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STRENGTHENED MAXIMAL FUNCTION AND POINTWISE CONVERGENCE IN $\mathbf{R}^n$, II

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1. Introduction. In the first article [**1**] of this title, we introduced a new method for dealing with problems of pointwise convergence in $\mathbf{R}^n$. The simplest problem of this sort is the differentiation problem for integrals, which may be phrased as follows: Find conditions on measurable functions f and sequences of sets $\{E_k(x)\}$ which guarantee that the corresponding sequence of averages of f over each $E_k(x)$ has $f(x)$ for its limit, either at particular values of x or at almost every x. The literature on this problem is extensive and dates from Lebesgue; Guzman [**4**] gives a comprehensive survey of developments through the early seventies. Our method in [**1**] led to a precise relationship between a type of regularity condition on the sequences of sets and integrability properties of the function needed for these problems in $\mathbf{R}^n$. Here we pursue a similar program in an abstract setting. This approach not only allows us to apply our methods in other situations but also gives a more precise analysis of the special case we treated in [**1**]. It also exposes some interesting points which were previously concealed.

For the problem of almost everywhere convergence, the difficulties we encounter are purely technical. Our earlier work was based on the Hardy–Littlewood maximal function; its precise continuity properties are reflected in norm inequalities involving well–known spaces. Consequently, it seemed natural to give norm inequalities for our strengthened maximal function in [**1**]. Such estimates here would require more hypotheses than we care to assume and would involve unduly complicated norms. Instead, we construct a kernel on the multiplicative group R_+ and show that the averaged decreasing rearrangement of our strengthened maximal function is bounded by the convolutions of this kernel with the decreasing rearrangement of the original function. In specific applications, the appropriate norm inequalities in rearrange-

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ment invariant spaces follow readily by Minkowski's inequality.

For the problem of convergence at specific points, the obstacles we encounter are fundamental. In contrast to the simple and natural localization results in [**1**], there is no principle of localization here for unbounded functions. There are families of sequences of sets such that, for each $f \in L^p$, the sequence of averages of f converges to f almost everywhere, yet the exceptional set may contain points away from the support of f.

To give a concrete meaning to our general result, we examine the case when Lebesgue's differentiation theorem is replaced by the strong differentiation theorem in $\mathbf{R}^2$ proved by Jessen, Marcinkiewicz, and Zygmund [**6**]. This amounts to replacing the discs (or squares) usually employed by rectangles of arbitrary eccentricity but fixed orientation. The differences are rather striking. Indeed, it was precisely this case which motivated our general construction; it seems to contain a little bit of everything that can go wrong.

Using the methods we developed earlier [**1**, §6], we can also use our strengthened maximal function to control certain integral operators and establish almost everywhere pointwise convergence for approximate identities. Since we have no significantly new techniques to add here, these topics will be mentioned only briefly.

For the most part, we shall follow the notation of [**1**]. New terminology will be introduced as needed.

2. Strengthened maximal functions in a general setting. Let X be a locally compact Hausdorff space whose topology has a countable base, and let μ be a non–atomic σ–finite regular Borel measure on X. We assume that the topology on X has a base $\mathcal{B}$ consisting of sets of finite positive measure, and we define

$$M_{\mathcal{B}} f(x) = \sup\{\frac{1}{\mu B}\int_B f d\mu : \quad x \in B \in \mathcal{B}\} \tag{2.1}$$

for each nonnegative measurable function f on X. We further assume that there is a strictly increasing function ϕ mapping $(0,\infty)$ onto itself such that, for each measurable set $E \subset X$,

$$\mu\{x : M_{\mathcal{B}}\chi_E(x) > 1/t\} \le \phi(t)\mu E, \ 1 < \infty \tag{2.2}$$

This last inequality is commonly used to define the halo function of $\mathcal{B}$; see de Guzman [**4**], for example.

DEFINITION 2.3. For $x \in X$ and E an arbitrary measurable subset of X, set

$$R(x,t,E) = \inf\{\mu(E \sim \bigcup B_i) : x \in B_i \in \mathcal{B}, \Sigma\mu B_i \leq t\}$$

and

$$m(x,\theta,E) = \sup_{0<t<\infty} t^\theta R(x,t,E)^{1-\theta}, \quad 0 < \theta < 1.$$

Choosing $t = \mu E/2$ shows $m(x,\theta,E) \geq \mu E/2$. Note that in the definition of $R(x,t,E)$, either finite or countable subcollections of $\mathcal{B}$ can be considered.

