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ON THE PICARD GROUP OF A COMPACT COMPLEX NILMANIFOLD-II

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ABSTRACT. Let G be a complex simply connected nilpotent Lie group and Γ be a lattice subgroup of G. Then the compact complex nilmanifold G/Γ fibres holomorphically over the complex torus $T = G/[G, G]/\pi(\Gamma)$ where $\pi: G \to G/[G, G]$ denotes the quotient map and the fibre is the nilmanifold $[G, G]/\Gamma \cap [G, G]$. Let $\mathrm{Pic}(G/\Gamma)$ denote the Picard group of G/Γ. Then under certain assumptions on T, we are able to obtain a partial generalization of the classical Appell-Humbert Theorem, and in addition, describe $\mathrm{Pic}(G/\Gamma)$ in terms of $\mathrm{Pic}(T)$. Many detailed examples are presented illustrating the nature of G/Γ and its Picard group. See pages 631–638 of the Rocky Mountain J. of Math. Vol. 13, Number 4, Fall 1983 for previous results on this subject.

1. Introduction. Wang [8] showed that compact complex parallelizable manifolds are homogeneous spaces up to analytic equivalence. As interesting examples of such spaces, consider the coset spaces G/Γ where G is a complex simply connected nilpotent Lie group and Γ is a lattice in G. The nilmanifold G/Γ is a natural generalization of the complex torus. Moreover, from the analytic point of view, such spaces provide natural examples of non-Kähler manifolds. In fact, G/Γ is Kähler if and only if it is a complex torus. Further, any such G/Γ has a canonically associated complex torus T given by

$$T = G/[G, G]/\pi(\Gamma) \tag{1.1}$$

where $\pi(\Gamma)$ is a lattice in the vector space $G/[G, G]$ and $\pi: G \to G/[G, G]$ denotes the quotient map. In fact, G/Γ fibres holomorphically over T with fibre the nilmanifold $N_1 = [G, G]/\Gamma_1$, $\Gamma_1 = \Gamma \cap [G, G]$. Let $(G/\Gamma, \pi, T, N_1)$ denote this fibration. See [6] and [7] for details.

This paper deals mainly with the Picard group of G/Γ, denoted $\mathrm{Pic}(G/\Gamma)$. Specifically, we extend some earlier results presented in [2]. As per habit, $\mathrm{Pic}(G/\Gamma)$ is the group of isomorphism classes of holomorphic line bundles on G/Γ. Under a certain condition (see Proposition 2.1), we construct holomorphic maps of T into G/Γ, and we use these same

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maps to guarantee the rationality of the first Chern class of any line bundle class on G/Γ (see Theorem 1). In this way, the earlier work of Sakane [7] is now generalized. At the same time, the rigid nature of a lattice in a complex vector space is uncovered. Ultimately, Proposition 2.1 proves to be a key tool for establishing the main result Theorem 2, which relates $\mathrm{Pic}(G/\Gamma)$ to $\mathrm{Pic}(T)$, and thus obtains for us a partial generalization of the classical Appell-Humbert Theorem.

In sections 3 and 5 of the paper, we present some examples of the aforementioned work, which hopefully illuminate the nature of things. There are a couple of interesting points that are made by all this business. Firstly, under the hypothesis of Theorem 2 we note that from the Pic point of view, the non-Kähler G/Γ is analyzed by the torus T. Secondly, Theorem 2 along with an example in section 5 further proves that the analytic difference between such spaces is not detected by the Picard group.

Finally, we would like to point out that in a previous paper [2], we gave a description of $\mathrm{Pic}^\tau(G/\Gamma) = \ker c_1$. In some recent work of K. B. Lee and F. Raymond, it has been shown that this object can be described via holomorphic Seifert fibrations. See [3] for details.

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2. Canonical Coordinates of the second kind and holomorphic maps of T into G/Γ. Let $\mathfrak{g}$ denote the Lie algebra of right invariant holomorphic vector fields on G; I denotes the complex structure of $\mathfrak{g}$, and $\mathfrak{g}^+$ (resp. $\mathfrak{g}^-$) denotes the vector space of $\sqrt{-1}$ (resp. $-\sqrt{-1}$) eigenvectors of I in the complexification $\mathfrak{g}^{\mathfrak{C}}$. In the usual way, identify $\mathfrak{g}^+$ with the complex Lie algebra $(\mathfrak{g}, I)$. Since G is a complex simply connected nilpotent Lie group, then relative to any basis $\{X_1, \ldots, X_n\}$ for $\mathfrak{g}^+$ we obtain a biholomorphic map $\psi\colon \mathfrak{g}^+ \to G$ given by

$$\psi\left(\sum_{j=1}^{n} z_j(g)\, X_j\right) = \prod_{j=1}^{n} (\exp z_j(g) X_j) = g. \tag{2.1}$$

In particular, $(z_1, \ldots, z_n)$ define a system of global coordinates for G referred to as canonical coordinates of the second kind. Next, we note that the lattice Γ has a canonical Malcev basis; that is, a set $\{d_1, d_2, \ldots, d_{2n}\} \subset \Gamma$ such that

(a) $\gamma = d_1^{m_1} d_2^{m_2} \cdots d_{2n}^{m_{2n}}$ for each $\gamma \in \Gamma$ where $m_j \in \mathbf{Z}$;

(b) $\{d_{2r+1}, \cdots, d_{2n}\}$ has property (a) for the lattice Γ_1 of $[G, G]$.

