than 2^κ non–empty clopen subsets of $U(\kappa)$ has a C^*–embedded selection. For the case $\kappa = \omega$, the comments above show that we used Solovay's Lemma at stage t to find a set X_t such that, for each $C \in \mathcal{U}_t, X_t^* \cap C^*$ is non–empty and, for each $D \in V_t, D^* - X_t^*$ is non–empty. In the model obtained by adding more than 2^ω Cohen reals, Solovay's Lemma is destroyed. However, given fewer than 2^ω clopen subsets of ω^*, there is a Cohen real generic over this family, satisfying the requirements of X_t.

The requirement in Theorem 2.1 that the family of clopen sets must be pairwise disjoint can be dropped subject to a cardinality restriction on λ. This is shown by means of a "disjoint refinement" result. This next result also follows easily from 1.5 of [**2**].

THEOREM 2.2. *If $\lambda^+ < c(U(\kappa))$ (cellularity number of $U(\kappa)$), and if $\{A_i : i < \lambda\}$ is a family of clopen subsets of $U(\kappa)$, then $U(\kappa)$ has a* *cellular family of non–empty clopen subsets* $\{B_i : i < \lambda\}$ *such that, for each* $i < \lambda, B_i \subset A_i$ *(i.e., the* B_i*'s form a pairwise disjoint refinement of the* A_i*'s).*

PROOF. Since A_0 is non–empty and clopen in $U(\kappa)$, it is homeomorphic to $U(\kappa)$, and hence A_0 admits a family of λ^+ pairwise disjoint non–empty clopen subsets, say $\{B_{0,j} : j < \lambda^+ = I_0^0\}$. Let j_1^0 be the first $j < \lambda^+$ such that $B_{0,j_1^0} \cap A_1 \neq \phi$. Let $\{B_{1,j} : j < \lambda^+ = I_1^1\}$ be a pairwise disjoint family of non–empty clopen subsets of $B_{0,j_1^0} \cap A_1$ and let $I_0^1 = \lambda^+ - \{j_1^0\}$. If none of the $B_{0,j}$'s meet A_1 in a non–empty set, let the $B_{1,j}$'s be a pairwise disjoint family of subsets of A_1, and let $I_0^1 = \lambda^+$. In either case, any $B_{0,j} (j \in I_0^1)$ is disjoint from any $B_{1,j}$. Our objective is to inductively find, for each $t < \lambda$, a family of λ^+ pairwise disjoint clopen subsets of A_t such that any member of the family found for t_0 will be disjoint from any member of the family found for t_1.

Suppose that $t < \lambda$, and suppose that, for each $s < t$ and $r \leq s$ there is a family $\{B_{r,j} : j \in I_r^s\}$ such that

(i) $|I_r^s| = \lambda^+$:

(ii) For r fixed, the $B_{r,j}$'s form a pairwise disjoint family of subsets

(iii) If $s_1 < s_2 < t$ and r is fixed, then $I_r^{s_1} \supset I_r^{s_2}$ and $|I_r^{s_1} - I_r^{s_2}| < \lambda$; and

(iv) If $r_1 < r_2 \leq s < t$, then any $B_{r_1,j}$ is disjoint from any $B_{r_2,j'}$.

We wish to construct the family of $B_{t,j}$'s and the sets I_r^t for all $r < t$. Let r_0 be the first $r < t$ such that some $B_{r_0,j}$ meets A_t in a non–empty set. Let this index j be $j_{r_0}^t$. Let $\{B_{t,j} : j < \lambda^+ = I_t^t\}$ be a pairwise disjoint family of non–empty clopen subsets of $B_{r_0,j_{r_0}^t} \cap A_t$, and let $I_{r_0}^t = \cap_{r_0 \leq s < t} I_{r_0}^s - \{j_{r_0}^t\}$. For all $r \neq r_0$, let $I_r^t = \cap_{r \leq s < t} I_r^s$. If $B_{r,j}$ is disjoint from A_t for all $r < t$, and j, then let the $B_{t,j}$'s be any pairwise disjoint family of λ^+ non–empty clopen subsets of A_t. Also, let $I_r^t = \cap_{r \leq s < t} I_r^s$ for each $r < t$ and $I_t^t = \lambda^+$. In either case it can be clearly seen that the families $\{B_{r,j} : j \in I_r^s\}$ for $r \leq s \leq t$ (extended to $s = t$) satisfy conditions (i) to (iv) above. Therefore we can construct by induction sets I_r^t for $r \leq t < \lambda$, satisfying conditions (i) to (iv) above. Since, for each $r < \lambda, \cap_{r \leq t < \lambda} I_r^t \neq \phi$ (in fact the intersection has cardinality λ^+), let i_r be any element of that intersection. Then the family $\{B_{i_r} : r < \lambda\}$ is the disjoint refinement of the A_i's mentioned in the statement of this theorem.