PROPOSITION 2.4. *For each fixed E and θ, $m(\cdot,\theta,E)$ is a measurable function.*

PROOF. In the supremum used to define m, only a countable set of values of t need be considered. Thus, it suffices to prove $R(\cdot,t,E)$ is measurable for each $t > 0$.

Suppose $R(x_0,t,E) < \alpha$. Then there are sets $B_1,\cdots,B_n$ such that $x_0 \in B_i \in \mathcal{B}, \sum \mu B_i \leq t$, and $\mu(E \ \bigcup B_i) < \alpha$. We may choose the same B_i to show $\alpha > R(x,T,E)$, all $x \in B_1 \bigcap \cdots \bigcap B_n$. Thus $R(\cdot,t,E)$ is semicontinuous and hence measurable.

PROPOSITION 2.5. *Given x,θ,E with $0 < m(x,\theta,E) < \infty$, and $K > 1$, there is an open set U with $\mu(e \sim U) = 0$ and $m(x,\theta,U) < Km(x,\theta,E)$.*

PROOF. Set $\lambda = K^{1/4}$, and choose a positive integer k such that

$$1 + \sum_{j=k}^{\infty} \lambda^{-j} \leq \lambda.$$

Then define $\varepsilon_j > 0$ such that

$$\Big(\sum_{j=n-k+1}^{\infty} \varepsilon_j\Big)^{1-\theta} = (\lambda-1)\lambda^{1-n\theta} m(x,\theta,E),$$

for all integers n.

For each n, we may choose an open set V_n such that

$$V_n = B_1 \cup \cdots \cup B_m \quad (m \text{ arbitrary but finite}),$$

with $x \in B_1 \in \mathcal{B}$, each $i, \sum \mu B_i \le \lambda^n$, and $\mu(E \sim V_n)$ approximating $R(x, \lambda^n, E)$ well enough that

$$\lambda^{n\theta}\mu(E \sim V_n)^{1-\theta} < \lambda m(x, \theta, E).$$

By the regularity of μ, we can pick open sets U_n such that

$$E \bigcap V_n \subset U_n \subset V_n \text{ with } \mu U_n < \mu(E \bigcap V_n) + \varepsilon_n.$$

We then take $U = \bigcup_{n=-\infty}^{\infty} U_n$. Since $E \sim U \subset E \sim V_n$ for every n and $\mu(E \sim V_n)^{1-\theta} \le \lambda^{1-n\theta} m(x, \theta, E)$, we must have $\mu(E \sim U) = 0$. We finish the proof by showing that, for all n,

$$\lambda^{(n+2)\theta} R(x, \lambda^{n+1}, U)^{1-\theta} \le K m(x, \theta, E).$$

Let us define $W_n = V_n \bigcup (\bigcup_{j \le n-k} V_j)$. By the definition of V_n, we may then express $W_n = \bigcup B_i$ with $x \in B_i \in \mathcal{B}$ and

$$\sum \mu B_i \le \lambda^n + \sum_{j \le n-k} \lambda^j = \lambda^n (1 + \sum_{j=k}^{\infty} \lambda^{-j}) \le \lambda^{n+1}.$$

Hence

$$\begin{aligned} R(x, \lambda^{n+1}, U) \le \mu(U \sim W_n) &\le \mu(\bigcup_{j=n-k+1}^{\infty} U_j \sim V_n) \\ &\le \mu(\bigcup_{j=n-k+1}^{\infty} U_j \sim E) + \mu(E \sim V_n) \end{aligned}$$

since $(A \sim B) \subset (A \sim C) \bigcup (C \sim B)$ for any A, B, C.

Thus

$$\begin{aligned} R(x, \lambda^{n+1}, U)^{1-\theta} &\le (\sum_{j=n-k+1}^{\infty} \mu(U_j \sim E))^{1-\theta} + \mu(E \sim V_n)^{1-\theta} \\ &\le (\sum_{j=n-k+1}^{\infty} \varepsilon_j)^{1-\theta} + \lambda^{1-n\theta} m(x, \theta, E) \\ &= \lambda^{2-n\theta} m(x, \theta, E). \end{aligned}$$

Hence

$$\lambda^{(n+2)\theta} R(x, \lambda^{n+1}, U)^{1-\theta} \le \lambda^{2+2\theta} m(x, \theta, E) < K m(x, \theta, E).$$

DEFINITION 2.6. For f a nonnegative measurable function on χ and $0 < \theta < 1$, define

$$M_{\mathcal{B}}^{(\theta)} f(x) = \sup \frac{1}{m(x, \theta, E)} \int_E f \, d\mu,$$

where the supremum is taken over all measurable sets. We interpret the quantity in the supremum as zero if either $\mu E = 0$ or $m(x, \theta, E) = \infty$.