See [1], [4], and [6] for details. Since $\exp\colon \mathfrak{g}^+ \to G$ is a biholomorphic map, let $Y_j \in \mathfrak{g}^+$ be given by $d_j = \exp Y_j$, $1 \leqq j \leqq n$. Note that $\{Y_1, \ldots, Y_{2n}\}$ is a real basis for $\mathfrak{g}^+$ such that $\{Y_{2r+1}, \ldots, Y_{2n}\}$ is a real basis for $[\mathfrak{g}^+, \mathfrak{g}^+]$.

If we now assume that T is completely reducible, that is, $T = T_1 \times \cdots \times T_r$ where each T_j is a one-dimensional complex torus and $r = \dim T$, then Γ admits a canonical Malcev basis $\{d_1, \ldots, d_{2n}\}$ such that

$$d_{2j-1} = \exp X_{2j-1},\ d_{2j} = \exp \tau_j X_{2j-1}$$

for $1 \leqq j \leqq r$, where $\tau_j \in \mathfrak{C}$ with $\operatorname{Im} \tau_j > 0$ and where $\{X_1, \ldots, X_r\} \in \mathfrak{g}^+$ are $\mathfrak{C}$-linearly independent; and in addition, they descend to vector fields on $G/[G, G]$. Extend $\{X_1, \ldots, X_r\}$ to a $\mathfrak{C}$-basis for $\mathfrak{g}^+$, say $\{X_1, \ldots, X_n\}$. Then following (2.1) we have the biholomorphic map ψ and a system of coordinates $(z_1, \ldots, z_n)$ where $z_j \in \operatorname{Hom}(G, \mathfrak{C})$ for $1 \leqq j \leqq r$. See Proposition 3.6 in [7] for details. In particular, it follows that for $2r + 1 \leqq j \leqq 2n$,

$$d_j = \psi\left(\sum_{r+1}^{n} y_{kj} X_k\right) \tag{2.2}$$

for $y_{kj} \in \mathfrak{C}$.

For notational convenience, let $(\exp X)^z = \exp zX$ where $z \in \mathfrak{C}$ and $X \in \mathfrak{g}^+$. Given the above data, define the holomorphic map s_j: $G \to G$ by

$$s_j(g) = (\exp X_j)^{z_j(g)}. \tag{2.3}$$

If $1 \leqq j \leqq r$, then s_j is also a homomorphism of G. Clearly, $[G, G] \subset \ker s_j$, and so we have an induced homomorphism s_j: $G/[G, G] \to G$. Next, we point out that since $[X, IX] = 0$, then

$$d_{2j-1}^{m_j} d_{2j}^{n_j} = (\exp X_j)^{m_j + \tau_j n_j} \quad (1 \leqq j \leqq r),$$

and hence it follows that $z_j(\Gamma) = \mathbf{Z} \oplus \tau_j \mathbf{Z}$ for $1 \leqq j \leqq r$. In particular, $T_j \simeq \mathfrak{C}/z_j(\Gamma)$, $1 \leqq j \leqq r$. In addition, we get that $s_j(\pi(\Gamma)) \subset \Gamma$. Explicitly, let $\pi(\gamma) \in \pi(\Gamma)$ with representative $\gamma \in \Gamma$. Relative to our Malcev basis, we have

$$\gamma = d_1^{m_1} d_2^{m_2} \cdots d_{2n}^{m_{2n}} \quad (m_j \in \mathbf{Z}),$$

from which it follows that

$$\begin{aligned} s_j(\pi(\gamma)) &= (\exp X_j)^{z_j(\gamma)} \\ &= (\exp X_j)^{m_{2j-1}} (\exp \tau_j X_j)^{m_{2j}} \\ &= d_{2j-1}^{m_{2j-1}} d_{2j}^{m_{2j}}, \end{aligned}$$

i.e., $z_j(\gamma) = m_{2j-1} + \tau_j m_{2j}$. Thus, by definition of Malcev basis $s_j(\pi(\Gamma)) \in \Gamma$. It follows that if p: $G \to G/\Gamma$ is the quotient map then $\hat{s}_j = p \circ s_j$ defines a holomorphic map of T into G/Γ for each $1 \leqq j \leqq r$. In general we have the following situation.

