ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
VOLUME 15, Number 2, Spring 1985

SOME ISOMORPHISM INVARIANTS OF INTEGRAL GROUP RINGS

STEVE KMET AND SUDARSHAN SEHGAL

Dedicated to the memory of E.G. Straus and R.A. Smith

1. Introduction. Let ZG be the ingegral group ring of a group G. Denote by $\{\gamma_i(G)\}$, and $\{\delta_i(G)\}$ the lower central series, and the derived series of G, respectively. Let us denote by $D_i(G)$ the ith dimension subgroup

$$D_i(G) = G \cap (1 + \Delta^i(G)),$$

where $\Delta(G)$ is the augmentation ideal of ZG. Suppose that the torsion elements of G form a subgroup $T = T(G)$. Then we write $T_1 = T$ and for $i \geqq 1$ we write

$$T_{i+1} = T_{i+1}(G) = [G, T_i(G)],$$

the group generated by all commutators $(g, t) = g^{-1}t^{-1}gt$, $g \in G$, $t \in T_i$. Our main result is

THEOREM A. *Suppose that G and H are groups such that the torsion elements $T(G)$ and $T(H)$ of G and H respectively form subgroups. Suppose $ZG \simeq ZH$. Then we have*

(1) $T_i(G)/T_{i+j}(G) \simeq T_i(H)/T_{i+j}(H)$ for $1 \leqq j \leqq i + 2$,

(2) $D_i(G) \cap T(G)/D_{i+j}(G) \cap T(G) \simeq D_i(H) \cap T(H)/D_{i+i}(H) \cap T(H)$ for $1 \leqq j \leqq i + 2$,

(3) $\gamma_i(T(G))/\gamma_{i+j}(T(G)) \simeq \gamma_i(T(H))/\gamma_{i+j}(T(H))$ for $1 \leqq j \leqq i$,

(4) $\delta_i(T(G))/\delta_{i+1}(T(G)) \simeq \delta_i(T(H))/\delta_{i+1}(T(H))$ for all i,

(5) $\delta_i(T(G))/[G, \delta_i(T(G))]' \simeq \delta_i(T(H))/[G, \delta_i(T(H))]'$ for all i.

As a special case we have the following result.

THEOREM B. *Suppose that G and H are torsion groups such that $ZG \simeq ZH$. Then we have*

Research supported by NSERC of Canada
Received by the editors on January 17, 1984.

(1) $\gamma_i(G)/\gamma_{i+j}(G) \simeq \gamma_i(H)/\gamma_{i+j}(H)$ for $1 \leqq j \leqq i + 2$,

(2) $D_i(G)/D_{i+j}(G) \simeq D_i(H)/D_{i+j}(H)$ for $1 \leqq j \leqq i + 2$,

(3) $\delta_i(G)/\delta_{i+1}(G) \simeq \delta_i(H)/\delta_{i+1}(H)$ for all i,

(4) $\delta_i(G)/[G, \delta_i(G)]' \simeq \delta_i(H)/[G, \delta_i(H)]'$.

Furukawa **[4]** has proved (1) and (2) with $1 \leqq j \leqq i$. He also proved (3). By taking $i = 2$ and $j = 4$ in (1) we have

COROLLARY. *If G and H are torsion groups with $D_6(G) = 1$ and $\mathbf{Z}G \simeq \mathbf{Z}H$, then $G' \simeq H'$.*

This result was proved by Ritter-Sehgal **[6]** with the further restriction that G' is of exponent p.

For some more notation; we shall write $\mathscr{U}(R)$ and $T\mathscr{U}(R)$ for the unit group and the set of torsion units of a ring R. We shall denote by $\Delta(G, A)$ the kernel of the map $\mathbf{Z}G \to \mathbf{Z}(G/A)$ if A is a normal subgroup of G.

2. Some torsion free subgroups. It is well known **[1]** that if A is an abelian normal subgroup of a finite group G then $\mathscr{U}(1 + \Delta(G, A)^2)$ is torsion free. We shall need an extension of this result.

