ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 17, Number 3, Summer 1987

ON SUMS OF SIXTEEN SQUARES

JOHN A. EWELL

ABSTRACT. The author shows that the function r_{16}, which counts the totality of representations of a natural number by sums of sixteen squares, is expressible entirely in terms of real divisor–functions.

1. The main result. It is the purpose of this paper to prove the following formula for the number $r_{16}(n)$ of ways of representing a positive integer n by sums of sixteen squares:

(1)

$$\begin{aligned} r_{16}(n) = \frac{32}{17}\big[\sigma_7(n) - 2\sigma_7(n/2) + 2^8\sigma_7(n/4) \\ + (-1)^{n-1}16\big(2^{3b(n)}\sigma_3(0(n)) \\ + 16\sum_{d=1}^{n-1}(-1)^d d^3 \sum_{k=1} 2^{3b(n-kd)}\sigma_3(0(n-kd))\big)\big], \end{aligned}$$

where for positive integers $r, m, \sigma_r(m)$ denotes the sum of the rth powers of all positive divisors of m, otherwise $\sigma_r(x) := 0; b(n)$ denotes the exponent of the highest power of 2 dividing n; and, $0(n)$ is then defined by the equation $n = 2^{b(n)}0(n)$. (By convention the sum on the right side of (1), indexed by k, extends over all positive integral values of k for which $n - kd > 0$.)

Proof of (1): We, first of all, recall that the modular function f is defined on the open unit disk of the complex plane (i.e. $x \in C\ |x| < 1$) by:

$$f(x) = x^{1/24}\prod_{1}^{\infty}(1 - x^n).$$

Received by the editor on September 27, 1983 and in revised form on August 1, 1985.

Our point of departure is then the following statement of Van der Pol [**4**, p. 359].

$$r_{16}(n) = \frac{32}{17}[\sigma_7(n) - 2\sigma_7(n/2) + 2^8\sigma_7(n/4) \\ + \text{coeff. of } x^n \text{ in } 16f^8(x^2)f^8(-x)].$$

(Here it is tacitly assumed that $n > 0$, since $r_{16}(0) = 1$.) What identity (1) accomplishes is a closed-form expression for the coefficient of x^n in $16f^8(x^2)f^8(-x)$. Our argument is further based on the following three identities, each of which is valid for each complex number x such that $|x| < 1$.

(2) $$\prod_1^\infty (1 - x^{2n})(1 + x^{2n-1})^2 = \sum_{-\infty}^\infty x^{n^2},$$

(3) $$\prod_1^\infty \frac{1 - x^{2n}}{1 - x^{2n-1}} = \sum_0^\infty x^{n(n+1)/2},$$

(4) $$x\Big(\sum_0^\infty x^{n(n+1)/2}\Big)^8 = \sum_1^\infty \frac{n^3 x^n}{1 - x^{2n}}.$$

Identities (2) and (3), due to Gauss, are now classical, e.g., see [**3**, p. 282-284]. Identity (4) is not as familiar as the other two; see [**5** p.144]. Identity (2) is especially important for our discussion; for, the eighth power of the right (hence also the left) side of (2) generates $r_8(n)$, the number of representations of a nonnegative integer n by sums of eight squares.

We temporarily suppress the factor 16 in Van der Pol's statement, and write:

$$
\begin{aligned}
&f^8(x^2)f^8(-x)\\
&\quad = (x^{1/12}\prod_1^\infty(1-x^{2n}))^8((-x)^{1/24}\prod_1^\infty(1-(-x)^n))^8\\
&\quad = x\prod_1^\infty(1-x^{2n})^{16}(1+x^{2n-1})^8\\
&\quad = x\prod_1^\infty\frac{(1-x^{2n})^8}{(1+x^{2n-1})^8}\cdot\prod_1^\infty(1-x^{2n})^8(1+x^{2n-1})^{16}\\
&\quad = x(\sum_0^\infty(-x)^{n(n+1)/2})^8\cdot\sum_0^\infty r_8(n)x^n \qquad ((\text{by } (2) \text{ and } (3))\\
&\quad = -\sum_1^\infty\frac{n^3(-x)^n}{1-(-x)^{2n}}\cdot\sum_0^\infty r_8(n)x^n \qquad (\text{by } (4)).
\end{aligned}
$$

But,

$$
\begin{aligned}
\sum_1^\infty\frac{n^3x^n}{1-x^{2n}} &= \sum_{n=1}^\infty n^3x^n\sum_{k=0}^\infty x^{2nk} = \sum_{n=1}^\infty\sum_{k=0}^\infty n^3x^{n(2k+1)}\\
&= \sum_{m=1} x^m \sum_{\substack{d\mid m\\ d \text{ odd}}} (m/d)^3 = \sum_{m=1}^\infty 2^{3b(m)}\sigma_3(0(m))x^m.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&f^8(x^2)f^8(-x)\\
&\quad = -\sum_{i=1}^\infty s^{3b(i)}\sigma_3(0(i))(-x)\sum_{j=0}^\infty r_8(j)x^j\\
&\quad = \sum_{n=1}^\infty(-1)^{n-1}x^n\sum_{j=0}^{n-1}2^{3b(n-j)}\sigma_3(0(n-j))(-1)^j r_s(j).
\end{aligned}
$$

