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SUPERNORMAL CONES AND FIXED POINT THEORY

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1. Recently, we defined in [**5**] an interesting class of convex cones imposed by the theory of Pareto optimization, by the study of critical points of dynamical systems and by the study of conical support points. This convex cone was called, in [**5**], "nuclear cone" since every normal cone in a nuclear space is a nuclear cone in our sense. We also note that in [**8**], this cone was called "supernormal" cone, since we observed that the nuclear cone, by its properties, seems to be a reinforcement of concept of normal cone. We adopt in this note the concept of supernormal cone. Applications of supernormal cones may be found in [**5, 6, 7, 8**].

In this paper, we give an interesting application of supernormal cones to fixed point theory.

2. We will use the concept of locally convex space defined by Treves [**14**], that is, a couple $(E, \mathrm{Spec}(E))$, where E is a real vector space and $\mathrm{Spec}(E)$ is a set of seminorms on E such that

(1°) $\forall \lambda \in \mathbf{R}_+$ and $\forall p \in \mathrm{Spec}(E)$, $\lambda p \in \mathrm{Spec}(E)$;

(2°) If $p \in \mathrm{Spec}(E)$ and q is a seminorm on E such that $q \leq p$, then $q \in \mathrm{Spec}(E)$; and

(3°) $\forall p_1, p_2 \in \mathrm{Spec}(E)$, $\sup(p_1, p_2) \in \mathrm{Spec}(E)$, where $\sup(p_1,p_2)(x) = \sup(p_1(x), p_2(x))$, for any $x \in E$.

If $\mathrm{Spec}(E)$ is given, then there exists a locally convex topology τ on E such that $E(\tau)$ is a locally convex vector space and a seminorm p on E is τ-continuous if and only if $p \in \mathrm{Spec}(E)$.

A subset $\mathcal{B} \subset \mathrm{Spec}(E)$ is called a basis of $\mathrm{Spec}(E)$ if and only if, for every $p \in \mathrm{Spec}(E)$, there exists $q \in \mathcal{B}$ and a real number $\lambda > 0$ such that $p \leq \lambda q$. We suppose that the $\mathrm{Spec}(E)$ has a Hausdorff basis, that is, $\ker \mathcal{B} = \{0\}$, where

$$\ker \mathcal{B} = \{\, x \in E | p(x) = 0, \forall p \in \mathcal{B} \,\}.$$

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For our terminology on convex cones, we refer to [**10,12**].

If $(E, \mathrm{Spec}E)$ is a locally convex space, we denote by E' the topological dual of E.

A subset $\mathbf{K} \subset E$ is called a convex cone if,

(i) $\mathbf{K} + \mathbf{K} \subset \mathbf{K}$,

(ii) $\forall \lambda \in \mathbf{R}_+, \lambda \mathbf{K} \subset \mathbf{K}$.

If $\mathbf{K} \subset E$ is a convex cone, we denote by $\mathbf{K}'$ the dual cone of $\mathbf{K}$ with respect to the duality $< E, E' >$, that is, $\mathbf{K}' = \{ y \in E' | < x, y > \geq 0 : \forall x \in \mathbf{K} \}$.

Let τ be the locally convex topology defined on E by the set $\mathrm{Spec}(E)$. We say that the convex cone $\mathbf{K} \subset E$ is normal (with respect to τ), if one of the following equivalent assertions are satisfied:

(n_1) There exists a basis $\mathcal{B}$ of $\mathrm{Spec}(E)$ such that $\forall p \in \mathcal{B}$ and $\forall x, y \in \mathbf{K}$, $x \leq y \Rightarrow p(x) \leq p(y)$,

(n_2) if $\{x_i\}_{i \in I}, \{y_i\}_{i \in I}$ are two nets of $\mathbf{K}$ such that $\forall i \in I$, $0 \leq x_i \leq y_i$ and $\lim_{i \in I} y_i = 0$, then $\lim_{i \in I} x_i = 0$. For other properties of normal cones see [**10, 12**].