THEOREM 2.7. *$M_{\mathcal{B}}^{(\theta)} f$ is a measurable function for each nonnegative measureable f. If $f \in L^\infty(\mu)$,*

$$M_{\mathcal{B}}^{(\theta)} f(x) \le 2 ||f||_\infty^{1-\theta} (M_{\mathcal{B}} f(x))^\theta.$$

PROOF. Set $F_E(x) = \frac{1}{m(x,\theta,E)} \int_E f d\mu$. By (2.4), F_E is measurable for each E. Thus, to prove $M_{\mathcal{B}}^{(\theta)} f$ is measurable we need only show that the supremum in (2.6) need be taken only over a countable set. We have assumed that X has a countable base; let $\mathcal{U}$ denote the collection of all open sets which are finite unions of sets in this base. Then $\mathcal{U}$ is countable, and we claim

$$M_{\mathcal{B}}^{(\theta)} f(x) = \sup\{F_U(x) : U \in \mathcal{U}\}.$$

The point of (2.5) was to show $F_E(x)$ can be approximated by $F_V(x)$ with V open. But we can write $V = \bigcup U_n$ with $U_n \subset U_{n+1}$ and each $U_n \in \mathcal{U}$, and it is clear that

$$F_V(x) \le \liminf_{n \to \infty} F_{U_n}(x).$$

Now assume $f \in L^\infty(\mu)$. For $x \in B_i \in \mathcal{B}$ and $\sum \mu B_i \le t$ we have

$$\int_E f d\mu \le \sum \int_{B_i} f d\mu + \int_{E \sim B_i} f d\mu \le (\Sigma(\mu B_i) M_{\mathcal{B}} f(x) + ||f||_\infty \mu(E \sim B_i).$$

Taking the infimum over all such B_i gives

$$\int_E f\ d\mu \le tM_{\mathcal{B}}f(x) + ||f||_\infty R(x,t,E)$$
$$\le tM_{\mathcal{B}}f(x) + ||f||_\infty [t^{-\theta} m(x,\theta,E)]^{\frac{1}{1-\theta}}.$$

Choosing $t = ||f||_\infty^{1-\theta} m(x,\theta,E)(M_{\mathcal{B}}f(x))^{\theta-1}$ gives

$$\int_E f d\mu \le 2m(x,\theta,E)||f||_\infty^{1-\theta}(M_{\mathcal{B}}f(x))^{\theta}.$$

REMARK 2.8. As in [**1**; Lemma 4.4], we could also prove

$$\int_E f d\mu \le c(M_{\mathcal{B}}(f^{1/\theta})(x))^{\theta} \int_0^\infty t^{n\theta} R(x,t,E)^{1-\theta} t^{-1} dt.$$

However, without more specific estimates for $M_{\mathcal{B}}$ this gives no additional information. The estimate in (2.7) has little direct application for the same reason. We use it to estimate the decreasing rearrangement $(M_{\mathcal{B}}^{(\theta)}\chi_E)^*$ and then use this to bound the average rearrangement $(M_{\mathcal{B}}^{(\theta)}f)^{**}$ for an arbitrary function.

THEOREM 2.9. *Let* $F(t) = \min\{1, 1/\phi^{-1}(t)\}$ *and set*

$$K_\theta(t) = \int_0^1 (F(rt)^\theta - F(t)^\theta) dr, \text{ where } 0 < \theta < 1,$$

and ϕ *is the function in (2.2). Then* $0 \le K^\theta(t) \le 1$ *with* $K_\theta(t) = 0$ *if and only if* $t \le \phi^{-1}(1)$. *Moreover, for any nonnegative measurable function* f *we have*

$$M_{\mathcal{B}}^{(\theta)} f)^{**}(t) \le \int_0^\infty f^*(s) K_\theta(t/s) s^{-1} ds, \qquad 0 < t < \infty.$$

PROOF. Since ϕ is strictly increasing, F is 1 on $(0, \phi^{-1}(1)]$ and strictly decreasing on $[\phi^{-1}(1), \infty)$. This gives $K_\theta = 0$ on $(0, \phi^{-1}(1)]$ and $0 < K_\theta \le 1$ on $(\phi^{-1}(1), \infty)$.

The positivity of $K_\theta(t/s)$ for small s shows that we may assume f^* is finite-valued. We first consider $f = \chi_E, 0 < \mu E < \infty$; this is the only case where we have explicitly assumed any knowledge of $M_{\mathcal{B}} f$.

