ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 15, Number 4, Fall 1985

p-VALENT CLASSES RELATED TO CONVEX FUNCTIONS OF COMPLEX ORDER

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ABSTRACT. Let $C(b, p)$ ($b \neq 0$ complex, $p \geqq 1$) denote the class of functions $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ analytic in $U = \{z: |z| < 1\}$ which satisfy, for $z = re^{i\theta} \in U$,

$$\operatorname{Re}\left\{p + \frac{1}{b}\left(1 + \frac{zf''(z)}{f'(z)} - p\right)\right\} > 0.$$

From $C(b, p)$, we can obtain many interesting known subclasses including the class of convex functions of complex order, the class of p-valent convex functions and the class of p-valent functions f for which zf' is λ-spirallike in U. In this paper we investigate certain properties of the above mentioned class.

1. Introduction. Let $A_p(p \geqq 1)$ denote the class of functions $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ which are analytic in $U = \{z: |z| < 1\}$. Let Ω denote the class of bounded analytic functions $\omega(z)$ in U, satisfying the conditions $\omega(o) = o$ and $|\omega(z)| \leqq |z|$, for $z \in U$. Also, let $P(p)$ (with p a positive integer) denote the class of functions with positive real parts that have the form $P(z) = p + \sum_{k=1}^{\infty} c_k z^k$, which are analytic in U and satisfy the conditions $P(o) = p$ and $\operatorname{Re}\{P(z)\} > o$ in U.

For $f \in A_p$, we say that f belongs to the class $C(b, p)$ ($b \neq 0$ complex, $p \geqq 1$) if

$$\operatorname{Re}\left\{p + \frac{1}{b}\left(1 + \frac{zf''(z)}{f'(z)} - p\right)\right\} > 0, \qquad z \in U. \tag{1.1}$$

It is noticed that, by giving specific values to b and p, we obtain the following important subclasses studied by various authors in earlier works:

(i) $C(1, 1) = C$ is the well known class of convex functions;

(ii) $C(b, 1) = C(b)$, is the class of univalent convex functions introduced by Wiatrowski **[11]** and investigated in **[8]** and **[9]**;

(iii) $C(1, p) = C(p)$, is the class of p-valent convex functions considered by Goodman **[3]**;

Received by the editors on November 18, 1983 and in revised form on February 21, 1984.

(iv) $C(\cos\lambda\, e^{-i\lambda}, p)$, $|\lambda| < \pi/2$, is the class of p-valent functions satisfying

$$\mathrm{Re}\left\{e^{i\lambda}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0, \qquad z \in U,$$

i.e., the class of functions $f(z)$ for which $zf'(z)$ is λ-spiral-like in U; and

(v) $C(\cos\lambda\, e^{-i\lambda}, 1) = C^{\lambda}$, $|\lambda| < \pi/2$, is the class of functions for which $zf'(z)$ is λ-spirall-like introduced by Robertson **[10]** and studied by Libera and Ziegler **[6]**, Bajpai and Mehrok **[1]** and Kulshrestha **[5]**.

In [7] Nasr and Aouf introduced the class of starlike functions of order b ($b \neq 0$ complex) defined as follows.

DEFINITION. A function $f \in A_1$ is said to be starlike function of order b ($b \neq 0$ complex), that is $f \in S(b)$ if and only if $f(z)/z \neq 0$ in U and

$$\mathrm{Re}\left\{1 + \frac{1}{b}\left(\frac{(zf'(z)}{f(z)} - 1\right)\right\} > 0, \qquad z \in U.$$

We state below some lemmas that are needed in our investigation.