DEFINITION 1. A convex cone $\mathbf{K} \subset E$ is called τ-supernormal (or nuclear) if and only if there exists a basis $\mathcal{B}$ of $\mathrm{Spec}(E)$ such that

$$(1) \qquad \forall p \in \mathcal{B},\ \exists f_p \in \mathbf{K}' \text{ such that } \forall x \in \mathbf{K},\ p(x) \leq f_p(x).$$

PROPOSITION 1. *If $(E(\tau), Spec(E))$ is a Hausdorff locally convex space, then any τ-supernormal convex cone $\mathbf{K} \subset E$ is a τ- normal cone.*

PROOF. Indeed, let $\{x_i\}_{i \in I}, \{y_i\}_{i \in I}$ be two nets of $\mathbf{K}$, such that, for any $i \in I$, $0 \leq x_i \leq y_i$, and $\lim_{i \in I} y_i = 0$. Since $\mathbf{K}$ is a τ-supernormal cone, there exists a basis $\mathcal{B}$ of $\mathrm{Spec}(E)$ satisfying formula (1), and we have

$$0 \leq p(x_i) \leq f_p(x_i) \leq f_p(y_i),$$

for any $p \in \mathcal{B}$ and $i \in I$. Now, since $\lim_{i \in I} f_p(y_i) = 0$, we obtain that $\lim_{i \in I} p(x_i) = 0$, and, because $\mathcal{B}$ is a basis of $\mathrm{Spec}(E)$, we have $\lim_{i \in I} x_i = 0$, that is, $\mathcal{B}$ is a τ-normal cone.

PROPOSITION 2. *If $(E(\tau), Spec(E))$ is a Hausdorff locally convex space, then any τ-normal convex cone $\mathbf{K} \subset E$ is $\sigma(E, E')$-supernormal.*

PROOF. If p is a seminorm $\sigma(E, E')$-continuous, then there exists a constant $C > 0$ and $f_1, f_2, \dots, f_n \in E'$ such that $p(x) \leq$

$C \sup_{i=1}^n (|f_i|(x))$ for any $x \in E$.

Since $\mathbf{K}$ is a τ-normal cone, there exist for every $i = 1, 2, \ldots, n$, h_i and $g_i \in \mathbf{K}'$ such that $f_i = h_i - g_i$; hence, for any $x \in \mathbf{K}$, we have

$$p(x) \le C \sup_{i=1}^{n} (|f_i|(x)) \le C \sup_{i=1}^{n} ((|h_i| + |g_i|)(x))$$
$$= C \sup_{i=1}^{n} (h_i(x) + g_i(x)) \le C \sum_{i=1}^{n} (h_i + g_i)(x),$$

that is, $\mathbf{K}$ is $\sigma(E, E')$-supernormal.

COROLLARY. *A convex cone* $\mathbf{K} \subset E$ *is* $\sigma(E, E')$*-supernormal if and only if it is* $\sigma(E, E')$*-normal.*

EXAMPLES. (1°). A convex cone $\mathbf{K} \subset E$ is called "well-based" if there exists a closed, convex, bounded set $A \subset E$ such that

(b_1) $0 \notin \overline{A}$;

(b_2) $\mathbf{K} = \bigcup_{\lambda \in \mathbf{R}_+} \lambda A$.

In any locally convex space $E(\tau)$, any well based convex cone is τ-supernormal [**5**].

(2°). In a normed space $(E, \|\|)$, a convex cone $\mathbf{K} \subset E$ is supernormal if and only if it is well based.

(3°). In a locally convex space $(E(\tau), \mathrm{Spec}(E))$, a locally compact (or weakly locally compact) convex cone $\mathbf{K} \subset E$ is τ-supernormal.

(4°). In a nuclear space $(E(\tau), \mathrm{Spec}(E))$, a convex cone $\mathbf{K} \subset E$ is τ-supernormal if and only if it is τ-normal. For nuclear space, see [**13, 15**].

(5°). Let $(E(\tau), \mathrm{Spec}(E))$, be a locally convex space and let $\{f_n\}_{n \in N}$ be a sequence of continuous linear forms.

Consider $\mathbf{K} \subset E$ a convex cone and suppose that f_n, for any $n \in N$, is positive with respect to the order defined by $\mathbf{K}$.

The convex cone $\mathbf{K}$ is called semicomplete with respect to $\{f_n\}$ if and only if, for any sequence $\{x_m\}_{m \in N} \subset \mathbf{K}$ such that $\sum_{n=1}^{\infty} f_n(x_m) < +\infty$, $\forall n \in N$, we have that $\{x_m\}_{m \in N}$ is summable and $\sum_{m=1}^{\infty} x_m \in \mathbf{K}$.

In [**5**], we proved that if $\mathbf{K} \subset E(\tau)$ is a semicomplete convex cone with respect to $\{f_n\}_{n \in N}$, then $\mathbf{K}$ is τ-supernormal.