Theorems 2.1 and 2.2 relate to the C^*–embedding of selections for families of clopen subsets of $U(\kappa)$. Relating to the question of C^*–embedding of unions of clopen subsets of $U(\kappa)$, it is shown in [**4**] that, for any singular cardinal κ, $U(\kappa)$ contains a family of fewer than κ pairwise disjoint clopen subsets whose union is not C^*–embedded in $U(\kappa)$. This contrasts with what was called known result (1) in the introduction, which pointed out that if κ is regular then the union of κ clopen subsets of $U(\kappa)$ is C^*–embedded in $U(\kappa)$.

For completeness we present a proof of the result obtained in [**4**] for the case $\kappa = \omega_\omega$. Let us write ω_ω as a disjoint union $\cup_{i<\omega} A_i$, where, for each $i < \omega, |A_i| = \omega_i$. Let F be any subset of ω. Define G_F to be the subset of $\omega_\omega : G_F = \cup_{i \in F} A_i$. As pointed out in the introduction to this paper (known result (3) concerning the C^*–embedding of unions of clopen sets) it is an exercise of [**6**] to show that $U(\omega)$ has a family of ω_1 disjoint clopen subsets whose union is not C^*–embedded. Let this family be $\{K_i : i < \omega_1\}$. For each $i < \omega_1, K_i$ is of the form $K_i = \mathrm{cl}_{\beta\omega} F_i - \omega$. The family $G = \{\mathrm{cl}_{\beta\omega_\omega} G_{F_i} \cap U(\omega_\omega) : i < \omega_1\} = \{K_{G_i} : i < \omega_1\}$, a pairwise disjoint family of non–empty clopen subsets of $U(\omega_\omega)$ (since the K_i's are pairwise disjoint, the F_i's have pairwise finite intersections, and hence the G_{F_i}'s have pairwise intersections with cardinality strictly less than ω_ω). We claim that the union of the members of the family G is not C^*–embedded in $U(\omega_\omega)$. This can be seen in the following way. Suppose that $\cup G$ is C^*–embedded in $U(\omega_\omega)$. Let M be any subset of ω_1. We will show that $\cup\{K_i : i \in M\}$ and $\cup\{K_i : i \in \omega_1 - M\}$ can be separated by a clopen subset of $U(\omega)$. Since $\cup G$ is C^*–embedded in $U(\omega_\omega)$, it follows that there is a set $W \subset \omega_\omega$ such that if $i \in M$, then $|G_{F_i} - W| < \omega_\omega$ and if $i \notin M$, then $|G_{F_i} \cap W| < \omega_\omega$. Define the set $U_M \subset \omega$ as follows: $n \in U_M$ if and only if $|A_n - W| < \omega_n$. Then, for each $i \in M$, $F_i - U_M$ is finite, and, for each $i \notin M$, $U_M \cap F_i$ is finite. Thus, $\mathrm{cl}_{\beta\omega} U_M$ contains $\cup\{K_i : i \in M\}$ and is disjoint from $\cup\{K_i : i \notin M\}$. Since this is true for any $M \subset \omega_1$, it implies that $\cup\{K_i : i < \omega_1\}$ is C^*–embedded in $\beta\omega$, which is a contradiction. Thus, $\cup G$ cannot be C^*–embedded in $U(\omega_\omega)$.

We complete this section of the paper by posing the following question: is it true in Theorem 2.1, that any selection for the family of A_i's is C^*–embedded in $U(\kappa)$?

In the next section we address a problem which should, perhaps, be discussed in conjunction with Theorem 2.1. However, several results are obtained that are interesting in their own right, and it seems appropriate to mention them in a distinct part of this paper.

3. Clopen families for which no selections are C^*–embedded. In contrast to Theorem 2.1, it can be shown that it is consistent with $2^\omega = 2^{\omega_1}$ that there is a family of ω_1 pairwise disjoint non–empty clopen subsets of $U(\omega)$ for which no selection is C^*–embedded. It follows from Exercise A10 on page 289 of [**6**] that it is consistent with $2^\omega = 2^{\omega_1}$ that there is a P–point x, with character ω_1 in $U(\omega)$. This point has, therefore, a descending neighborhood base of clopen sets of cardinality ω_1, say $\{B_i : i < \omega_1\}$. For each $i < \omega_1$, let $A_i = B_i - B_{i+1}$. Then no selection for the A_i's can be C^*–embedded in $U(\omega)$ (any selection for the A_i's converges to x).

In the remainder of this section we are concerned with the following question: is it consistent with $2^{\omega_1} = 2^{\omega_2}$ that there is a sequence of ω_2 pairwise disjoint clopen subsets of $U(\omega_1)$ such that no selection for this sequence is C^*–embedded in $U(\omega_1)$? This is the generalization to ω_1 of the result discussed in the preceding paragraph.