Proposition 2.1. *There exists a system of coordinates $(z_1, \ldots, z_n)$ for G such that for each $j = 1, \ldots, r$ and some positive integer $d \geqq 1$, $dz_j(\Gamma)$*

is a lattice in $\mathfrak{C}$ *and* $s_j^d(\pi(\Gamma)) < \Gamma$ *(see (2.3)) if and only if there is an isogeny* ϕ: $T_1 \times \cdots \times T_r \to T$ *where each* T_j *is a one-dimensional complex torus.*

PROOF. Suppose firstly that ϕ: $T_1 \times \cdots \times T_r \to T$ is an isogeny; that is, a holomorphic homomorphism which is a finite sheeted covering map. Clearly, ϕ is induced by a $\mathfrak{C}$-linear isomorphism $\tilde{\phi}$: $\mathfrak{C}^r \to G/[G, G]$ such that

$$\begin{array}{ccc} \mathfrak{C}^r & \xrightarrow{\tilde{\phi}} & G/[G, G] \\ \downarrow q & & \downarrow q' \\ T_1 \times \cdots \times T_r & \xrightarrow{\phi} & T \end{array} \tag{2.3}$$

commutes. The vertical q maps are the natural quotient maps. Writing $T_j = \mathfrak{C}/L_j$ where L_j is its defining lattice in $\mathfrak{C}$, let $L = L_1 \times \cdots \times L_r$. Now since ϕ is an isogeny onto T, $\tilde{\phi}(L)$ is a lattice in $G/[G, G]$ of rank $2r$ which has finite index in $\pi(\Gamma)$. The following commutative diagram of exact sequences gives the complete picture.

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \longrightarrow & L & \xrightarrow{\tilde{\phi}} & \pi(\Gamma) & \longrightarrow & \operatorname{coker}(\tilde{\phi}|_L) \longrightarrow 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathfrak{C}^r & \xrightarrow{\tilde{\phi}} & G/[G, G] & & \\ & & & & \downarrow q & & \downarrow q' & & \\ 0 & \longrightarrow & \ker \phi & \longrightarrow & \mathfrak{C}^r/L & \xrightarrow{\phi} & T & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

By the snake lemma, $\operatorname{coker}(\tilde{\phi}|_L) \simeq \ker\phi$ and hence $[\pi(\Gamma)\colon \tilde{\phi}(L)] = d$ where $d = |\ker \phi|$. In particular, $\pi(\gamma)^d \in \tilde{\phi}(L) \subset \pi(\Gamma)$.

One can choose a canonical Malcev basis for L, say $\{\ell_1, \ldots, \ell_{2r}\}$, such that $\ell_{2j} = \tau_j \ell_{2j-1}$ for $1 \leqq j \leqq r$ where $\tau_j \in \mathfrak{C}$ with $\operatorname{Im} \tau_j > 0$. Moreover, each pair is arranged so that it is a basis for L_j. Although $\{\tilde{\phi}(\ell_1), \ldots, \tilde{\phi}(\ell_{2r})\}$ is not in general a canonical Malcev basis for $\pi(\Gamma)$, it does have the following property: $\forall\, \gamma \in \Gamma$ there exist integers $m_j \in \mathbf{Z}$ such that

$$\pi(\gamma)^d = \prod_{j=1}^{2r} \tilde{\phi}(\ell_j)^{m_j}.$$

Next, choose $d_j \in \Gamma$ such that $\pi(d_j) = \tilde{\phi}(\ell_j)$ and then adjoin to $\{d_1, \ldots, d_{2r}\}$ a canonical Malcev basis for Γ_1, say $\{d_{2r+1}, \ldots, d_{2n}\}$. The set $\{d_1, \ldots, d_{2n}\}$ generates a lattice Γ' in G such that $[\Gamma\colon \Gamma'] = d$. Consider the nilmanifold G/Γ', and let $p'\colon G \to G/\Gamma'$ denote the quotient map. Identifying

L with $\tilde{\phi}(L)$, then G/Γ' fibres holomorphically over $T_1 \times \cdots \times T_r$. We are now in the completely reducible situation described earlier. So there are holomorphic homomorphisms s_j: $G/[G, G] \to G$ (see (2.3)) for $1 \leqq j \leqq r$ such that $s_j(L) \subset \Gamma'$ and hence $\hat{s}_j = p' \circ s_j$ yields a holomorphic map of $T_1 \times \cdots \times T_r$ into G/Γ'. Since $\pi(\gamma)^d \in L$, it follows that $s_j^d(\pi(\gamma)) = s_j(\pi(\gamma)^d) \in \Gamma' < \Gamma$ and hence $s_j^d(\pi(\Gamma)) < \Gamma$. Thus, $\hat{s}_j^d = p \circ s_j^d$ defines a holomorphic map of T into G/Γ.

Suppose now we are given the converse hypothesis. Consider the $\mathfrak{C}$-linear isomorphism ϕ: $\mathfrak{C}^r \to G/[G, G]$ given by $\phi(z) = \pi(g)$ where $z = (z_1, \ldots, z_n)$ and $g \in G$ with $z_j(g) = z_j$. Let $L = L_1 \times \cdots \times L_r$ where $L_j = dz_j(\Gamma)$. Since $s_j(\pi(\gamma))^d = (\exp X_j)^{dz_j(\gamma)} \in \Gamma$ and $\phi(e_j) = \pi(\exp X_j) = \exp \pi_* X_j$ where e_j denotes the j^{th} unit vector, then it follows that for each $\ell_j \in L_j$, $\phi(\ell_j e_j) = \ell_j \phi(e_j) \in \pi(\Gamma)$ and hence $\phi(L) \subset \pi(\Gamma)$. Clearly, $[\pi(\Gamma)$: $\phi(L)] = d$ and ϕ induces an isogeny of $T_1 \times \cdots \times T_r$ onto T where $T_j = \mathfrak{C}/L_j$.