THEOREM 1. *Let A be a nilpotent normal torsion subgroup of a group G. Let $\mathscr{Z}$ be the centre of A. Then*

$$T\mathscr{U}(1 + \Delta(A)\Delta(\mathscr{Z})\mathbf{Z}G) = 1.$$

We shall first obtain the next result from which Theorem 1 will easily follow.

THEOREM 2. *Let A be a nilpotent p-group of bounded exponent, where p is a fixed prime. Let $\mathscr{Z}$ be a central subgroup of A. Suppose that I is an ideal in $\mathbf{Z}A$ (written $I \lhd \mathbf{Z}A$). Then*

$$I \subseteq \Delta(A)\Delta(\mathscr{Z}),\ pI \subseteq I^p \Rightarrow I = 0.$$

PROOF. Let us first suppose that $\mathscr{Z}$ is finite. We prove the result in this case by induction on the order of $\mathscr{Z}$. If $|\mathscr{Z}| = 1$ there is nothing to prove. We choose an element z of $\mathscr{Z}$ of order p and conclude by induction that

$$I \subseteq (1 - z)\mathbf{Z}A = \Delta(A, \langle z \rangle).$$

We claim that

(*) $$(1 - z)\mathbf{Z}A \cap \Delta(A)\Delta(\mathscr{Z}) = (1 - z)\Delta(A).$$

To see this, let α be an element in the intersection. Write $\alpha = (1 - z)\gamma$, for $\gamma \in \mathbf{Z}A$. Then

$$\alpha \equiv c(1 - z) + \delta, \qquad c \in \mathbf{Z}, \qquad \delta \in (1 - z)\Delta(A)$$
$$\equiv (1 - z^c) \bmod (1 - z)\Delta(A).$$

Since $\alpha \in \Delta(A)\Delta(\mathscr{Z})$ it follows that $1 - z^c \in \Delta(A)\Delta(\mathscr{Z})$. We conclude by [7, p. 102] that

$$1 - z^c \in \Delta(\mathscr{Z})^2$$

and thus by [7, p. 75] $1 - z^c = 0$ as $\mathscr{Z}$ is abelian. We have proved that $\alpha \in (1 - z)\Delta(A)$ and (*) is established. Hence

$$I \subseteq \Delta(A)\Delta(\mathscr{Z}) \cap (1 - z)\mathbf{Z}A = (1 - z)\Delta(A).$$

Suppose that $I \subseteq (1 - z)\Delta(A)^{\ell}$, where ℓ is a natural number. Then by hypothesis $pI \subseteq I^p \subseteq (1 - z)^p\Delta(A)^{p\ell}$. Since z has order p we have $1 = (1 - z + z)^p = 1 + \binom{p}{1}(1 - z)z^{p-1} + \cdots + (1 - z)^p$. This implies that $(1 - z)^p \in p(1 - z)\mathbf{Z}A$. We conclude that

$$pI \subseteq p(1 - z)\Delta(A)^{p\ell}.$$

Thus $I \subseteq (1 - z)\Delta(A)^{p\ell}$. Hence $I \subseteq \Delta(A)^{\omega}$, which is zero by a theorem of Hartley [5]. This completes the proof of the theorem in the case that $\mathscr{Z}$ is finite. Now suppose that $\mathscr{Z}$ is infinite. But it is a commutative group of bounded exponent. It follows by [2, p. 88] that $\mathscr{Z}$ is a direct sum of cyclic groups. Let B_ν be a subgroup of $\mathscr{Z}$ obtained by dropping a finite number of factors. Then applying what we have proved to A/B_ν we conclude that $I \subseteq \Delta(G, B_\nu)$. To finish the proof we only have to observe that $\bigcap_\nu \Delta(G, B_\nu) = 0$.