Since $M_{\mathcal{B}} \chi_E \leq 1$, (2.2) and the definition of decreasing rearrangement show

$$(M_{\mathcal{B}} \chi_E)^*(t) \leq \min\{1, 1/\phi^{-1}(t/\mu E)\} = F(t/\mu E).$$

Hence, by (2.7),

$$\begin{aligned}(M_{\mathcal{B}}^{(\theta)} \chi_E)^*(t) &\leq 2((M_{\mathcal{B}} \chi_E)^\theta)^*(t) \\ &= 2(M_{\mathcal{B}} \chi_E)^*(t)^\theta \leq 2F(t/\mu E)^\theta\end{aligned}$$

Thus

$$\begin{aligned}(M_{\mathcal{B}}^{(\theta)} (\chi_E)^{**}(t) &= \frac{1}{t} \int_0^t (M_{\mathcal{B}}^{(\theta)} (\chi_E)^*(s) ds \\ &\leq \frac{2}{t} \int_0^t F(s/\mu E)^\theta ds \\ &= \frac{2\mu E}{t} \int_0^{t/\mu E} F(s)^\theta ds.\end{aligned}$$

Next we assume f is a function such that $E_\lambda = \{x : f(x) > \lambda\}$ has $\mu E_\lambda < \infty$, all $\lambda > 0$. The general case then follows by monotone convergence. Since

$$\begin{aligned}f(x) &= \int_0^\infty \chi_{E_\lambda}(x) d\lambda \\ M_{\mathcal{B}}^{(\theta)} f(x) &\leq \int_0^\infty M_{\mathcal{B}}^{(\theta)} \chi_{E_\lambda}(x) d\lambda\end{aligned}$$

and

$$\begin{aligned}(M_{\mathcal{B}}^{(\theta)} f)^{**}(t) &\leq \int_0^\infty (M_{\mathcal{B}}^{(\theta)} \chi_{E_\lambda})^{**}(t) d\lambda \\ &\leq \frac{2}{t} \int_0^\infty \mu E \lambda (\int_0^{t/\mu E\lambda} F(s)^\theta ds) d\lambda.\end{aligned}$$

Here we interpret the entire integrand as 0 if $\mu E_\lambda = 0$; alternatively, we may regard the λ–integration as taking place only over those λ with $\mu E_\lambda > 0$. Using the definition $f^*(s) = \inf\{\lambda : \mu E_\lambda \leq s\}$ and changing the order of integration yields

$$\begin{aligned}(M_{\mathcal{B}}^{(\theta)} f)^{**}(t) &\leq \frac{2}{t} \int_0^\infty F(s)^\theta (\int_{f^*(t/s)}^\infty \mu E_\lambda d\lambda) ds \\ &= 2 \int_0^\infty F(t/s)^\theta (\int_{f^*(s)}^\infty \mu E_\lambda d\lambda) s^{-2} ds.\end{aligned}$$

Since

$$\int_{f^*(s)}^{\infty} \mu E_\lambda d\lambda = \int\int_{f^*(s)<\lambda<f^*(r)} dr d\lambda = \int_0^s (f^*(r) - f^*(s))dr,$$

we have

$$(M_{\mathcal{B}}^{(\theta)} f)^{**}(t) \leq 2\int_0^\infty F(t/s)^\theta (\int_0^s (f^*(r) - f^*(s))dr)s^{-2}ds.$$

We claim that, for $t > 0$ and for arbitrary finite–valued, nonnegative, nonincreasing functions F and f^* which vanish at infinity,

$$(*) \qquad \begin{aligned} &\int_0^\infty F(t/s)^\theta (\int_0^s (f^*(r) - f^*(s))drs^{-2}ds \\ &= \int_0^\infty (\int_s^\infty (F(t/r)^\theta - F(t/s)^\theta)r^{-2}dr)f^*(s)ds. \end{aligned}$$

We shall justify $(*)$ presently. The definition of K_θ and a change of variables gives

$$\begin{aligned} s^{-1}K_\theta(t/s) &= \frac{2}{s}\int_0^1 (F(rt/s)^\theta - F(t/s)^\theta)dr \\ &= 2\int_s^\infty (Ft/r)^\theta - F(t/s)^\theta)r^{-2}dr. \end{aligned}$$

Combining this with $(*)$ then gives

$$(M_{\mathcal{B}}^{(\theta)} f)^{**}(t) \leq \int_0^\infty K_\theta(t/s)f^*(s)s^{-1}ds.$$