In closing we make some observations about complex tori. Following Mumford [5] p. 174, we say that a complex torus $T = \mathfrak{C}^r/L$ is simple if it does not contain a subcomplex torus distinct from itself and zero. The following gives a useful criterion for determining the simplicity of T.

PROPOSITION 2.2. *The complex torus $T = \mathfrak{C}^r/L$ is simple if and only if given any lattice basis $\mathfrak{B} = \{\ell_1, \ldots, \ell_{2r}\}$ for L then any set of $2k$ vectors from $\mathfrak{B}$ with $0 < k < r$ do not span a k-dimensional $\mathfrak{C}$-linear subspace of $\mathfrak{C}^r$.*

PROOF. If $\{\ell_{j_1}, \ldots, \ell_{j_{k2}}\}$ is a set of $2k$ distinct vectors from $\mathfrak{B}$ which span a k-dimensional $\mathfrak{C}$-linear subspace W of $\mathfrak{C}^r$ $(k \leqq r)$, then $\{\ell_{j_1}, \ldots, \ell_{j_{2k}}\}$ forms an $\mathfrak{R}$-basis for W_R. It follows that the $\mathbf{Z}$-span of $\{\ell_{j_1}, \ldots, \ell_{j_{2k}}\}$ forms a lattice, call it L_1, in W and hence W/L_1 is a sub-complex torus of T. So if T is simple then $k = 0$ or $k = r$. The converse is immediate.

REMARK. From the above lemma, the homomorphisms s_j, $1 \leqq j \leqq r$, induce holomorphic maps from T into G/Γ provided one can choose a lattice of finite index in $\pi(\Gamma)$ which admits a basis $\{\ell_1, \ldots, \ell_{2r}\}$ such that each pair ℓ_{2j-1}, ℓ_{2j}, $1 \leqq j \leqq r$, spans a one-dimensional complex subspace of $G/[G, G]$. At the G/Γ level, this means that there exists a lattice Γ' of finite index d in Γ such that the following diagram commutes:

(2.5)
$$\begin{tikzcd} G/\Gamma' \arrow[r, "\phi'"] \arrow[d, "\pi'"'] & G/\Gamma \arrow[d, "\pi"] \\ T' \arrow[r, "\phi"] & T \end{tikzcd}$$

where $T' = (G/[G, G])/\pi'(\Gamma'')$ and ϕ' is a finite sheeted covering map induced by the isogeny ϕ.

3. Some Examples. In this section we present some explicit examples of the material from the previous section.

(1) Let

$$G = \left\{ g = \begin{bmatrix} 1 & g_1 & g_3 \\ 0 & 1 & g_2 \\ 0 & 0 & 1 \end{bmatrix} \middle| g_j \in \mathfrak{C} \right\}$$

and let

$$\Gamma = \left\{ \gamma = \begin{bmatrix} 1 & \gamma_1 & \gamma_3 \\ 0 & 1 & \gamma_2 \\ 0 & 0 & 1 \end{bmatrix} \middle| \gamma_j \in \mathbf{Z} \oplus i\mathbf{Z} \right\}.$$

Then G/Γ is the well known Iwasawa manifold. The Lie algebra

$$\mathfrak{g} = \left\{ X = \begin{bmatrix} 0 & g_1 & g_3 \\ 0 & 0 & g_2 \\ 0 & 0 & 0 \end{bmatrix} \middle| g_j \in \mathfrak{C} \right\}.$$

Identifying $\mathfrak{g}$ with $\mathfrak{g}^+$ in the usual way, then relative to the standard basis $\{X_1 = E_{12}, X_2 = E_{23}, X_3 = E_{13}\}$ for $\mathfrak{g}$, the canonical coordinates of the second kind are given by $z_j(g) = g_j$, $1 \leqq j \leqq 3$. In this case, it is clear that T is a product of two one-dimensional complex tori; that is, $T \simeq \mathfrak{C}/L_1 \times \mathfrak{C}/L_2$ where $L_j = z_j(\Gamma) = \mathbf{Z} \oplus i\mathbf{Z}$, $j = 1, 2$. Consequently, it can be checked directly that

$$s_j(g) = \exp z_j(g)X_j = I + z_j(g)X_j \qquad (j = (1, 2)$$

has the property that $s_j(\pi(\Gamma)) \subset \Gamma$.