LEMMA 1 **[8]**. *$h(z) \in C(b)$ if and only if*

(i) *there exists $q \in C$ such that $h'(z) = (q'(z))^b$*; and

(ii) $h'(z) = \exp\{\int_0^{2\pi} - 2b \log(1 - ze^{-it})d\mu(t)\}$, *where $\mu(t)$ is a real—valued non-negative non-decreasing function defined for $t \in [0, 2\pi]$ with total variation $\int_0^{2\pi} d\mu(t) = 1$.*

LEMMA 2 **[8]**. *Suppose $h(z) \in C(b)$. Then $H(z)$, defined by $H(0) = 0$ and*

$$H'(z) = \frac{h'\left(\dfrac{z+a}{1+\bar{a}z}\right)}{h'(a)(1+\bar{a}z)^{2b}},$$

for $|a| < 1$ and $z \in U$, also belongs to $C(b)$.

LEMMA 3 **[9]**. *If $h(z) = z + \sum_{n=2}^{\infty} b_n z^n \in C(b)$, then*

$$|b_2| \leqq |b|,$$

$$|b_3| \leqq \frac{|b|}{3}\max\{1, |2b+1|\}.$$

These bounds are attained by the function $h^*(z)$ defined by

$$h'^*(z) = (1-z)^{-2b} = 1 + \sum_{n=1}^{\infty}\prod_{m=0}^{n-1}\frac{2b+m}{m+1}z^n. \tag{1.2}$$

LEMMA 4 **[7]**. *$G(z) \in S(b)$ if and only if there is a function $g(z) \in S^*$ (the well known class of starlike functions) such that*

$$G(z) = z\left(\frac{g(z)}{z}\right)^b.$$

LEMMA 5 [4]. *Let* $\omega(z) = \sum_{k=1}^{\infty} d_k z^k$ *be analytic with* $|\omega(z)| < 1$ *in* U. If ν is any complex number, then

$$|d_2 - \nu d_1^2| \leqq \max\{1, |\nu|\}. \tag{1.3}$$

Equality may be attained with functions $\omega(z) = z^2$ and $\omega(z) = z$.
We also need the following lemma.

LEMMA 6. *The function* $P \in P(p)$ *if and only if*

$$P(z) = p\left(\frac{1 - \omega(z)}{1 + \omega(z)}\right),$$

where $\omega \in \Omega$.

It follows from the definition of $C(b, p)$ and Lemma 6 that $f \in C(b, p)$ if and only if, for $z \in U$,

$$\frac{zf''(z)}{f'(z)} = \frac{\omega(z)\,(p - 2pb - 1) + (p - 1)}{1 + \omega(z)}, \qquad \omega \in \Omega. \tag{1.4}$$

2. Representation formulas for the class $C(b, p)$.

LEMMA 7. *The function* $f \in C(b, p)$, where $p \geqq 1$, *if and only if*

$$f'(z) = pz^{p-1}(h'(z))^p \tag{2.1}$$

for some $h \in C(b)$.

PROOF. Let $f'(z) = pz^{p-1}(h'(z))^p$ for $h(z) = z + \sum_{n=2}^{\infty} b_n z^n \in C(b)$, $z \in U$. By direct computation, we obtain

$$\operatorname{Re}\left\{p + \frac{1}{b}\left(1 + \frac{zf''(z)}{f'(z)} - p\right)\right\} = p \operatorname{Re}\left\{1 + \frac{1}{b}\frac{zh''(z)}{h'(z)}\right\} > 0$$

and the result follows from (1.1).

An immediate consequence of Lemmas 7, 1 and 4 is

THEOREM 1. $f(z) \in C(b, p)$, where $p \geqq 1$, *if and only if*
(i) $f'(z) = pz^{p-1} \exp\{-2pb \int_0^{2\pi} \log(1 - ze^{-it})d\mu(t)\}$;
(ii) *there exists a starlike function* $g \in S^*$ such that

$$f'(z) = pz^{p-1}\left(\frac{(g(z)}{z}\right)^{pb};$$

and
(iii) *there exists a starlike function of order* b ($b \neq 0$, *complex*), $G \in S(b)$, *such that*

$$f'(z) = pz^{p-1}\left(\frac{(G(z)}{z}\right)^{p}.$$

3. Coefficient estimates for the class $C(b, p)$.

THEOREM 2. *If* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in C(b, p)$ *and* μ *is any complex number, then*

$$(3.1)\qquad |a_{p+2} - \mu\, a_{p+1}^2| \leqq \left(\frac{p^2}{p+2}\right)|b| \max\left\{1, \left|(2pb+1) - \frac{4p^2b(p+2)}{(p+1)^2}\mu\right|\right\}.$$

This inequality is sharp for each μ.