To eliminate $r_s(j)$ from this expression, we use the well–known formula [**3** p. 314]:

$$
r_8(n) = 16(-1)^n\sum_{d\mid n}(-1)^d d^3 \qquad (n \in Z^+).
$$

First, however, we define $\varepsilon(j,d)$ for $j \in Z^+$ and $d \in \{1,2,...,j\}$ by:

$$\varepsilon(j,d) = \begin{cases} 1, & \text{if } d|j, \\ 0, & \text{if } d \nmid j. \end{cases}$$

Then,

$$\begin{aligned}
&\sum_{j=0}^{n-1} 2^{3b(n-j)}\sigma_3(0(n-j))(-1)^j r_8(j) \\
&= 2^{3b(n)}\sigma_3(0(n)) + 16\sum_{j=1}^{n-1}\sum_{d=1}^{j} 2^{3b(n-j)}\sigma_3(0(n-j))\varepsilon(j,d)(-1)^d d^3 \\
&= 2^{3b(n)}\sigma_3(0(n)) + 16\sum_{d=1}^{n-1}(-1)^d d^3 \sum_{j=d}^{n-1} \varepsilon(j,d) 2^{3b(n-j)}\sigma_3(0(n-j)) \\
&= 2^{3b(n)}\sigma_3(0(n)) + 16\sum_{d=1}^{n-1}(-1)^d d^3 \sum_{k=1} 2^{3b(n-kd)}\sigma_3(0(n-kd)).
\end{aligned}$$

Hence, the coefficient of x^n in $16f^8(x^2)f^8(-x)$ is:

$$16(-1)^{n-1}(2^{3b(n)}\sigma_3(0(n)) + 16\sum_{d=1}^{n-1}(-1)^d d^3 \sum_{k=1} 2^{3b(n-kd)}\sigma_3(0(n-kd))).$$

2. Concluding remarks. The author has also established the following result.

THEOREM. *For each nonnegative integer m,*

$$\begin{aligned}
r_{12}(2m+1) &= 8\sigma_5(2m+1) + 16(\sigma_1(2m+1) \\
&\quad + 16\sum_{d=1}^{m}(-1)^d d^3 \sum_{k=1} \sigma_1(2m-2kd+1)), \\
r_{12}(2m+2 &= 8(\sigma_5(2m+2) - 64\sigma_5((m+1)/2)).
\end{aligned}$$

Following the notation of Hardy [**2**, p.136], we write

$$r_{2s}(n) = \delta_{2s}(n) + e_{2s}(n) \quad (s, n \in Z^+),$$

where $r_{2s}(n)$ denotes the cardinality of the set

$$\{(x_1, x_2, ...x_{2s}) \in Z^{2s} | x_1^2 + x_2^2 + ... + x_{2s}^2 = n\},$$

$\delta_{2s}(n)$ is a divisor–function and $e_{2s}(n)$ is much smaller than $\delta_{2s}(n)$ for large n, so that

$$r_{2s}(n) \sim \delta_{2s}(n)$$

when n tends to infinity. Jacobi studied the functions r_{2s} for $2s = 2, 4, 6, 8$, and showed for these cases $e_{2s}(n) = 0$, for each positive integer n. (These results are now part of the folklore.) According to Hardy and Wright [**3**, p.316], Liouville gave the formulas for r_{10} and r_{12}. Glaisher [**1**, p.479-490] studied r_{2s} up to $2s = 18$, and Ramanujan [**5**, p.157-162] continued Glaisher's table up to $2s = 24$. Up to the present time most workers in this field have held the view "whenever $s > 4, e_{2s}(n)$ cannot for all values of n be expressed entirely in terms of real divisors." (Here, the quoted statement means that for some value of n and some $k \in \{1, 2, ..., n\}$, complex divisors of k are required to express $e_{2s}(n)$.) Our results contradict this view for $s = 6, 8$.

REFERENCES

1. J.W.L. Glaisher, *On the number of representations of a number as a sum of 2r squares, where 2r does not exceed eighteen,* Proc. London Math. Soc. (2), **5** (1907), 479-490.

2. G.H. Hardy, *Ramanujan,* Chelsea, New York, 1959.

3. G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers,* 4th ed., Clarendon Press, Oxford, 1960.

4. Balth. Van der Pol, *The representation of numbers as sums of eight, sixteen, and twenty-four squares,* Proc. Kon. Nederl. Akad. Wetensch. Ser A57. Indagationes Math, **16** (1954), 349-361.

5. Ramanujan, *Collected papers,* Chelsea, New York, 1962.

DEPARTMENT OF MATHEMATICAL SCIENCES NORTHERN ILLINOIS UNIVERSITY DEKALB, ILLINOIS 60115