(2) Let G be the same as in (1), but let Γ'' be the lattice generated by the following elements:

$$t_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ t_2 = \begin{bmatrix} 1 & i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ t_3 = \begin{bmatrix} 1 & \sqrt{2} & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$t_4 = \begin{bmatrix} 1 & \sqrt{2}i & 0 \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix},\ t_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ t_6 = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A typical element of Γ'' has the form

$$\gamma = \prod_{j=1}^{6} t_j^{n_j} = \begin{bmatrix} 1 & (n_1 + \sqrt{2}n_3) + i(n_2 + \sqrt{2}n_4) & n_5 + in_6 \\ 0 & 1 & n_3 + in_4 \\ 0 & 0 & 1 \end{bmatrix}$$

where $n_j \in \mathbf{Z}$. Let $\{X_1, X_2, X_3\}$ denote the standard basis for $\mathfrak{g}$ as in (1). We obtain a new basis for $\mathfrak{g}$ by defining $Y_1 = X_1$, $Y_2 = X_2 + \sqrt{2}X_1$, and $Y_3 = X_3$. It can be shown that the canonical coordinates of the second kind with respect to $\{Y_1, Y_2, Y_3\}$ are given by

$$w_1(g) = z_1(g) - \sqrt{2}z_2(g)$$
$$w_2(g) = z_2(g)$$
$$w_3(g) = z_3(g) + z_2(g)(z_1(g) - (\sqrt{2}/2)z_2(g))$$

for each $g \in G$ where the z_j are as defined in (1). In particular, $w_j(\Gamma'') = \mathbf{Z} \oplus i\mathbf{Z}$ for $j = 1, 2$, from which it follows that the maps $s_j(g) = \exp w_j(g)Y_j$, $j = 1, 2$, have the property $s_j(\Gamma'') \subset \Gamma''$. Moreover, $T \simeq \mathfrak{C}/\mathbf{Z} \oplus i\mathbf{Z} \times \mathfrak{C}/\mathbf{Z} \oplus i\mathbf{Z}$. Finally, we point out that the set $z_1(\Gamma'') = \{(n_1 + \sqrt{2}n_3) + i(n_2 + \sqrt{2}n_4) \mid n_j \in \mathbf{Z}\}$ is not a lattice in $\mathfrak{C}$.

(3) Let G be the same as in (1), but let Γ be the lattice generated by the following elements:

$$t_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; t_2 = \begin{bmatrix} 1 & i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; t_3 = \begin{bmatrix} 1 & \frac{1}{2}i & 0 \\ 0 & 1 & \frac{1}{2}\tau \\ 0 & 1 & 0 \end{bmatrix},$$

$$t_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \; t_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; t_6 = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\tau \in \mathfrak{C}$ with $\operatorname{Im} \tau > 0$. Then G/Γ is a three-dimensional complex nilmanifold where T is analytically equivalent to the complex torus $\mathfrak{C}^2/L$ where L is the lattice generated by $(1, 0)$, $(i, 0)$, $((1/2i, (1/2)\tau)$, $(0, 1)$. Using the data $\{X_1, X_2, X_3\}$ and (z_1, z_2, z_3) from (1) one can see directly that $T \simeq \mathfrak{C}^2/L$. In this example, T is not a product of two complex tori. This follows from the observation that pairing $({}^i/_2, {}^\tau/_2)$ with any other generator for L from above gives a basis for $\mathfrak{C}^2$. However, T is isogenous to $T' = \mathfrak{C}^2/L'$ where L' is the lattice generated by $(1, 0)$, $(i, 0)$, $(0, 1)$, $(0, \tau)$. T' is clearly a product of two complex tori. Moreover, in T', $({}^i/_2, {}^\tau/_2)$ represents a point of order two. So letting H denote the subgroup of T' generated by the class of $({}^i/_2, {}^\tau/_2)$, then $T = T'/H$ with the isogeny $\phi\colon T' \to T$ being the quotient map. Clearly, the degree of ϕ is $d = 2$, $s_j(\Gamma) \not\subset \Gamma$ but $s_j^2(\Gamma) \subset \Gamma$, $j = 1, 2$.

(4) For our final example in this section we take G as in (1) but take Γ to be the lattice generated by

$$t_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ t_2 = \begin{bmatrix} 1 & \sqrt{3}i & 0 \\ 0 & 1 & \sqrt{7}i \\ 0 & 0 & 1 \end{bmatrix},\ t_3 = \begin{bmatrix} 1 & \sqrt{7}i & 0 \\ 0 & 1 & \sqrt{5}i \\ 0 & 0 & 1 \end{bmatrix},$$

$$t_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ t_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ t_6 = \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this case, $T \in \mathfrak{C}^2/L$ where L is the lattice generated by (1, 0), (0, 1), $(\sqrt{3}i, \sqrt{7}i)$, $(\sqrt{7}i, \sqrt{5}i)$. Since any two of these generators are a $\mathfrak{C}$-basis for $\mathfrak{C}^2$, T is not isogenous to a product of two complex tori. Hence, by Proposition 2.1 there are no non-trivial holomorphic maps of T into G/Γ arising from a canonical coordinate system of the second kind on G.