LEMMA 3. *Let A be a normal nilpotent subgroup of exponent n contained in G. Then $\mathscr{U}(1 + \Delta(G, A))$ has no torsion elements of order relatively prime to n.*

PROOF. Suppose $\mathscr{U}(1 + \Delta(G, A))$ has a torsion element $(1 + \delta)$ with $(1 + \delta)^q = 1$ where q is a prime not dividing n. It suffices to prove that $\delta = 0$. We use induction on n. If n has at least two distinct prime factors we can write $A = A_1 \times A_2$ where A_1 is p-Sylow subgroup. By induction $\delta \in \Delta(G, A_1)$. Thus we may assume to begin with that A is a p-group. We have

$$1 = (1 + \delta)^q = 1 + q\delta + \binom{q}{2}\delta^2 + \cdots + \delta^q.$$

It follows that $q\delta \in \Delta(G, A)^2$. Moreover, for any $a \in A$,

$$o(a)(1 - a) \in \Delta(G, A)^2.$$

Thus there exists m such that $p^m\delta \in \Delta(G, A)^2$. Since $(p, q) = 1$ we deduce that $\delta \in \Delta(G, A)^2$. Repeating this argument we conclude that $\delta \in \Delta(G, A)^\omega$ which is zero by a theorem of Hartley [5].

PROOF OF THEOREM 1. Suppose that $(1 + \delta)^p = 1$ where p is a prime and $\delta = \sum_1^n x_i\delta_i \in \Delta(A)\Delta(\mathscr{Z})\mathbf{Z}G$ where $\delta_i \in \Delta(A)\Delta(\mathscr{Z})$ and x_i are different coset representatives of G/A. Note that $\delta_1, \ldots, \delta_n$ involve a finite subset X of elements of A in their supports; and in their expressions as elements of $\Delta(A)\Delta(\mathscr{Z})$ they involve a finite set Y of elements of A. We replace A by the normal subgroup generated by $\langle X, Y\rangle$, which is a nilpotent group of bounded exponent. This fact is well known and may be deduced from a theorem of Schur [7, p. 39]. So we may assume that A is a normal nilpotent subgroup of G of bounded exponent. We use induction on the number of primes in this exponent. If $A = A_1 \times B$ where $A_1 \neq 1$ is a p-group and $B \neq 1$ is a p'-group, we conclude that $\delta \in \Delta(G, B)$. It follows by Lemma 3 that $\delta = 0$. Thus we may assume to begin with that A is a p-group. Let I be the smallest ideal of $\mathbf{Z}A$ containing $\delta_1, \ldots, \delta_n$ and invariant under conjugation by G. Then $I \subseteq \Delta(A)\Delta(\mathscr{Z})$. We claim that $pI \subseteq I^p$. The equality $(1 + \delta)^p = 1$ gives

$$p\delta + \binom{p}{2}\delta^2 + \cdots + \delta^p = 0.$$

This implies that $p\delta \in \delta^p\mathbf{Z}G \subseteq I^p\mathbf{Z}G$. We have

$$\sum_i px_i\delta_i \in I^p\mathbf{Z}G.$$

Hence, $p\delta_i \in I^p$ and $pI \subseteq I^p$ as claimed. It follows by Theorem 2 that $I = 0$ and thus $\delta_i = 0$, $\delta = 0$.

3. Some Lemmas.

LEMMA 4. *Let N be a torsion central subgroup of G. Then*
(1) $T\mathscr{U}(1 + \Delta(G, N)) = N$, and
(2) $T\mathscr{U}(1 + \Delta(G)\Delta(N)) = 1$.

PROOF. (1) is contained in [6, p. 34]. To prove the second part observe that $1 + \Delta(G)\Delta(N) \subseteq 1 + \Delta(G, N)$ and thus $T\mathscr{U}(1 + \Delta(G)\Delta(N)) \subseteq N$. Therefore, if $n - 1 \in \Delta(G)\Delta(N)$ and $n \in N$ we get $n \in N'$ [7, p. 102 & 75]. Hence $n = 1$.

As usual we assume, without loss of generality that all group ring isomorphisms are augmentation preserving.