PROOF. Since $f \in C(b, p)$, we get from (1.4), after expanding and equating coefficients, that

$$(3.2)\qquad a_{p+1} = -\left(\frac{2p^2b}{p+1}\right)d_1,$$

$$(3.3)\qquad a_{p+2} = -\left(\frac{p^2b}{p+2}\right)d_2 + \frac{(p+1)^2(2pb+1)}{4p^2b(p+2)}\, a_{p+1}^2.$$

Using (3.2), (3.3) and (1.3), we get

$$|a_{p+2} - \mu a_{p+1}^2| \leqq \left(\frac{p^2}{p+2}\right)|b| \max\left\{1, \left|(2pb+1) - \frac{4p^2b(p+2)}{(p+1)^2}\mu\right|\right\}$$

and since (1.3) is sharp, (3.1) is also sharp.

COROLLARY 1. *If* $f \in C(b, p)$, *then*

$$(3.4)\qquad |a_{p+1}| \leqq \left(\frac{2p^2}{p+1}\right)|b|$$

and

$$(3.5)\qquad |a_{p+2}| \leqq \left(\frac{p^2}{p+2}\right)|b| \max\{1, |2pb+1|\}.$$

The bounds in (3.4) and (3.5) are attained by the function $f^*(z)$ defined by

$$(3.6)\qquad f'^*(z) = pz^{p-1}(h'^*(z))^p,$$

where $h^*(z)$ is defined by (1.2).

PROOF. The inequalities (3.4) and (3.5) follow directly from (3.2) and (3.1), respectively.

The bounds on the modulus of the second and third coefficients for functions in $C(b, p)$ are obtained by another method as follows.

THEOREM 3. *If* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in C(b, p)$, $p \geqq 1$, *then*

$$(3.7)\qquad |a_{p+1}| \leqq \left(\frac{2p^2}{p+1}\right)|b|$$

and

$$(3.8)\qquad |a_{p+2}| \leqq \left(\frac{p^2}{p+2}\right)|b|\, (\max\{1, |2b+1|\} + 2(p-1)\,|b|).$$

These results are sharp with equality for $f^*(z)$ defined by (3.6).

PROOF. By Lemma 7, there exists an $h(z) = z + \sum_{n=2}^{\infty} b_n z^n \in C(b)$, such that

$$
\begin{aligned}
f'(z) &= pz^{p-1} + \sum_{n=p+1}^{\infty} na_n z^{n-1} \\
&= pz^{p-1}\left(1 + \sum_{n=2}^{\infty} nb_n z^{n-1}\right)^p.
\end{aligned} \tag{3.9}
$$

Expanding the right hand side of (3.9), we obtain

$$f'(z) = pz^{p-1} + 2p^2 b_2 z^p + p(3pb_3 + 2p(p-1)b_2^2)\, z^{p-1} + \dots . \tag{3.10}$$

Equating coefficients from (3.9) and (3.10), we have

$$
\begin{aligned}
(p+1)a_{p+1} &= 2p^2 b_2, \\
(p+2)a_{p+2} &= p(3pb_3 + 2p(p-1)b_2^2).
\end{aligned} \tag{3.11}
$$

Thus, the result follows from Lemma 3.

REMARK. Comparing the results in Corollary 1 and Theorem 3 we see that:

(1) when $\max\{1, |2b+1|\}$ in Theorem 3 is $|2b+1|$, Corollary 1 is a better result; and

(2) when $\max\{1, |2b+1|\}$ in Theorem 3 is 1, Theorem 3 is a better result.