(6°). Let $(E(\tau), \mathrm{Spec}(E))$ be a Hausdorff locally convex space and suppose that $\mathbf{K} \subset E$ is a convex cone such that $\mathbf{K} \bigcap (-\mathbf{K}) = \{0\}$.

If $\mathbf{K}$ is $\sigma(E, E')$-complete, $\sigma(E, E')$-normal, and there exists a countable fundamental system of weak neighborhoods of zero with respect to $\mathbf{K}$, then $\mathbf{K}$ is a τ-supernormal cone.

(7°). The convex cone $\mathcal{H}^+$ of positive harmonic functions on a locally compact space Ω, with respect to an axiomatic theory (Bauer, Brelot, Constantinescu - Cornea or Mokobodzki - Sibony) is a supernormal cone.

We can also obtain other examples of supernormal cones using the order topology [**10, 12, 16**].

Let $< E, F >$ be a dual system of vector spaces and suppose that E is ordered by $\mathbf{K}$ and F by $\mathbf{K}'$(the dual cone of $\mathbf{K}$). If E is generated by $\mathbf{K}$, then the locally convex topology defined by the basis $\{[-f, f]^\circ\}_{f\in\mathbf{K}'}$ on E is called order topology. We denote this topology by $\mathcal{O}(E, F)$.

If $F = E'$ and E is generated by $\mathbf{K}$, then the topology $\mathcal{O}(E, E')$ is defined by the family of seminorms $\{p_f\}_{f\in\mathbf{K}'}$, where

$$p_f(x) = \sup\{|g(x)||g \in [-f, f]\} \quad \forall x \in E.$$

For other details on $\mathcal{O}(E, F)$, we refer to [**16**].

PROPOSITION 3. *If $E(\tau)$ is a locally convex lattice* [**12**], *then* $\mathbf{K} = \{x \in E|x \geq 0\}$ *is $\mathcal{O}(E, E')$-supernormal.*

PROOF. Indeed, in this case $\{p_f\}_{f\in\mathbf{K}'}$ is a basis of Spec$(\mathcal{O}\ (E, E'))$ and it is well known [**12**, Corollary 2.6] that $\mathcal{O}(E, E')$ is consistent with the duality $< E, E' >$. Moreover, the definition of p_f implies that $\forall x \in \mathbf{K}\ p_f(x) \leq f(x)$.

PROPOSITION 4. *Suppose that E is a regularly ordered vector space (that is, the points of E are separated by the order dual E^+). If $\mathbf{K} \subset E$ is a generating convex cone, then it is $\mathcal{O}(E, E^+)$-supernormal.*

PROOF. In this case $\mathcal{O}(E, E^+)$ is well defined and consistent with dual system $< E, E^+ >$ [**12**].

Let $E(\tau)$ be an ordered locally convex space for which $\mathbf{K} = \{x \in E|x \geq 0\}$ is a generating cone and let p be a seminorm on E. We say that p is a (PL)-seminorm if there exists an $f \in \mathbf{K}'$ such that

(*) $p(x) \leq \inf\{f(w)|w \pm x \in \mathbf{K}\ ,\ \forall x \in E.$

PROPOSITION 5. *Let $(E(\tau), Spec(E))$ be a locally solid space [**12**] and $\mathbf{K} = \{x \in E|x \geq 0\}$. If every equicontinuous subset of E' is ordered bounded, then $\mathbf{K}$ is τ-supernormal.*

PROOF. From [**16**; Corollary 3.1.4, p. 109] we have in this case that any continuous seminorm p on E is a (PL)-seminorm, and hence f

satisfies the formula (*).

A locally convex space $E(\tau)$ is an (L)-space if, it is a locally convex lattice possessing a Spec(E) with a basis $\mathcal{B}$ such that, for any $p \in \mathcal{B}$, the following properties are satisfied:

(ℓ_1) $\forall x, y \in E,\ |x| \leq |y|, \Rightarrow p(x) \leq p(y)$;

(ℓ_2) $\forall x, y \in \mathbf{K},\ p(x+y) = p(x) + p(y)$. The (L)-spaces were studied in [**4, 9**].