4. A Structure Theorem for Pic(G/Γ)**.** Let $\mathfrak{L} \in \mathrm{Pic}(G/\Gamma)$. As is demonstrated by Propositions 3.4 and 3.5 of [7] there is a unique real right invariant 2-form $\alpha \in \Lambda^2(\mathfrak{g}^+)^*$ of type (1, 1) representing $c_1(\mathfrak{L})$, and it is given by

$$\alpha = \frac{1}{2i} \sum_{j,k=1}^{r} h_{jk} dz_j \wedge d\bar{z}_k \tag{4.1}$$

where (h_{jk}) is an $r \times r$-hermitian matrix and $r = \dim \mathfrak{g}^+/[\mathfrak{g}^+, \mathfrak{g}^+]$. We remark here that the uniqueness of α is relative to the coordinates $(z_1, \ldots, z_n)$; that is, relative to the basis $\mathfrak{B} = \{X_1, \ldots, X_n\}$ for $\mathfrak{g}$. If $\mathfrak{B}' = \{Y_1, \ldots, Y_n\}$ is another basis for $\mathfrak{g}^+$ and $(w_1, \ldots, w_n)$ the corresponding canonical coordinates of the second kind so that Proposition 3.6 in [7] is true, then as above one has a real (1, 1) form $\alpha' = (1/2i) \sum_{j,k=1}^{r} h'_{jk} dw_j \wedge d\bar{w}_k$ also representing $c_1(\mathfrak{L})$. Although α' is cohomologous to α, α' need not equal α. We show the usefulness of this remark in the following lemma.

LEMMA 4.1. *Let $\mathfrak{L} \in \mathrm{Pic}(G/\Gamma)$. Then there exists a system of coordinates for G relative to which the (unique) $r \times r$ hermitian matrix representing $c_1(\mathfrak{L})$ is a diagonal matrix.*

PROOF. Let $\mathfrak{L} \in \mathrm{Pic}(G/\Gamma)$ and let α defined by (4.1) represent $c_1(\mathfrak{L})$. As is well known, the hermitian matrix (h_{jk}) is unitarily equivalent to a diagonal matrix D; i.e., $D = P^{-1}(h_{jk})P$ where P is unitary. Let S: $\mathfrak{g}^+/[\mathfrak{g}^+, \mathfrak{g}^+] \to \mathfrak{g}^+/[\mathfrak{g}^+, \mathfrak{g}^+]$ be the linear transformation whose matrix relative to $\{X_1^*, \ldots, X_r^*\}$ is (h_{jk}). If $\{Y_1^*, \ldots, Y_r^*\}$ is an eigenbasis for S then the matrix of S relative to the basis is D with P being the change of basis matrix from $Y_1^*, \ldots, Y_r^*$ to $X_1^*, \ldots, X_r^*$. One can choose a basis $\mathfrak{B}' = \{Y_1, \ldots, Y_n\}$ for $\mathfrak{g}^+$ such that $\pi_* Y_j = Y_j^*$ for $1 \leqq j \leqq r$ and $Y_j = X_j$ for $r + 1 \leqq j \leqq n$. Define the matrix

$$A = \left[\begin{array}{c|c} P & 0 \\ \hline 0 & I_{n-r} \end{array}\right]. \tag{4.2}$$

If $(w_1, \ldots, w_n)$ denotes the system of canonical coordinates of the second kind corresponding to $\mathfrak{B}'$ then Proposition 3.6 in [7] holds for this set-up. In particular, let $\psi_w \colon \mathfrak{g} \to G$ be the biholomorphic map given by

$$\psi_w\Big(\sum_j w_j(g) Y_j\Big) = \prod_j (\exp Y_j)^{w_j(g)} = g.$$

The matrix $A = (a_{ij})$ defines a $\mathfrak{C}$-linear isomorphism $T_A \colon \mathfrak{g} \to \mathfrak{g}$ by the recipe

$$T_A\Big(\sum_{j=1}^{n} c_j X_j\Big) = \sum_{i,j=1}^{n} c_j a_{ij} Y_i. \tag{4.3}$$

In turn T_A induces a biholomorphic map of G by the diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{T_A} & \mathfrak{g} \\ \Big\downarrow{\scriptstyle \psi_w} & & \Big\downarrow{\scriptstyle \psi} \\ G & \xrightarrow{T_A} & G \end{array} \tag{4.4}$$

As elements of $\mathrm{Pic}(G/\Gamma)$, $T_A^*\mathfrak{L} = \mathfrak{L}$; i.e., $T_A^*\mathfrak{L}$ is analytically equivalent to $\mathfrak{L}$. Hence $c_1(\mathfrak{L}) = c_1(T_A^*\mathfrak{L})$. Since D clearly represents $c_1(T_A^*\mathfrak{L})$ (i.e., $\bar{D} = D$), we are done.

THEOREM 1. *Let G/Γ be a compact complex nilmanifold such that T is isogenous to a product of r one-dimensional complex tori* (*see Proposition* 2.1). *Let $\mathfrak{L} \in \mathrm{Pic}(G/\Gamma)$. Then there exists a real invariant form $\beta \in \Lambda^2(\mathfrak{g}^+)^*$ of type* (1, 1) *representing $c_1(\mathfrak{L})$ which is rational.*