We now prove the following

LEMMA 8. *If integers p and m and greater than zero and $b \neq 0$ is complex, then*

$$
\begin{aligned}
&\prod_{j=0}^{m-1} \frac{|2pb + j|^2}{(j+1)^2} \\
&= \frac{4p}{m^2}\left\{p\,|b|^2 + \sum_{k=1}^{m-1} (p|b|^2 + k\operatorname{Re}\{b\}) \prod_{j=0}^{k-1} \frac{|2pb + j|^2}{(j+1)^2}\right\}.
\end{aligned} \tag{3.12}
$$

PROOF. We prove the lemma by induction on m. For $m = 1$, the lemma is obvious.

Next suppose that the result is true for $m = q - 1$.

We have

$$
\begin{aligned}
&\frac{4p}{q^2}\left\{p|b|^2 + \sum_{k=1}^{q-1} (p|b|^2 + k\operatorname{Re}\{b\}) \prod_{j=0}^{k-1} \frac{|2pb + j|^2}{(j+1)^2}\right\} \\
&\quad = \frac{4p}{q^2}\left\{p|b|^2 + \sum_{k=1}^{q-2} (p|b|^2 + k\operatorname{Re}\{b\}) \prod_{j=0}^{k-1} \frac{|2pb + j|^2}{(j+1)^2}\right. \\
&\qquad\qquad \left. + (p|b|^2 + (q-1)\operatorname{Re}\{b\}) \prod_{j=0}^{q-2} \frac{|2pb + j|^2}{(j+1)^2}\right\}
\end{aligned}
$$

$$= \frac{(q-1)^2}{q^2}\prod_{j=0}^{q-1}\frac{|2pb+j|^2}{(j+1)^2} + \frac{(4p^2|b|^2+4p(q-1)\operatorname{Re}\{b\})}{q^2}\prod_{j=0}^{q-2}\frac{|2pb+j|^2}{(j+1)^2}$$

$$= \prod_{j=0}^{q-2}\frac{|2pb+j|^2}{(j+1)^2}\left\{\frac{(q-1)^2+4p(q-1)\operatorname{Re}\{b\}+4p^2|b|^2}{q^2}\right\}$$

$$= \prod_{j=0}^{q-1}\frac{|2pb+j|^2}{(j+1)^2},$$

showing that the result is valid for $m = q$. This proves the lemma.

THEOREM 4. *If* $f \in C(b, p)$, *then*

$$|a_n| \leqq \frac{p}{n}\prod_{k=0}^{n-(p+1)}\frac{|2pb+k|}{(k+1)}, \tag{3.13}$$

for $n \geqq p+1$, *and these bounds are sharp for each* n.

PROOF. Since $f \in C(b, p)$, from (1.4) we have

$$(zf''(z) + (2pb - p + 1)f'(z))\omega(z) = ((p-1)f'(z) - zf''(z)).$$

Hence

$$\begin{aligned}&\Big\{p(p-1)z^{p-1} + \sum_{k=1}^{\infty}(p+k)(p+k-1)a_{p+k}z^{p+k-1}\\ &\quad + (2pb-p+1)(pz^{p-1} + \sum_{k=1}^{\infty}(p+k)a_{p+k}z^{p+k-1})\Big\}\omega(z)\\ &= \Big\{(p-1)(pz^{p-1} + \sum_{k=1}^{\infty}(p+k)a_{p+k}z^{p+k-1})\\ &\quad - p(p-1)z^{p-1} - \sum_{k=1}^{\infty}(p+k)(p+k-1)a_{p+k}z^{p+k-1}\Big\},\end{aligned}$$

or

$$\begin{aligned}&\Big(p(p-1) + (2p^2b - p^2 + p) + \sum_{k=1}^{\infty}\Big\{(p+k)(p+k-1)\\ &\quad + (2pb-p+1)(p+k)\Big\}a_{p+k}z^k\Big)\omega(z)\\ &= \sum_{k=1}^{\infty}((p-1)(p+k) - (p+k)(p+k-1))a_{p+k}z^k,\end{aligned}$$

which may be written as

$$\begin{aligned}&\sum_{k=0}^{\infty}(((p+k)(p+k-1) + (2pb-p+1)(p+k))a_{p+k}z^k)\omega(z)\\ &\quad = \sum_{k=0}^{\infty}((p-1)(p+k) - (p+k)(p+k-1))a_{p+k}z^k,\end{aligned} \tag{3.14}$$

where $a_p = 1$ and $\omega(z) = \sum_{k=0}^{\infty} d_{k+1}z^{k+1}$.