We conclude the proof by demonstrating $(*)$. Let us note that we can write

$$F(s)^\theta = \sum_{n=1}^\infty \alpha_n(s) \quad \text{and} \quad f^*(s) = \sum_{n=1}^\infty \beta_n(s),$$

where α_n and β_n are bounded, nonnegative, nonincreasing functions which vanish for large s. Such decompositions are easily visualized by slicing the regions under the graphs of F^θ and f^* horizontally. For each m and n we have

$$\begin{aligned}\int_0^\infty \alpha_m(t/s)(\int_0^s \beta_n(r)dr)s^{-2}ds &= \int_0^\infty \beta_n(r)\int_r^\infty \alpha_m(t/s)s^{-2}dsdr\\ &= \int_0^\infty \beta_n(s)\int_s^\infty \alpha_m(t/r)r^{-2}drds\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty \alpha_m(t/s)(\int_0^s \beta_n(s)dr)s^{-2}ds &= \int_0^\infty \beta_n(s)\alpha_m(t/s)s^{-1}ds\\ &= \int_0^\infty \beta_n(s)\int_s^\infty \alpha_m(t/s)r^{-2}drds.\end{aligned}$$

All of these integrals are convergent, so subtracting gives

$$\begin{aligned}&\int_0^\infty \alpha_m(t/s)(\int_0^s (\beta_n(r)-\beta_n(s)dr)s^{-2}ds\\ &= \int_0^\infty \beta_n(s)\int_s^\infty (\alpha_m(t/r)-\alpha_m(t/s))r^{-2}drds.\end{aligned}$$

The monotonicity of α_m and β_n shows that these integrands are nonnegative. Summing with respect to m and then n gives $(*)$ by monotone convergence.

3. Problems of almost everywhere convergence. First we consider the differentiation problem for indefinite integrals. For convenience, we restrict our attention to nonnegative functions; this clearly suffices. We take θ to be a fixed parameter in $(0,1)$. First we define the class of sequences of sets we will consider.

DEFINITION 3.1. We say a sequence of sets $\{E_k\}$ is a *differentiating sequence* at x if for each $k, 0 < \mu E_k < \infty$, and, for each open set U containing x,

$$\lim_{k\to\infty} \frac{\mu(E_k \sim U)}{\mu E_k} = 0.$$

We say a differentiating sequence at x is θ-regular if there is a constant N such that, for all $k, m(x,\theta,E_k) \leq N\mu E_k$.

As in [**1**], we see that if f is a bounded continuous function on X and $\{E_k\}$ is a differentiating sequence at x, then

$$\lim_{k\to\infty} \frac{1}{\mu E_k} \int_{E_k} f d\mu = f(x).$$

THEOREM 3.2. *Suppose $\int_0^\infty K_\theta(t_0/s) f^*(s) s^{-1} ds < \infty$ for some $t_0 > 0$. Then for all x outside an exceptional set of measure zero,*

$$\lim_{k\to\infty} \frac{1}{\mu E_k} \int_{E_k} f d\mu = f(x),$$

for every θ-regular differentiating sequence at x.

PROOF. If f vanishes almost everywhere, there is nothing to be proved, so we may as well assume the hypothesis is satisfied for some nontrivial f. Since $K_\theta(t_0/s)$ is bounded and vanishes for large s, we then have

$$\int_0^\infty K_\theta(t_0/s) s^{-1} ds < \infty.$$

Let $B_{\varepsilon,N}(f)$ denote the set of all x such that, for some differentiating sequence $\{E_k\}$ at x with $m(x,\theta,E_k) \leq N\mu E_k$, all k, we have

$$\lim_{k\to\infty} \sup \frac{1}{\mu E_k} \int_{E_k} |f(y) - f(x)| d\mu(y) \geq \varepsilon.$$

As in [**1**], it suffices to prove $\mu B_{\varepsilon,N}(f) = 0$ for all ε and $N > 0$.

For any finite-valued measurable function g we have

$$\begin{aligned}
&\frac{1}{\mu E_k} \int_{E_K} |f(y) - f(x)| d\mu(y) \\
&\leq \frac{1}{\mu E_k} \int_{E_k} \{|g(x) - f(x)| + |f(y) - g(y)| + |g(y) - g(x)|\} d\mu \\
&\leq |g(x) - f(x)| + N M_{\mathcal{B}}^{\theta} |f - g|(x) + \frac{1}{\mu E_k} \int_{E_k} |g(y) - g(x)| d\mu.
\end{aligned}$$

Thus

$$\begin{aligned}
\mu B_{\varepsilon,N}(f) \leq &\mu\{x : |g(x) - f(x)| \geq \varepsilon/3\} \\
&+ \mu\{x : M_{\mathcal{B}}^{(\theta)} |f - g|(x) > \varepsilon/3N\} + \mu B_{\varepsilon/3,N}(g).
\end{aligned}$$

Note that for any F, $F^*(t) < \delta$ implies $\mu\{x : |F(x)| \geq \delta\} \leq t$. Hence it suffices to show there is a sequence $\{g_n\}$ such that

(i) $\mu B_{\varepsilon,N}(g_n) = 0$, all ε, N, n.