PROOF. Let $(z_1, \ldots, z_n)$ be a system of coordinates for G subject to the conditions of Proposition 2.1. By Lemma 4.1 we can if necessary apply a unitary change of coordinates to obtain a diagonal form representing $c_1(\mathfrak{L})$. Moreover, such a change of coordinates preserves the conditions of Proposition 2.1. So we may assume that

$$\beta = \frac{1}{2i} \sum_{j=1}^{r} d_j \, dz_j \wedge d\bar{z}_j$$

is the unique (relative to $(z_1, \ldots, z_n)$) real invariant form in $\Lambda^2(\mathfrak{g}^+)^*$ of type (1, 1) representing $c_1(\mathfrak{L})$. We will prove the theorem by showing that the hermitian form H defined by β on G has an imaginary part A which is rational valued when restricted to $\Gamma \times \Gamma$. We define H: $G \times G \to \mathfrak{C}$ by

$$H(g, h) = \sum_{j=1}^{r} d_j z_j(g)\overline{z_j(h)}. \tag{4.5}$$

Then H is a hermitian bihomomorphism. Let $\hat{s}_j^d$: $T \to G/\Gamma$, $1 \leqq j \leqq r$, be the holomorphic maps induced from the coordinates $(z_1, \ldots, z_n)$ as described in Proposition 2.1. Consider next the line bundle $(\hat{s}_j^d)^*\mathfrak{L}$ on T; i.e., the pullback of $\mathfrak{L}$ by $\hat{s}_j^d$. By the Appell-Humbert Theorem (see [2], [5]), one knows that relative to the coordinates $(z_1, \ldots, z_r)$ for $G/[G, G]$, $c_1[(s_j^d)^*\mathfrak{L}]$ is represented by a unique hermitian form on $G/[G, G]$ for which the imaginary part is integral on $\pi(\Gamma)$. Explicitly, this form is given by

$$\begin{aligned} H_j(g, h) &:= (s_j^d)^* H(g, h) \\ &= \sum_{k=1}^{r} d_k z_k(s_j^d(g))\overline{z_k(s_j^d(h))} \\ &= d^2 d_j z_j(g)\overline{z_j(h)}. \end{aligned}$$

Clearly, $H = 1/d^2(H_1 + \cdots + H_r)$. Continuing, the imaginary part of H_j is then given by

$$A_j(g, h) = \frac{d^2 d_j}{2i}(z_j(g)\overline{z_j(h)} - \overline{z_j(g)}z_j(h)).$$

Moreover, since $A = 1/d^2(A_1 + \cdots + A_r)$ and since $A_j(\pi(\Gamma) \times \pi(\Gamma)) \subset \mathbf{Z}$, it follows that $A(\Gamma \times \Gamma) \subset (1/d^2)\mathbf{Z}$ and the theorem is proved.

We now state and prove the main theorem.

THEOREM 2. *Let G/Γ be a compact complex nilmanifold such that T is isogenous to a product of r one-dimensional complex tori (see Proposition 2.1). Let $\mathfrak{L} \in \mathrm{Pic}(G/\Gamma)$. Then there exists $\mathfrak{L}' \in \mathrm{Pic}(T)$ such that*

$$\mathfrak{L}^{d^2} = \pi^*\mathfrak{L}',$$

where d is the integer deffined in Proposition 2.1.

PROOF. Let $\mathfrak{L} \in \mathrm{Pic}(G/\Gamma)$. By Theorem 1, choose a system of canonical coordinates of the second kind for G, say $(z_1, \ldots, z_n)$, relative to which $c_1(\mathfrak{L})$ is represented by the Hermitian form H in (4.5). Define

$$\mathfrak{L}_j = \pi^*(\hat{s}_j^d)^*\mathfrak{L} \tag{4.6}$$

where $1 \leqq j \leqq r$ and π: $G/\Gamma \to T$ is the fibre map. It can be shown directly that $c_1(\mathfrak{L}_1 \otimes \cdots \otimes \mathfrak{L}_r) = d^2 c_1(\mathfrak{L})$ and so $\mathfrak{L}^{d^2} \otimes \mathfrak{L}_1^{-1} \otimes \cdots \otimes \mathfrak{L}_r^{-1} \in$

$\mathrm{Pic}^0(G/\Gamma)$. By Proposition 3.1 of [2] (or see section 4.8 of [3]) there exists $\mathfrak{L}'' \in \mathrm{Pic}^0(T)$ such that

$$\mathfrak{L}^{d^2} = \pi^*\mathfrak{L}'' \otimes \mathfrak{L}_1 \otimes \cdots \otimes \mathfrak{L}_r.$$

Taking $\mathfrak{L}' = \mathfrak{L}'' \otimes (\hat{s}_1^d)^* \mathfrak{L} \otimes \cdots \otimes (\hat{s}_r^d)^* \mathfrak{L}$, the proof is complete.

5. More Examples and Concluding Remarks. For examples 1 and 2 of section 3, Theorem 2 says that $\mathrm{Pic}(G/\Gamma) \simeq \mathrm{Pic}(G/\Gamma')$. However, one gets this directly since the map $(z_1(g), z_2(g), z_3(g)) \to (w_1(g), w_2(g), w_3(g))$ induces an analytic isomorphism of G/Γ onto G/Γ'. On the other hand, let $G(n)$ be the simply connected complex nilpotent Lie group defined by

$$G(n) = \left\{ \begin{bmatrix} 1 & z_1 & z_2 \cdots\cdots & z_{n-1} & w \\ & 1 & 0 \cdots\cdots & 0 & y_{n-1} \\ & & & \vdots & \vdots \\ & & & 0 & \vdots \\ & & & 1 & y_1 \\ & & & & 1 \end{bmatrix} \middle| \begin{array}{l} z_j, y_j, w \in \mathbb{C} \\ j = 1, 2, \ldots, n-1 \end{array} \right\} \text{for } n \geqq 2,$$