LEMMA 5. *Suppose that $\mathbf{Z}G \simeq^\theta \mathbf{Z}H$. Suppose that A and B are normal torsion subgroups of G and H respectively with $\theta\Delta(G, A) = \Delta(H, B)$. Then $\theta\Delta(G, [G, A]) = \Delta(H, [H, B])$.*

PROOF. We first observe that

$$\Delta(G, [G, A]) \subseteq \Delta(G)\Delta(G, A) + \Delta(G, A)\Delta(G).$$

This follows because for $g \in G$, $a \in A$, $g^{-1}a^{-1}ga - 1 = g^{-1}a^{-1}[(g - 1) \cdot (a - 1) - (a - 1)(g - 1)]$. Let $a \in [G, A]$. Then $a - 1 \in \Delta(G)\Delta(G, A) + \Delta(G, A)\Delta(G)$. Applying θ we get $\theta(a) \in 1 + \Delta(H)\Delta(H, B) + \Delta(H, B)\Delta(H)$. Factoring by $[H, B]$ we conclude

$$\overline{\theta(a)} \in 1 + \Delta(\bar{H})\Delta(\bar{H}, \bar{B}), \qquad [\bar{H}, \bar{B}] = 1.$$

It follows by Lemma 4 that $\overline{\theta(a)} = 1$. Thus

$$\theta(a) \in 1 + \Delta(H, [H, B])$$

and

$$\theta\Delta(G, [G, A]) \subseteq \Delta(H, [H, B]).$$

The reverse inclusion follows by symmetry, proving the lemma.

The next result is due to Furukawa [3].

LEMMA 6. *Suppose that $\mathbf{Z}G \simeq^{\theta} \mathbf{Z}H$. Suppose that G_1 and H_1 are subgroups of G and H respectively. Suppose that $I \lhd \mathbf{Z}G$ such that $\theta(G_1(1 + I)) = H_1(1 + \theta(I))$. Then*

$$G_1/G_1 \cap (1 + I) \simeq H_1/H_1 \cap (1 + \theta(I)).$$

PROOF. Define $\gamma: G_1 \to H_1/H_1 \cap (1 + \theta(I)) = \bar{H}_1$ by

$$\gamma(g_1) = \bar{h}_1 \text{ if } \theta(g_1) = h_1(1 + \theta(i)), \qquad g_1 \in G_1, h_1 \in H_1, i \in I.$$

It is easy to check that γ is an homomorphism with kernel $G_1 \cap (1 + I)$.

LEMMA 7. *Suppose that $\mathbf{Z}G \simeq_{\theta} \mathbf{Z}H$. Suppose that $A_1 \leqq A_2$ and $B_1 \leqq B_2$ are normal torsion subgroups of G and H respectively. Suppose that $\theta\Delta(G, A_i) = \Delta(H, B_i)$ for $i = 1, 2$. Further suppose that A_2/A_1 and B_2/B_1 are nilpotent. Then*

$$\theta\Delta(G, [A_1, A_2]) = \Delta(H, [B_1, B_2]).$$

PROOF. From the fact $[A_1, A_2] \subseteq 1 + \Delta(G, A_1)\Delta(G, A_2) + \Delta(G, A_2) \cdot \Delta(G, A_1)$, it follows, for $a \in [A_1, A_2]$, that

$$\theta(a) \in 1 + \Delta(H, B_1)\Delta(H, B_2) + \Delta(H, B_2)\Delta(H, B_1).$$

Factoring by $[B_1, B_2]$ we conclude $\overline{\theta(a)} \in 1 + \Delta(\bar{H}, \bar{B}_1)\Delta(\bar{H}, \bar{B}_2)$. Since B_2/B_1 is nilpotent, so is $B_2/[B_1, B_2] = \bar{B}_2$. Applying Theorem 1 we deduce that $\overline{\theta(a)} = 1$. Thus

$$\theta(a) \in 1 + \Delta(H, [B_1, B_2]).$$

The lemma is proved due to symmetry.

The next lemma is a crucial result from which our Theorem A will follow easily.