(ii) $(f - g_n)^*(t) \to 0$, all $t > 0$.

(iii) $(M_{\mathcal{B}}^{\theta}|f - g_n|)^*(t) \to 0$, all $t > 0$.

For any F, t, and $t_0, F^*(t) \leq F^{**}(t_0) \cdot \max(1, t_0/t)$. Consequently, whenever we can invoke dominated convergence we see that (2.9) and (ii) imply (iii).

First we suppose $f = \chi_E$, where $0 < \mu E < \infty$. Since μ is a regular Borel measure, for each n there is a compact set K and an open set U such that $K \subset E \subset U$ and $\mu(U \sim K) < 1/n$. Hence, by Urysohn's lemma, there is a continuous bounded function g_n such that $|\chi_E - g_n|^* \leq \chi_{(0,1/n)}$; the sequence $\{g_n\}$ then satisfies (i), (ii), and (iii)).

Next we remove the restriction $\mu E < \infty$. Since μ is a σ-finite regular Borel measure, $X = \bigcup V_n$ where each V_n is open and satisfies $\mu V_n < \infty$. If $x \in V_n$ and $\{E_k\}$ is any differentiating sequence at x, we have

$$\begin{aligned} 0 &\leq \frac{1}{\mu E_k} \int_{E_k} (|\chi_E(y)| - \chi_E(x)| - |\chi_{E \bigcap V_n}(y) - \chi_{E \bigcap V_n}(x)|) d\mu(y) \\ &\leq \frac{\mu(E_k \sim V_n)}{\mu E_k} \to 0 \quad \text{as } k \to \infty. \end{aligned}$$

Hence $\mu B_{\varepsilon,N}(\chi_E) \leq \sum_{n=1}^{\infty} \mu B_{\varepsilon,N}(\chi_{E \bigcap V_n}) = 0$ for all measurable sets E, and for all $N, \varepsilon > 0$.

If g is a simple function (not assumed to be integrable), then $g = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ implies

$$\frac{1}{\mu E_k} \int_{E_k} |g(y) - g(x)| du \leq \sum_{i=1}^{n} \frac{|\alpha_i|}{\mu E_k} \int_{E_k} |\chi_{A_i}(y) - \chi_{A_i}(x)| d\mu$$

so that $\mu B_{\varepsilon,N}(g) = 0$ in this case also.

In general, any nonnegative measurable function f is the limit of an increasing sequence of simple functions $\{g_n\}$; moreover, the finiteness of $f^*(t)$ implies (ii). Since in this case, $0 \leq (f - g_n)^*(t) \leq f^*(t)$, (iii) is satisfied as well and the proof is complete.

Now we turn our attention to problems involving integral operators on X. For each ς in some set of parameters, let $G_\varsigma(x, y)$ be a nonnegative measurable function on $X \times X$. For f a nonnegative measurable function on X, let us set $T_\varsigma f(x) = \int G_\varsigma(x,y)f(y)d\mu(y)$ and $\tilde{T}f(x) = \sup_\varsigma T_\varsigma f(x)$. Calling $E_\varsigma(x,\lambda) = \{y \in X : G_\varsigma(x,y) > \lambda\}$, we assume $w_\theta(x) = \sup_\varsigma \int_0^\infty m(x,\theta,E_\varsigma(x,\lambda))d\lambda$ defines a measurable function which is finite almost everywhere.

THEOREM 3.3. *For $\tilde{T}$ and w_θ as defined above and K_θ as in Theorem 3.2, we have*

$$(\tilde{T}f)^{**}(t) \le \frac{1}{t}\int_0^\infty \Big(\int_0^t w_\theta^*(s)K_\theta(s/\tau)ds\Big)f^*(\tau)\tau^{-1}d\tau.$$

PROOF. As in [**1**, §6] we may write

$$\begin{aligned} T_\varsigma f(x) &= \int_0^\infty \Big(\int_{E_\varsigma(x,\lambda)} f d\mu\Big)d\lambda \\ &\le \int_0^\infty m(x,\theta,E_\varsigma(x,\lambda))M_{\mathcal{B}}^{(\theta)}f(x)d\lambda \\ &\le w_\theta(x)M_{\mathcal{B}}^{(\theta)}f(x). \end{aligned}$$