Equating coefficients of z^m on both sides of (3.14), we obtain

$$\sum_{k=0}^{m-1}((p+k)(p+k-1)+(2pb-p+1)(p+k)a_{p+k}d_{m-k})$$
$$=((p-1)(p+m)-(p+m)(p+m-1))a_{p+m},$$

which shows that a_{p+m} on the right-hand side depends only on a_p, a_{p+1}, $\ldots$, $a_{p+(m-1)}$ of the left-hand side. Hence, for $k \geqq 0$, we write

$$\sum_{k=0}^{m-1}((k(p+k)+2pb(p+k))a_{p+k}z^k)\omega(z)$$
$$=\sum_{k=0}^{m}(-k(p+k))a_{p+k}\,z^k+\sum_{k=m+1}^{\infty}A_k z^k,$$

for $m = 1, 2, 3, \ldots$ and a proper choice of A_k.

Let $z = \mathrm{r}e^{i\theta}$, $o < r < 1$, $o \leqq \theta \leqq 2\pi$. Then

$$\sum_{k=0}^{m-1}|k(p+k)+2pb(p+k)|^2\,|a_{p+k}|^2\,r^{2k}$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=0}^{m-1}(k(p+k)+2pb(p+k))a_{p+k}r^k\,e^{i\theta k}\right|^2 d\theta$$
$$\geqq\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=0}^{m-1}(k(p+k)+2pb(p+k))a_{p+k}\,r^k\,e^{i\theta k}\right|^2|\omega(re^{i\theta})|^2\,d\theta$$
$$\geqq\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=0}^{m}(-k(p+k))a_{p+k}r^ke^{i\theta k}+\sum_{k=m+1}^{\infty}A_kr^ke^{i\theta k}\right|^2 d\theta$$
$$\geqq\sum_{k=0}^{m}k^2(p+k)^2\,|a_{p+k}|^2\,r^{2k}+\sum_{k=m+1}^{\infty}|A_k|^2\,r^{2k}$$
$$\geqq\sum_{k=0}^{m}k^2(p+k)^2\,|a_{p+k}|^2r^{2k}.$$

(3.15)

Setting $r \to 1$ in (3.15), the inequality (3.15) may be written as

$$(3.16)\quad \sum_{k=0}^{m-1}(|k(p+k)+2pb(p+k)|^2-k^2(p+k)^2)|a_{p+k}|^2\geqq m^2(p+m)^2|a_{p+m}|^2.$$

Simplification of (3.16) leads to

$$\sum_{k=0}^{m-1}4p(p+k)^2(p|b|^2+k\mathrm{Re}\{b\})\,|a_{p+k}|^2\geqq m^2(p+m)^2|a_{p+m}|^2,$$

i.e.,

$$(3.17)\quad |a_{p+m}|^2\leqq\frac{4p}{m^2(p+m)^2}\sum_{k=0}^{m-1}(p+k)^2(p|b|^2+k\mathrm{Re}\{b\})\,|a_{p+k}|^2.$$

Replacing $p + m$ by n in (3.17), we are led to

$$(3.18)\quad |a_n|^2 \leqq \frac{4p}{n^2(n-p)^2}\cdot \sum_{k=0}^{n-(p+1)} (p+k)^2(p|b|^2 + k\mathrm{Re}\{b\})|a_{p+k}|^2,$$

where $n \geqq p + 1$.

For $n = p + 1$, (3.18) reduces to

$$|a_{p+1}|^2 \leqq \left(\left(\frac{2p^2}{p+1}\right)|b|\right)^2$$

or

$$(3.19)\qquad |a_{p+1}| \leqq \left(\frac{2p^2}{p+1}\right)|b|,$$

which is equivalent to (3.13).