More specifically, we consider the question: is there a point $x \in U(\omega_1)$ and a sequence of clopen sets $\{A_i : i < \omega_2\}$ in $U(\omega_1)$ such that if, for each $i < \omega_2, x_i \neq y_i$ and $\{x_i, y_i\} \subset A_i$, then $x \in \operatorname{cl}\{x_i : i < \omega_2\} \cap \operatorname{cl}\{y_i : i < \omega_2\}$?

It is interesting to note that we cannot have a situation exactly analogous to that for $U(\omega)$. Indeed, in $U(\omega)$ we were able to obtain a point $x \in U(\omega)$ and clopen sets $\{A_i : i < \omega_1\}$ such that the A_i's were disjoint and each neighborhood of x contained all but countably many of the A_i's. We will show that this behaviour is not exhibited by $U(\omega_1)$.

LEMMA 3.1. *There is a family $\{B_n : n < \omega\}$ of clopen subsets of $U(\omega_1)$ such that, for each non–empty clopen subset B of $U(\omega_1)$, there is an $n < \omega$ such that $B_n \cap B \neq \phi$ and $B - B_n \neq \phi$ (i.e., the B_n's form a splitting family; the minimum size of a splitting family for $U(\omega)$ is ω_1, and it varies with the model assumed).*

PROOF. Let f be a one–to–one map of ω_1 into the unit interval. Let $B = \{f^{\leftarrow}([r, s]) : r, s \text{ rational numbers }\}$, and let $\{B_n : n < \omega\}$ be an enumeration of $\{\operatorname{cl}_{\beta\omega_1} B \cap U(\omega_1) : B \in B\}$.

COROLLARY 3.2. *$U(\omega_1)$ can be partitioned into 2^ω nowhere dense zero sets. Furthermore, if the reals contain a Luzin set, then countably many of these zero sets may have dense union in $U(\omega_1)$.*

PROOF. The map f in the proof of Lemma 3.1 can be extended to a continuous map from $\beta\omega_1$ to the unit interval. The restriction to $U(\omega_1)$, say g, of this extension is a continuous map. The range of g is a separable space, and $U(\omega_1) = \cup\{g^{\leftarrow}(x) : x \in g[U(\omega_1)]\}$. It suffices to show that $g^{\leftarrow}(x)$ is nowhere dense for each x in the unit interval.

If $\mathrm{int}(g^{\leftarrow}(x)) \neq \phi$, then there is set $A \subset \omega_1$ such that $|A| = \omega_1$ and $\mathrm{cl}_{\beta\omega_1} A \cap U(\omega_1) \subset g^{\leftarrow}(x)$. This is impossible since B_n splits this set for some $n < \omega$.

If the unit interval contains a Luzin set, say X, then f can be chosen to map one–to–one into X. In this case, if S is a countable dense subset of X, then $\cup_{s\in S} g^{\leftarrow}(s)$ is dense in $U(\omega_1)$. Indeed, if A is an uncountable subset of ω_1, then $\mathrm{int}_X(\overline{f[A]}) \neq \phi$ and therefore there is an $s \in S$ such that s is a complete accumulation point of $f[A]$. It follows that if we extend $\{f^{\leftarrow}[(s - \frac{1}{n}, s + \frac{1}{n})] : n < \omega\}$ to a uniform ultrafilter on ω_1, say p, then $p \in \mathrm{cl}_{\beta\omega_1} A \cap g^{\leftarrow}(s)$.

This result is again in contrast to the situation for $U(\omega)$. First of all, each zero set of $U(\omega)$ has dense interior and any countable union of nowhere dense subsets of $U(\omega)$ is again nowhere dense. Furthermore, it has been shown in [**1**] that MA implies that fewer than 2^ω nowhere dense subsets of $U(\omega)$ has nowhere dense union, and that $U(\omega)$ cannot be covered by 2^ω nowhere dense subsets.

PROPOSITION 3.3. *If $p \in U(\omega_1)$ and $\{A_i : i < \omega_2\}$ is a family of pairwise disjoint clopen subsets of $U(\omega_1)$, then, for some neighborhood U of p, $|\{i < \omega_2 : A_i - U \neq \phi\}| = \omega_2$.*

PROOF. Let $\{B_n : n < \omega\}$ be chosen as in Lemma 3.1. For each $i < \omega_2$, there is an $n < \omega$ such that $A_i - B_n \neq \phi$ and $A_i \cap B_n \neq \phi$. It follows that there is a set $S \subset \omega_2$ with $|S| = \omega_2$ such that, for some fixed $n < \omega, A_i - B_n \neq \phi$ and $A_i \cap B_n \neq \phi$ for all $i \in S$. Since $p \in B_n \cup (U(\omega_1) - B_n)$ we see that p has a neighborhood U as desired.