Since this last estimate is independent of ς, it must also hold for $\tilde{T}f(x)$. We may now estimate $(\tilde{T}f)^{**}(t)$ by estimating its integral over sets of finite measure, making use of a fundamental ineqaulity for products given in Hunt [**5**].

$$\begin{aligned} (\tilde{T}f)^{**}(t) &= \sup_{|E|\le t}\frac{1}{t}\int_E \tilde{T}f(x)d\mu(x) \\ &\le \sup_{|E|\le t}\frac{1}{t}\int_E w_\theta(x)M_{\mathcal{B}}^{(\theta)}f(x)d\mu(x) \\ &\le \frac{1}{t}\int_0^t w_\theta^*(s)(M_{\mathcal{B}}^{(\theta)}f)^*(s)ds \\ &\le \frac{1}{t}\int_0^t w_\theta^*(s)\Big(\int_0^\infty K_\theta(s/\tau)f^*(\tau)\tau^{-1}d\tau\Big)ds \end{aligned}$$

by (2.9) and the fact $F^* \le F^{**}$ for all F. Changing the order of integration concludes the proof.

Assuming pointwise almost everywhere convergence of $T_\varsigma(fx)$ as $\varsigma \to \varsigma_0$ holds for an appropriately dense class of functions, we can extend this result to all f such that the integral in (3.3) is finite. We omit the details.

4. Examples. First we consider a simple case, equivalent to that of [**1**]. We take $X = \mathbf{R}^n$, μ =Lebesgue measure, and we let $\mathcal{B}$ be the collection of all open cubes in $\mathbf{R}^n$ with edges parallel to the coordinate axes. In this case $\phi(t) = ct$, and so $F(t) = \min(1, c/t)$. A straightforward calculation then gives

$$K_\theta(t) = \frac{2\theta}{1-\theta}((c/t)^\theta - (c/t)), \quad t \geq c$$

with K_θ vanishing on $(0, c]$. Thus, in this case,

$$(M_{\mathcal{B}}^{(\theta)} f)^{**}(t) \leq \frac{2\theta}{1-\theta} \int_0^{t/c} ((\frac{cs}{t})^\theta - (\frac{cs}{t})) f^*(s) s^{-1} ds.$$

Dropping the negative term and simplifying,

$$t^\theta (M_{\mathcal{B}}^{(\theta)} f)^{**}(t) \leq \frac{2\theta c^\theta}{1-\theta} \int_0^{t/c} s^{\theta-1} f^*(s) ds.$$

This shows $M_{\mathcal{B}}^{(\theta)}$ is continuous from the Lorentz space $L(1/\theta, 1)$ to $L(1/\theta, \infty)$ as in [**1**], but with a better estimate as $t \to 0$.

In the above example, $M_{\mathcal{B}} f$ was one of the standard versions of the Hardy-Littlewood maximal function. Now we replace it by the strong maximal function of Jessen, Marcinkiewicz, and Zygmund [**6**]; for simplicity we take $n = 2$ so that $\mathcal{B}$ is the collection of all open bounded rectangles in the plane with edges parallel to the coordinate axes. The estimate

$$|\{x : M_{\mathcal{B}} f(x) > \lambda\}| \leq c \int \frac{|f|}{\lambda}(1 + \log_+ \frac{|f|}{\lambda})$$

is given in Cordoba and Fefferman [**3**]; a somewhat different and more detailed proof was obtained by the author [**2**] recently. While this estimate suggests taking $\phi(t) = ct(1 + \log_+ t)$, this renders $\phi^{-1}(t)$ intractible. Instead, we choose a larger function

$$\phi(t) = \psi^{-1}(t), \quad \text{with} \quad \psi(t) = \frac{t}{c}[1 + \log_+ \frac{t}{c}]^{-1}.$$

Thus $F(t) = \min\{1, \frac{c}{t}(1 + \log_+ \frac{t}{c})\}$.