PROPOSITION 6. *If $\big(E(\tau), Spec(E)\big)$ is an (L)-space, then $\mathbf{K} = \{x \in E | x \geq 0\}$ is a τ-supernormal cone.*

PROOF. In this case, $E(\tau)$ is a locally solid space, and if $p \in \mathcal{B}$, where $\mathcal{B}$ is a basis of the Spec(E) satisfying (ℓ_1) and (ℓ_2), then from [**16** Corollary 3.2.6, p.131], we have that p is a (PL)-seminorm, which implies (as in the proof of Proposition 5) that $\mathbf{K}$ is τ-supernormal.

COROLLARY. *If $(E, \|\|)$ is a normed (L)-space, then $\mathbf{K} = \{x \in E | x \geq 0\}$ is a well based convex cone.*

3. Now consider $\big(E, \text{Spec}(E)\big)$, a locally convex space, and $M \subset E$, a non-empty subset. We say that $\phi : M \to R$ is lower semicontinuous (lsc) if, for every $\lambda \in \mathbf{R}$, the set $S_\lambda = \{x \in M | \phi(x) \leq \lambda\}$ is closed with respect to M.

As in our paper [**5**], we use the following concept of a dynamical system. We say that Γ is a dynamical system on M if,

(s_1) $\Gamma : M \to 2^M$

(s_2) $\forall x \in M,\ \Gamma(x) \neq \Phi$.

A point $x_* \in M$ is called a critical point of Γ if $\Gamma(x_*) = \{x_*\}$. The concept of critical point is a very important concept in the theory of dynamical systems.

We use in this paper the following result proved in our papers [**5, 6**].

PROPOSITION 7. *Let $\big(E, Spec(E)\big) = \{p_\alpha\}_{\alpha \in \mathcal{A}}$ be a locally convex space. Consider $M \subset E$ a complete nonempty subset and $\Gamma : M \to 2^M$ a dynamical system. If, for each $\alpha \in \mathcal{A}$, there exist an lsc function $\Phi_\alpha : M \to \mathbf{R}_+$ and a constant $c_\alpha > 0$ such that*

$$\forall x \in M \text{ and } \forall u \in \Gamma(x), \quad c_\alpha p_\alpha(x-u) \leq \Phi_\alpha(x) - \Phi_\alpha(u),$$

then Γ has a critical point in M.

REMARKS.

(1°). This result is the Corollary 3 of Theorem 1 of our paper [**5**].

(2°). If Γ is a point-to-point mapping, then a critical point of Γ is a fixed point.

THEOREM 1. *Let $\big(E, Spec(E)\big) = \{p_\alpha\}_{\alpha\in\mathcal{A}})$ be a locally convex space ordered by a supernormal convex cone $\mathbf{K} \subset E$, and let $S \subset E$ be a subset. A mapping $f : S \to E$ has a fixed point in S if and only if there exist a complete subset $S_0 \subset S$ and a continuous mapping $\phi : S \to E$ such that*

(1°) $f(S_0) \subset S_0$;

(2°) $\phi(S_0)$ *is bounded;*

(3°) $\phi(f(x)) \geq \phi(x)$, $\forall x \in S_0$; *and*

(4°) $\forall\alpha \in \mathcal{A}$, $\exists\beta \in \mathcal{A}$ *and* $c_\beta > 0$ *such that* $\forall x \in S_0$,

$$p_\alpha\big(x - f(x)\big) \leq c_\beta p_\beta\Big(\phi\big(f(x)\big) - \phi(x)\Big).$$

PROOF. (a). Suppose that f has a fixed point in S, that is, there exists $x_* \in S$ such that $f(x_*) = x_*$. In this case the assumptions (1°) - (4°) are satisfied if we put, $S_0 = \{x_*\}, \phi = 1_S$, and if, for relation (4°), we consider, for every $\alpha \in \mathcal{A}$, $\beta = \alpha$ and $c_\beta = 1$.

(b). Suppose now, assumptions (1°) - (4°) are satisfied; we will prove that f has a fixed point in S. Indeed, since $\mathbf{K}$ is a supernormal cone, we have that, for every $\alpha \in \mathcal{A}$, there exist $\beta \in \mathcal{A}$, $c_\beta > 0$ and $f_\beta \in E'$ such that

$$\begin{aligned} p_\beta\big(x - f(x)\big) &\leq c_\beta p_\beta\Big(\phi\big(f(x)\big) - \phi(x)\Big) = c_\beta p_\beta(k_x) \\ &\leq c_\beta f_\beta(k_x) = c_\beta f_\beta\big(k_x + \phi(x) - \phi(x)\big) \\ &= c_\beta f_\beta\big(k_x + \phi(x)\big) - c_\beta f_\beta\big(\phi(x)\big), \end{aligned} \tag{1}$$

for any $x \in S_0$. In these relations, $k_x = \phi\big(f(x)\big) - \phi(x)$ and $k_x \in \mathbf{K}$.

