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APPENDIX

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This chapter is one of the most interesting in Ramanujan's Second Notebook and it shows Ramanujan's strengths and weaknesses better than any of the other chapters with the possible exception of the work on modular functions and forms that makes up much of the last part of this Notebook. First to the strengths, which can be illustrated both by reference to important work of Euler and Gauss, and also by some results that are unlikely to have been found by anyone else.

Gauss **[12]** defined two hypergeometric functions to be contiguous if they have the same power series variable, if two of the parameters are pairwise equal, and the third pair differ by one. Thus the functions contiguous to ${}_2F_1(a, b; c; x)$ are $F(a\pm) = {}_2F_1(a \pm 1, b; c; x)$, $F(b\pm)$ and $F(c\pm) = {}_2F_1(a, b; c \pm 1; x)$. Gauss showed that a hypergeometric function and any two contiguous to it are linearly related, and gave the fifteen formulas (actually nine different ones when the symmetry in a and b is used). These can be iterated, so any three hypergeometric functions whose parameters differ by integers are linearly related. Gauss used the linear relation between ${}_2F_1(a, b; c; x)$, ${}_2F_1(a, b + 1; c + 1; x)$ and ${}_2F_1(a + 1, b + 1; c + 2; x)$ to obtain the continued fraction in Entry 20.

Much earlier Euler **[10]** considered the integral

$$\int_0^1 (1 - xt)^{-a} t^{b-1}(1 - t)^{c-b-1}\, dt$$

and after integration by parts and a little algebra obtained a three term recurrence relation that he used to find a continued fraction. Later **[11**, vol. 2, §1, problem 130] he showed that

$${}_2F_1(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(c - b)\,\Gamma(b)} \int_0^1 (1 - xt)^{-a} t^{b-1}(1 - t)^{c-b-1}\, dt.$$

Using this it is easy to see that his continued fraction is also an expansion of ${}_2F_1(a, b + 1; c + 1; x)/{}_2F_1(a, b; c; x)$. Surprisingly this continued fraction is not the same as Gauss's. What Euler had done was to derive the three term recurrence between ${}_2F_1(a, b; c; x)$, ${}_2F_1(a, b + 1; c + 1; x)$

and ${}_2F_1(a, b + 2; c + 2; x)$. Ramanujan rediscovered Euler's continued fraction in Entry 22.

Many interesting continued fractions are special or limiting cases of these two expansions. Two types of limits are confluence

$$ {}_1F_1(a; c; x) = \lim_{b\to\infty} {}_2F_1(a, b; c; x/b) $$

$$ {}_0F_1(-; c; x) = \lim_{a\to\infty} {}_1F_1(a; c; x/a) $$

and differentiation in the form

$$ \lim_{a\to 0} [{}_2F_1(a, b; c; x) - 1]/a. $$

Ramanujan gave examples of both of these limits. He had a remarkable eye for interesting special cases.

Entry 17 is very impressive and shows Ramanujan's deep understanding of formulas. Entry 40 is also very impressive and I do not understand it yet. It is possibly related to Entries 20 and 22 in the following way.

At the end of his paper on hypergeometric series, Kummer **[16]** remarked that he had tried to extend his results to ${}_3F_2(a, b, c; d, e; x)$, but had only been successful when $x = 1$. He stated two results, a series transformation and the evaluation of a special series, but it is clear from the second of these results that he had more. Ramanujan considered this series and more general ones in Chapters 10 and 11, and found most of the series summations that are given in [7], and many of the transformations and other formulas given in this book. He rediscovered Dougall's sum of the very well poised two balanced ${}_7F_6$, but does not seem to have found Whipple's transformation between a very well-poised ${}_7F_6$ and a balanced ${}_4F_3$. However, with the exception of the recurrence relation in Entry 24 (see (24.3)), Ramanujan did not seem to realize there are very important three term recurrence relations for some higher hypergeometric series. He was not alone, since none are given in [7].

The series Kummer considered, ${}_3F_2(a, b, c; d, e; 1)$, behaves very much like the ${}_2F_1$, and this series and two contiguous to it are linearly related. This is also true for the balanced ${}_4F_3$; and so, by Whipple's transformation, for the very well poised ${}_7F_6$. Here contiguous needs to be modified slightly, to allow a minimal number of changes while keeping the type of the series. It is possible that an extension of Entry 40 can be obtained in this way, with Entry 40 arising when the very well poised ${}_7F_6$ series are two balanced, and so can be summed. If so, then it is unlikely Entry 40 holds in the nonterminating case, since termination is required to sum balanced series in general, i.e., without further restrictions on the parameters.

The existence of some of these recurrence relations was implicitly known over a hundred years ago, and some have been used extensively in angular momentum theory for the last forty years, but complete listings of the fundamental ones have only recently been published **[20]**, **[21]**, **[26]** or worked out **[27]**.