We see $F(t) = 1$ precisely on $(0, c]$, so the support of K_θ is $[c, \infty]$. For $t > c$, we see

$$\begin{aligned}
\int_0^1 F(rt)^\theta dr &= \frac{c}{t} + \int_{c/t}^1 (\frac{c}{rt}(1 + \log \frac{rt}{c}))^\theta dr \\
&= \frac{c}{t} + \frac{c}{t} \int_1^{t/c} (\frac{1}{r}(1 + \log r))^\theta dr \\
&\leq \frac{c}{t} + \frac{c}{t}(1 + \log \frac{t}{c})^\theta \int_0^{t/c} r^{-\theta} dr \\
&= \frac{c}{t} + \frac{1}{1-\theta} F(t)^\theta.
\end{aligned}$$

Thus

$$\begin{aligned}
K_\theta(t) &= 2\int_0^1 (F(rt)^\theta - F(t)^\theta) dr \\
&\leq 2(\frac{c}{t} + \frac{\theta}{1-\theta} F(t)^\theta) \\
&\leq \frac{2}{1-\theta} F(t)^\theta = \frac{2}{1-\theta}(\frac{c}{t}(1 + \log \frac{t}{c}))^\theta, \qquad t > c.
\end{aligned}$$

Hence our estimate for $M_B^{(\theta)} f$ is

$$(M_B^{(\theta)} f)^{**}(t) \leq \frac{2}{1-\theta} \int_0^{t/c} (\frac{cs}{t}(1 + \log \frac{t}{cs}))^\theta f^*(s) s^{-1} ds.$$

For $\theta > 1/p$, Minkowski's inequality and the definition of Lorentz space norms as in [**1**] gives

$$||M_B^{(\theta)} f||_{p,q} \leq c_{p,\theta} ||f||_{p,q}^*$$

with

$$c_{p,\theta} = \frac{2c^{1/p}}{1-\theta} \int_1^\infty s^{\frac{1}{p}-\theta}(1 + \log s)^\theta s^{-1} ds.$$

Choosing $p = 1/\theta$ gives no result in terms of Lorentz space norms, but the hypotheses of our differentiation theorem are satisfied for all f such that

$$\int_0^1 (s \log \frac{1}{s})^\theta f^*(s) s^{-1} ds < \infty.$$

This particular choice of $\mathcal{B}$ is useful for studying certain two-parameter approximate identities in the plane. For $\varsigma = (\varsigma_1, \varsigma_2)$ with $\varsigma_i > 0$, define $A_\varsigma : \mathbf{R}^2 \to \mathbf{R}^2$ by $A_\varsigma x = (\varsigma_1 x_1, \varsigma_2 x_2)$. We then have

$$m(0, \theta, A_\varsigma E) = \varsigma_1 \varsigma_2 m(0, \theta, E)$$

for all measurable sets $E \subset \mathbf{R}^2$. Consequently, we can use the technique of Theorem 3.3 to bound

$$T_\varsigma f(x) = \varsigma_1 \varsigma_2 \int g(A_\varsigma x - A_\varsigma y) f(y) dy$$

independently of ς, assuming the level sets E_λ of g satisfy

$$\int_0^\infty m(0, \theta, E_\lambda) d\lambda < \infty.$$

We conclude with an example which shows that if f is unbounded, the exceptional set in Theorem 3.2 may contain points away from the support of f. Of course, if f is bounded this is impossible; our definition of differentiating sequence precludes this. Our example involves a function in $L^p(\mathbf{R}^2)$ for all finite p, but it could be modified to satisfy any growth condition weaker than boundedness.

Define $f(x) = f(x_1, x_2)$ to be x_1 if $x_1 > 0$ and $0 < x_2 < \exp(-x_1)$, with $f(x) = 0$ otherwise. We then have

$$||f||_p = \Gamma(p+1)^{1/p}, \qquad 1 \le p < \infty.$$

We shall show that the exceptional set for f contains the origin; in fact, it contains the entire x_1 axis.

Define $E_k = ((0, k^a e^{-k/2}) \times (0, k^a e^{-k/2})) \bigcup ((0, k) \times (0, e^{-k}))$ where $1 < 2a < 2$. Then, for all k, we have

$$k^{2a} e^{-k} \le |E_k| \le 2k^{2a} e^{-k}.$$

Given any neighborhood of the origin, the square part of E_k will eventually lie inside it. Thus,

$$\lim \frac{|E_k \sim U|}{|E_k|} \le \lim \frac{k e^{-k}}{k^{2a} e^{-k}} = 0.$$

Since $R(0,t,E_k) \le |E_k|$ and vanishes for $t \ge 2k^{2a}e^{-k}$, we see $m(0,\theta,E_k) \le 2|E_k|$, all $k, 0 < \theta < 1$. Hence $\{E_k\}$ is a θ–regular differentiating sequence at the origin. But we have

$$\frac{1}{|E_k|}\int_{E_k} f(x)dx \ge \frac{1}{2k^{2a}e^{-k}}\int_0^k e^{-k}x_1 dx_1$$
$$= \frac{1}{4}k^{2-2a} \to \infty.$$

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