To establish (3.13) for $n > p + 1$, we will apply an induction argument. Fix n, $n \geqq p + 2$, and suppose (3.13) holds for $k = 1, 2, 3, \ldots, n - (p + 1)$. Then

$$(3.20)\quad |a_n|^2 \leqq \frac{p^2}{n^2}\left(\frac{4p}{(n-p)^2}\left(p|b|^2 + \sum_{k=1}^{n-(p+1)} (p|b|^2 + k\mathrm{Re}\{b\})\prod_{j=0}^{k-1}\frac{|2pb+j|^2}{(j+1)^2}\right)\right).$$

Thus, from (3.18), (3.20) and Lemma 8 with $m = n - p$, we obtain

$$|a_n|^2 \leqq \frac{p^2}{n^2}\prod_{j=0}^{n-(p+1)}\frac{|2pb+j|^2}{(j+1)^2}.$$

This completes the proof of (3.13). This proof is based on a technique found in Clunie [2].

For sharpness of (3.13) consider the function $f^*(z)$ defined by (3.6).

4. Properties of the class $C(b, p)$.

LEMMA 9. *If $f \in C(b, p)$, then the transformation F_a satisfying*

$$(4.1)\qquad F_a'(z) = \frac{pa^{p-1}z^{p-1} f'\left(\frac{z+a}{1+\bar{a}z}\right)}{f'(a)(z+a)^{p-1}(1+\bar{a}z)^{p(2b-1)+1}}, \qquad z \in U$$

and $F_a(o) = o$, is in $C(b, p)$, for all $|a| < 1$.

The proof of this lemma follows by using Lemmas 7 and 2.

LEMMA 10. *For $|z| \leqq r$ and f ranging over $C(b, p)$, the domain of values of $(zf''(z))/f'(z))$ is the disc with center $((p(2\mathrm{Re}\{b\} - 1) + 1)r^2 + (p-1))/((1 - r^2), (2p\,\mathrm{Im}\,\{b\}\,r^2)/(1 - r^2))$ and radius $(2p^2|b|r)/(1 - r^2)$.*

PROOF. Whenever $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in C(p, b)$, then $\lim_{z\to 0}((f''(z) - p(p - 1)z^{p-2}/z^{p-1}) = p(p + 1)a_{p+1}$. For $f \in C(b, p)$, let $F_a(z) = z^p + \sum_{n=p+1}^{\infty} A_n z^n \in C(b, p)$ be given by Lemma 9 for $|a| < 1$. By direct calculation we have

$$(4.2)\qquad p(p+1)A_{p+1}=p(1-|a|^2)\frac{f''(a)}{f'(a)}-p(p(2b-1)+1)|a|^2+p(p-1)/a.$$

Combining (3.4) and (4.2), we obtain

$$(4.3)\qquad \left|\frac{f''(a)}{f'(a)}-\frac{(p(2b-1)+1)\,|a|^2+(p-1)}{a(1-|a|^2)}\right|\leqq\frac{2p^2|b|}{(1-|a|^2)}.$$

From (4.3), it follws that, for $|z| = r < 1$,

$$(4.4)\qquad \left|\frac{zf''(z)}{f'(z)}-\frac{(p(2b-1)+1)\,r^2+(p-1)}{1-r^2}\right|\leqq\frac{2p^2|b|r}{1-r^2},$$

and the proof is completed.