Since ϕ and f_β for all $\beta \in \mathcal{A}$ are continuous and $\phi(S_0)$ is supposed bounded, the number $m_\beta = \sup_{x\in S_0} c_\beta f_\beta\big(\phi(x)\big)$ for all $\beta \in \mathcal{A}$ is well defined; we have, using (1),

$$\begin{aligned} p_\alpha(x - f(x)) &\leq c_\beta f_\beta(k_z + \phi(x)) - c_\beta f_\beta(\phi(x)) \\ &= \Big(m_\beta - c_\beta f_\beta(\phi(x))\Big) - \Big(m_\beta - c_\beta f_\beta(k_x + \phi(x))\Big) \\ &= \Big(m_\beta - c_\beta f_\beta(\phi(x))\Big) - \Big(m_\beta - c_\beta f_\beta(\phi(f(x)))\Big). \end{aligned} \tag{2}$$

Because β is dependent on $\alpha \in \mathcal{A}$, we put for all $\alpha \in \mathcal{A}$,

$$\Phi_\alpha(x) = m_{\beta(\alpha)} - c_{\beta(\alpha)} f_{\beta(\alpha)}\big(\phi(x)\big), \ \forall x \in S_0,$$

and from (2) we obtain

$$(3) \qquad p_\alpha\big(x - f(x)\big) \leq \phi_\alpha(x) - \Phi_\alpha\big(f(x)\big), \quad \forall \alpha \in \mathcal{A}, \ \forall x \in S_0.$$

Since Φ_α is a continuous function for all $\alpha \in \mathcal{A}$, the formula (3) proves that we can use Proposition 7 which implies that f has a fixed point in $S_0 \subset S$.

COROLLARY 1. *Let $(E, Spec(E) = \{p_\alpha\}_{\alpha \in \mathcal{A}})$ be a locally convex space ordered by a supernormal convex cone $\mathbf{K} \subset E$ and let $S \subset E$ be a complete subset. Consider a continuous mapping $g : S \to E$, such that*
(1°) *$g(S)$ is bounded;*
(2°) *$g(x) \leq g\big(x - g(x)\big)$, $\forall x \in S$; and*
(3°) *$\forall \alpha \in \mathcal{A}$, $\exists \beta \in \mathcal{A}$ and $c_\beta > 0$ such that $\forall x \in S$,*

$$p_\alpha\big(g(x)\big) \leq c_\beta p_\beta\Big(g\big(x - g(x)\big) - g(x)\Big).$$

If $f(S) \subset S$, where $f(x) = x - g(x)$, $\forall x \in S$, then there exists $x_0 \in S$ such that $g(x_0) = 0$.

PROOF. All assumptions of Theorem 1 are satisfied for $f(x) = x - g(x)$ if we consider $\phi(x) = x - f(x) = g(x)$, $\forall x \in S$. Thus we obtain that f has a fixed point $x_0 \in S$, that is, $g(x_0) = 0$.

COROLLARY 2. *Let $\big(E, Spec(E)\big) = \{p_\alpha\}_{\alpha \in \mathcal{A}})$ be a locally convex space ordered by a supernormal convex cone $\mathbf{K} \subset E$ and let $S \subset E$ be a complete subset. If, for a continuous mapping $f : S \to S$, there exist $r, p > 0$, $p > r$ such that*
(1°) *$\{f^p(x) - f^r(x) | x \in S\}$ is bounded;*
(2°) *$f^{p+1}(x) - f^p(x) \geq f^{r+1}(x) - f^r(x)$, $\forall x \in S$; and*
(3°) *$\forall \alpha \in \mathcal{A}$, $\exists \beta \in \mathcal{A}$ and $c_\beta > 0$ such that $\forall x \in S$,*

$$p_\alpha\big(x - f(x)\big) \leq c_\beta p_\beta\big(f^{p+1}(x) - f^{r+1}(x) - f^p(x) + f^r(x)\big),$$

then f has a fixed point in S.

PROOF. The corollary is a consequence of Theorem 1 if we consider $\phi(x) = f^p(x) - f^r(x),\ \forall x \in S$.

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