LEMMA 8. *Suppose that* $\mathbf{Z}G \simeq_\theta \mathbf{Z}H$. *Suppose that* $A \supseteq N$ *and* $B \supseteq M$ *are normal subgroups of G and H respectively. Further suppose that*

(1) A/N, B/M *are torsion,*

(2) $[[A, G], [A, G]] \subseteq N$, $[[B, H], [B, H]] \subseteq M$, and

(3) $\theta\Delta(G, N) = \Delta(H, M)$, $\theta\Delta(G, A) = \Delta(H, B)$.

Then $A/N \simeq B/M$.

PROOF. Write $G_1 = G/N$, $H_1 = H/M$. Then $\mathbf{Z}G_1 \simeq \mathbf{Z}H_1$ with $A_1 = A/N$, $B_1 = B/M$, and $\theta\Delta(G_1, A_1) = \Delta(H_1, B_1)$. Let $a_1 \in A_1$. Then $\theta(a_1) \in 1 + \Delta(H_1, B_1)$. Factoring by $[H_1, B_1]$ we have $\overline{\theta(a_1)} \in 1 + \Delta(\bar{H}_1, \bar{B}_1)$. But $\bar{B}_1$ is central in $\bar{H}_1$. It follows by Lemma 4 that $\overline{\theta(a_1)} \in \bar{B}_1$. Thus $\theta(a_1) = b_0(1 + \delta)$, for $\delta \in \Delta(H_1, [H_1, B_1])$ and $b_0 \in B_1$. It follows by a well known argument of Whitcomb [7, p. 103] that $1 + \delta \equiv b_2 \mod \Delta(H_1)\Delta([H_1, B_1])$ for some $b_2 \in [H_1, B_1]$. Thus $\theta(a_1) = b_1(1 + \delta_1)$, $b_1 = b_0 b_2 \in B_1$, $\delta_1 \in \Delta(H_1)\Delta([H_1, B_1])$. We have seen that

$$\theta(A_1) \subseteq B_1(1 + \Delta(H_1)\Delta(H_1, [H_1, B_1])).$$

It follows by Lemma 5 that

$$\theta(A_1(1 + \Delta(G_1)\Delta(G_1, [G_1, A_1]))) \subseteq B_1(1 + \Delta(H_1)\Delta(H_1, [H_1, B_1])).$$

By symmetry we get equality. Now we deduce by Lemma 6 that

$$A_1/A_1 \cap (1 + \Delta(G_1)\Delta(G_1, [G_1, A_1])) \simeq B_1/B_1 \cap (1 + \Delta(H_1)\Delta(H_1, [H_1, B_1])).$$

It follows by (2) of the hypothesis and [7, p. 75] that $A_1 \simeq B_1$.

4. Proof of Theorem A. (1) $\mathbf{Z}G/\Delta(G, T(G)) \simeq \mathbf{Z}(G/T(G))$ has no torsion units [7, p. 176] and $\Delta(G, T(G))$ is the smallest ideal I such that $\mathscr{U}(\mathbf{Z}G/I)$ is torsion free. Thus we conclude that

$$\theta\Delta(G, T(G)) = \Delta(H, T(H)).$$

It follows by Lemma 5 that $\theta\Delta(G, T_i(G)) = \Delta(H, T_i(H))$. We shall apply Lemma 8. Write

$$A = T_i(G),\ N = T_{i+j}(G),$$
$$B = T_i(H),\ M = T_{i+j}(H), \qquad 1 \leqq j \leqq i + 2.$$

Then $[[T_i(G), G], [T_i(G), G]] \subseteq T_{2i+2}(G) \subseteq N$. The hypothesis of Lemma 8 is satisfied. Thus $A/N \simeq B/M$.