THEOREM 5. *The sharp radius of convexity of the class $C(b, p)$ is*

$$(4.5)\qquad \{p|b| + (p^2|b|^2 - 2\mathrm{Re}\{b\} + 1)^{1/2}\}^{-1}.$$

PROOF. From (4.4), we have

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\geqq p\left(\frac{1-2p|b|r+(2\mathrm{Re}\{b\}-1)r^2}{1-r^2}\right),$$

where $|z| = r$. Thus $\mathrm{Re}(1 + (zf''(z)/f'(z))) > 0$, for

$$|z| = r < r_c = (p|b| + \sqrt{p^2|b|^2 - 2\mathrm{Re}\{b\} + 1})^{-1}.$$

To show that (4.5) is sharp, we let $f_*'(z) = pz^{p-1}[h_*'(z)]^p$, $h_*'(z) = (1-z)^{-2b}$ and $w = (r(r - p\sqrt{\bar{b}/b}))/(1 - rp\sqrt{\bar{b}/b})$ and obtain

$$1+\frac{wf_*''(w)}{f_*'(w)}=p\left(\frac{1-2p|b|r+(2b-1)r^2}{1-r^2}\right),$$

which has a zero real part at r given by (4.5). This completes the proof of the theorem.

5. Distortion and rotation theorems for the class C(*b*, *p*). It was shown [**8**] that, for $h(z) \in C(b)$,

$$(5.1)\qquad \Phi_2(r) \leqq \log|h'(z)| \leqq \Phi_1(r),$$

and

$$(5.2)\qquad \Psi_2(r) \leqq \arg(h'(z)) \leqq \Psi_1(r),$$

where

$$(5.3)\qquad \begin{aligned}\Phi_{1,2}(r) = {} & -2\mathrm{Re}\{b\}\log\left(\left(1-\left(\frac{r\mathrm{Im}\{b\}}{|b|}\right)^2\right)^{1/2}\mp\frac{r\mathrm{Re}\{b\}}{|b|}\right)\\ & \pm 2\mathrm{Im}\{b\}\sin^{-1}\left(\frac{r\mathrm{Im}\{b\}}{|b|}\right)\end{aligned}$$

and

$$\Psi_{1,2}(r) = -2\,\mathrm{Im}\{b\}\log\left(\left(1-\left(\frac{r\mathrm{Re}\{b\}}{|b|}\right)^2\right)^{1/2} \mp \frac{r\mathrm{Im}\{b\}}{|b|}\right) \pm 2\mathrm{Re}\{b\}\sin^{-1}\left(\frac{r\mathrm{Re}\{b\}}{|b|}\right). \tag{5.4}$$

The proof of the following two theorems follows from lemma 7 and the above bounds.

THEOREM 6. *If $f \in C(b, p)$, $p \geqq 1$, then for $|z| = r < 1$, we obtain*

$$p\,\Phi_2(r) \leqq \log\left|\frac{f'(z)}{pz^{p-1}}\right| \leqq p\,\Phi_1(r). \tag{5.5}$$

Equality is attained in the left and right of (5.5) for the function $f^*(z)$ defined by (3.6), for $z = re^{i\theta_j}$, $j = 1, 2$, where

$$\theta_{1,2} = \sin^{-1}\left(\frac{(r\mathrm{Im}\{b\}\mathrm{Re}\{b\}}{|b|^2} \pm \frac{\mathrm{Im}\{b\}}{|b|}\left(1 - \left(\frac{r\mathrm{Im}\{b\}}{|b|}\right)^2\right)^{1/2}\right).$$

THEOREM 7. *If $f \in C(b, p)$, $p \geqq 1$, then, for $|z| = r < 1$, we obtain*

$$(p-1)\theta + p\Psi_2(r) \leqq \arg(f'(z)) \leqq (p-1)\theta + \mathrm{p}\Psi_1(r). \tag{5.6}$$

Equality is attained in the left and right of (5.6) for the function $f^*(z)$ defined by (3.6) for $z = re^{i\theta_j}$, $j = 3, 4$, where

$$\theta_{3,4} = -\sin^{-1}\left(\frac{r\mathrm{Im}\{b\}\mathrm{Re}\{b\}}{|b|^2} \pm \frac{\mathrm{Re}\{b\}}{|b|}\left(1-\left(\frac{r\mathrm{Re}\{b\}}{|b|}\right)^2\right)^{1/2}\right).$$

Acknowledgment. In conclusion, I would like to thank Prof. D.K. Thomas for his kind encouragement and helpful guidance in preparing this paper.

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