and $\Gamma(n)$ be the lattice of $G(n)$ defined by

$$\Gamma(n) = \left\{ \begin{bmatrix} 1 & a_1 & a_2 \cdots\cdots & a_{n-1} & 0 \\ & 1 & 0 \cdots\cdots & 0 & b_{n-1} \\ & & & \vdots & \vdots \\ & & & 0 & \vdots \\ & & & 1 & b_1 \\ & & & & 1 \end{bmatrix} \middle| \begin{array}{l} a_j, b_j, c \in \mathbf{Z} \oplus i\mathbf{Z} \\ j = 1, \ldots, n-1 \end{array} \right\}.$$

Then, by Theorem 2, $\mathrm{Pic}(G(n)/\Gamma(n)) \simeq \mathrm{Pic}(T(n))$ for each $n \geqq 2$. More importantly, though, $\mathrm{Pic}(G(n)/\Gamma(n)) \simeq \mathrm{Pic}(G/\Gamma)$ for G/Γ from example 1. Explicitly, this follows from Theorem 2 and the observation that the base tori $T(n)$ and T are biholomorphic. There are two points to this last example. The first is that for $n \geqq 3$, $G(n)/\Gamma(n) \not\simeq G(2)/\Gamma(2) = G/\Gamma$, yet the respective Picard groups are isomorphic; and the second point is that the Picard group is not a good indicator of the analytic difference between two compact complex nilmanifolds.

Finally, in [2] (see Theorem 2) it was shown that $\mathrm{Pic}^\tau(G/\Gamma)$ is a compact complex manifold which is a finite sheeted disconnected covering of $\mathrm{Pic}^0(T)$. The following proof of the latter fact was suggested by K. Coombes, K. B. Lee and D. McCullough. Recall from [2] that $F := \Gamma_1/[\Gamma, \Gamma]$ where $\Gamma_1 = \Gamma \cap [G, G]$ is a finite abelian group, $\Gamma/\Gamma_1 \simeq \mathbf{Z}^{2r}$ and one has the short split exact sequence

$$0 \to F \to \Gamma/[\Gamma, \Gamma] \rightleftarrows \Gamma/\Gamma_1 \to 0.$$

In particular, $\Gamma/[\Gamma, \Gamma] \simeq \Gamma/\Gamma_1 \oplus F$ and so

$$\mathrm{Hom}(\Gamma, \mathfrak{C}_1^*) = \mathrm{Hom}(\Gamma/[\Gamma, \Gamma], \mathfrak{C}_1^*) \simeq \mathrm{Hom}(\Gamma/\Gamma_1, \mathfrak{C}_1^*) \oplus \mathrm{Hom}(F, \mathfrak{C}_1^*).$$

Using the results of section 3 in **[2]** and the above facts, we have isomorphic exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\Gamma/\Gamma_1, \mathfrak{C}_1^*) \arrow[r] \arrow[d, "\beta"', "\wr"] & \mathrm{Hom}(\Gamma/[\Gamma, \Gamma], \mathfrak{C}_1^*) \arrow[r] \arrow[d, "\beta"', "\wr"] & \mathrm{Hom}(F, \mathfrak{C}_1^*) \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(T) \arrow[r] & \mathrm{Pic}^\tau(G/\Gamma) \arrow[r] & F \arrow[r] & 0.
\end{tikzcd}$$

In particular, $\mathrm{Pic}^\tau(G/\Gamma) \simeq \mathrm{Pic}^0(T) \oplus F$. Next, Lemma 3.1 of **[2]** yields the exact sequence

$$0 \to \ker D \hookrightarrow \mathrm{Pic}^\tau(G/\Gamma) \xrightarrow{D} \mathrm{Pic}^0(T) \to 0$$

where it follows from above that $\ker D \simeq (\mathbf{Z}_k)^{2r} \oplus F$. Combining all of the above data, we obtain the following diagram of exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & (\mathbf{Z}_k)^{2r} \oplus F \arrow[dr] & & & 0 \\
& & \mathrm{Pic}^\tau(G/\Gamma) \arrow[r, shift left=2, "D"] \arrow[r, shift right=2, "f^*"'] \arrow[dl] & \mathrm{Pic}^0(T) \arrow[ur] \arrow[l, "\pi^*"'] & \\
0 & F \arrow[l] & & & 0 \arrow[ul]
\end{tikzcd}$$

All of this information summarizes as follows. D is a $2rk^2$ sheeted disconnected covering map. Moreover, π^* injectively maps $\mathrm{Pic}^0(T)$ onto the identity component of $\mathrm{Pic}^\tau(G/\Gamma)$, and $\mathfrak{L} \to \mathfrak{L}^k$ maps $\mathrm{Pic}^\tau(G/\Gamma)$ onto $\pi^*\mathrm{Pic}^0(T) = \mathrm{Pic}^0(G/\Gamma)$. Thus, $D(\mathfrak{L}) = (\pi^*)^{-1}\mathfrak{L}^k$.

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