(2) We wish to prove that $(D(_iG) \cap T(G))/(D_{i+j}(G) \cap T(G))$ with $1 \leqq j \leqq i + 2$ is an isomorphism invariant. We shall apply Lemma 8. We shall first prove that

$$\theta\Delta(G, D_i(G) \cap T(G)) = \Delta(H, D_i(H) \cap T(H)).$$

We use induction on i. For $i = 1$, this simply says $\theta\Delta(G, T(G)) = \Delta(H, T(H))$ which we know is true. Now, let $i \geqq 1$ and conclude by induction that

$$(*) \qquad \theta\Delta(G, D_{i+1}(G) \cap T(G)) \subseteq \Delta(H, D_i(H) \cap T(H)) \cap \Delta^{i+1}(H).$$

Factoring by $D_{i+1}(H) \cap T(H)$ we conclude

$$\overline{\theta(D_{i+1}(G) \cap R(G))} \subseteq 1 + \Delta(\bar{H}, \overline{D_i(H) \cap T(H)}) \cap \Delta^{i+1}(\bar{H}).$$

But $D_i(H) \cap T(H)$ is central modulo $D_{i+1}(H) \cap T(H)$. It follows by Lemma 4 that

$$\overline{\theta(D_{i+1}(G) \cap T(G))} \subseteq \overline{D_i(H) \cap T(H)}.$$

Thus if $g \in D_{i+1}(G) \cap T(G)$, then $\theta(g) = h(1 + \delta)$, for $h \in D_i(H) \cap T(H)$, and $\delta \in \Delta(H, D_{i+1}(H) \cap T(H))$. We know by (*) that $\theta(g) \in 1 + \Delta^{i+1}(H)$. Thus $h(1 + \delta) \in 1 + \Delta^{i+1}(H)$. It follows that $h \in D_{i+1}(H)$. Hence

$$\theta\Delta(G, D_{i+1}(G) \cap T(G)) \subseteq \Delta(H, D_{i+1}(H) \cap T(H)).$$

Equality follows by symmetry. Now apply Lemma 8 by taking

$$A = D_i(G) \cap T(G),\ N = D_{i+j}(G) \cap T(G), \qquad 1 \leqq j \leqq i + 2.$$

Then $[[A, G], [A, G]] \subseteq (D_{i+1}(G) \cap T(G))' \subseteq D_{2i+2}(G) \cap T(G) \subseteq N$. Hence $D_i(G) \cap T(G)/D_{i+j}(G) \cap T(G)$ is preserved as desired.

(3) We wish to prove that $\gamma_i(T(G))/\gamma_{i+j}(T(G))$, where $1 \leqq j \leqq i$, is preserved. We take

$$A = \gamma_i(T(G)),\ N = \gamma_{i+j}(T(G)), \qquad 1 \leqq j \leqq i.$$

We know $\theta\Delta(G, T(G)) = \Delta(H, T(H))$. Suppose that $\theta\Delta(G, \gamma_m(T(G))) = \Delta(H, \gamma_m(T(H)))$. Now apply Lemma 7, using the fact that $T(G)/\gamma_m(T(G))$ is nilpotent, to conclude that

$$\theta\Delta(G, \gamma_{m+1}(T(G)) = \Delta(H, \gamma_{m+1}(T(H)).$$

Notice that $[[A, G], [A, G]] \subseteq \gamma_{2i}(T(G)) \subseteq N$. The hypotheses of Lemma 8 are satisfied. The result follows.

(4) We wish to prove that $\delta_i(T(G))/\delta_{i+1}(T(G))$ is an isomorphism invariant. By Lemma 7 we know that

$$\theta\Delta(G, \delta_i(T(G))) = \Delta(H, \delta_i(T(H))).$$

Moreover, take $A = \delta_i(T(G))$, $N = \delta_{i+1}(T(G))$ so that

$$[[A, G], [A, G]] \subseteq [\delta_i(T(G)), \delta_i(T(G))] \subseteq N.$$

It follows that A/N is an isomorphism invariant.

(5) Now take $A = \delta_i(T(G))$, $N = [G, \delta_i(T(G))]'$. Then $[A, G]' \subseteq N$ and the result follows as above.

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UNIVERSITY OF ALBERTA EDMONTON, CANADA T6G 2G1