In continuing our discussion of C^*–embedded selections, we will require the following definition.

DEFINITION 3.4. $d_1 = \min\{\kappa$: there is a family $F \subset {}^{\omega_1}\omega_1$ with $|F| = \kappa$ such that, for every $g \in {}^{\omega_1}\omega_1$, there is an $f \in F$ such that $|\{\alpha : f(\alpha) \leq g(\alpha)\}| < \omega_1\}$.

It is not difficult to see that $d_1 \geq \omega_2$ and that SL_{ω_1} implies that $d_1 = 2^{\omega_1}$ (in fact see Exercise VIII A3 of [**6**]).

LEMMA 3.5. *If $d_1 = 2^{\omega_1}$, then there is a point $p \in U(\omega_1)$ and a family of pairwise disjoint clopen subsets of $U(\omega_1)$, say $\{A_i : i \in 2^{\omega_1}\}$, such that, for each neighborhood U of p, $|\{i \in 2^{\omega_1} : A_i \subset U\}| = 2^{\omega_1}$.*

PROOF. We shall prove the result for $U(\omega_1 \times \omega_1)$ which will be equivalent to the stated result. For any $\alpha \in \omega_1$ and any $g \in {}^{\omega_1}\omega_1$, let $S_{\alpha,g} = \{< \beta, \gamma >: \beta > \alpha \text{ and } \gamma > g(\beta)\}$. Let p be any ultrafilter which extends the filter generated by $\{S_{\alpha,g} : \alpha \in \omega_1, g \in {}^{\omega_1}\omega_1\}$. It is not difficult to verify that if $g \in {}^{\omega_1}\omega_1$ is given and U is any element of p, then there is an $h \in {}^{\omega_1}\omega_1$ such that $U \cap S_{\alpha,g} - S_{\alpha,h} \neq \phi$ for any α. Let $p = \{U_i : i \in 2^{\omega_1}\}$ be an indexing of the ultrafilter such that each member of the ultrafilter appears 2^{ω_1} times. Choose $g_0 \in {}^{\omega_1}\omega_1$ such that $U_0 \cap S_{\alpha,g_0} \neq \phi$ for each α. Suppose that $\beta < 2^{\omega_1}$ and we have chosen recursively $\{g_\gamma : \gamma < \beta\} \subset {\omega_1}_{\omega_1}$ such that (i) $\gamma < \xi < \beta$ implies $|\{\alpha : g_\gamma(\alpha) > g_\xi(\alpha)\}| < \omega_1$ and (ii) $\gamma + 1 < \beta$ implies $U_\gamma \cap S_{\alpha,g_\gamma} - S_{\alpha,g_{\gamma+1}} \neq \phi$ for each α.

We wish to find g_β satisfying conditions (i) and (ii). If β is a limit ordinal, we can choose g_β satisfying (i), since $d_1 = 2^{\omega_1}$. In the case that $\beta = \gamma + 1$, by the above fact, we may choose g'_β such that $U_\gamma \cap S_{\alpha,g_\gamma} - S_{\alpha,g'_\beta} \neq \phi$ for each α. It follows trivially that if we set $g_\beta = g'_\beta + g_\gamma$, then (i) and (ii) are satisfied.

When we have found $\{g_\beta : \beta \in 2^{\omega_1}\}$, we let

$$A_\beta = U(\omega_1 \times \omega_1) \cap \mathrm{cl}_{\beta\omega_1\times\omega_1}(U_\beta \cap S_{0,g_{\beta(\omega_1\times\omega_1)}} - S_{0,g_{\beta+1}}).$$

This completes the proof since each element of p was listed 2^{ω_1} times and the A_β's are pairwise disjoint.

COROLLARY 3.6. *$2^{\omega_1} = \omega_2$ implies that there is a point $p \in U(\omega_1)$ and a sequence $\{A_i : i \in \omega_2\}$ of clopen subsets of $U(\omega_1)$ such that if $\{x_i^0, x_i^1\} \subset A_i$ for $i \in \omega_2$, then $p \in cl\{x_i^t : i \in \omega_2\}$ for $t = 0, 1$.*

To remove the assumption $2^{\omega_1} = \omega_2$ from Corollary 3.6, one might start with a model of $2^{\omega_1} > \omega_2$ and introduce a sequence of clopen sets so that there is an ultrafilter satisfying the relationship in the corollary, or start with A's constructed and "blow up" 2^{ω_1} without destroying the closure property. We know, however, that adding Cohen subsets will not work. Also, the existence of a family such as in Lemma 3.1 seems to complicate the possibility of introducing a sequence with no C^*–embedded selection.

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