Ramanujan's greatest weakness was his poor education. As Hardy wrote, "So little was wanted, £60 a year for five years, occasional contact with almost anyone who had real knowledge and a little imagination, for the world to have gained another of its greatest mathematicians." Note Hardy wrote mathematicians, not number theorists. Ramanujan was a great number theorist, but he was more. Even without a good education he found a number of new results in other areas that no one else has found up to the present. To give an idea of what Ramanujan could have done in one other area, consider orthogonal polynomials. A set of polynomials $\{p_n(x)\}$ is orthogonal if there is a positive measure $d\mu(x)$ with finite moments of all orders so that

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)d\mu(x) = 0, \qquad m \neq n. \tag{A1}$$

Any set of orthogonal polynomials satisfies a three term recurrence relation:

$$\begin{aligned} &xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x), \\ &p_0(x) = 1, \qquad p_{-1}(x) = 0, \\ &A_n,\ B_n,\ C_{n+1} \text{ real},\ A_n C_{n+1} > 0, \qquad n = 0, 1, \cdots. \end{aligned} \tag{A2}$$

Conversely, any set of polynomials that satisfies (A2) is orthogonal with respect to some positive measure, which may not be unique. See the comments to paper [68–1] in [**25**, volume 3, 866–867] for references. There are many important orthogonal polynomials whose recurrence relations are contained in the contiguous relations of Gauss and their iterates. Among these are Jacobi polynomials, which include the spherical functions on spheres and projective spaces as special cases; Krawtchouk polynomials, which play an important role in coding theory; and Pollaczek polynomials, which among other uses are the random walk polynomials associated with birth and death processes with linear growth parameters. At the ${}_3F_2$ level there are two important sets of orthogonal polynomials. One that would have interested Ramanujan is

$$R_n(x^2) = R_n(x^2; a, b, c) = {}_3F_2\left({-n, a+ix, a-ix \atop a+b, a+c}; 1\right).$$

When a, b, $c > 0$ the orthogonality is

$$\int_0^{\infty} R_n(x^2)R_m(x^2)\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2 dx = 0, \qquad m \neq n. \tag{A3}$$

See Wilson **[28]**. These polynomials determine the behavior of a birth and death process whose birth and death rates are quadratic in the population size and grow at the same rate. See **[22]**. Ramanujan would have been interested in the integral

$$\int_0^\infty \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)} \right|^2 dx$$

since he evaluated a number of integrals like this **[18]**, but does not seems to have done this one.

For different conditions on the parameters these polynomials are orthogonal with respect to a discrete measure on a finite set of points, and the polynomials can be transformed into the series that occurs in the Clebsch-Gordan coefficients, or $3-j$ symbols, of angular momentum theory. The weight function in this case has a sum which Ramanujan evaluated. It is a special case of Dougall's summation of the very well poised ${}_5F_4$.

In addition to the obvious connection between orthogonal polynomials and continued fractions that comes from the recurrence relation (A2), there is a deeper one that goes back to Gauss in the case of Legendre polynomials and was given by Stieltjes and Markov in the general case. Let $q_n(x)$ denote the second solution to (A2) with $q_0(x)=0$, $q_1(x)=A_0^{-1}$. Then under suitable conditions.

$$\text{(A4)} \qquad \lim_{n\to\infty} \frac{q_n(x)}{p_n(x)} = A \int_{-\infty}^{\infty} \frac{d\mu(t)}{t-x}$$

and this is also the limit of the continued fraction generated by (A2). Some examples are given in **[4]**, **[6]**, **[9]**, **[17]**, as are references to earlier work. Ramanujan would have enjoyed doing some of these examples since it combined two topics he loved, continued fractions and definite integrals.

The recurrence relation (24.3) can be rewritten so it gives the recurrence relation for a set of orthogonal polynomials, which are called modified Lommel polynomials. See Ismail **[15]** for extensions and references to earlier work.

There is one very surprising omission in this chaper. It contains no continued fractions related to basic hypergeometric series. A generalized hypergeometric series, or a hypergeometric series for short, is a series $\sum c_n$ with c_{n+1}/c_n a rational function of n. A basic hypergeometric series has c_{n+1}/c_n a rational function of q^n for a fixed q. Heine **[14]** introduced a q-extension of the ${}_2F_1$:

$$\text{(A5)} \qquad {}_2\phi_1(a,b;c;q,x) = \sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n} x^n$$

where

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \qquad n = 1, 2, \cdots,$$
$$(a;q)_0 = 1.$$

Two of these series are contiguous if they are the same except for one parameter, and this pair differs by a factor of q. Heine worked out the three term contiguous relations and from iterates found an extension of Gauss's continued fraction. Ramanujan does not seem to have found this, which is surprising. He had considered some continued fractions that come from recurrence relations whose coefficients involve q^n. The continued fraction

$$\frac{1}{1+}\;\frac{q}{1+}\;\frac{q^2}{1+}\;\frac{q^3}{1+\cdots}$$

played an important role in Hardy's early appreciation of Ramanujan. Recall that Hardy wrote [**13**, p. 9]; "... (1.10)–(1.12) defeated me completely; I had never seen anything in the least like them before." These three identities all use this continued fraction. These identities are contained in later chapters in the Second Notebook, and there are a few other continued fractions whose elements are functions of q^n rather than n. Also Ramanujan extended some of the series identities in Chapter 10 to basic hypergeometric series in Chapter 16, so he was aware such extensions were possible. Well after the Second Notebook was written Ramanujan read a very important paper of L. J. Rogers [**23**]. This paper is the third in a series of four. The first one was written to try to explain some transformation formulas of Heine for (A5). Thus it is probable Ramanujan was aware of Heine's work, which might explain why he did not include it in the pages G. Andrews has called the "Lost Notebook". There is another connection with orthogonal polynomials in Rogers's paper [**23**], but no one was aware of it at the time. Rogers's derivation of the Rogers-Ramanujan identities comes from his determination of the expansion coefficients when a q-extension of Hermite polynomials and Chebychev polynomials are expanded in terms of each other. The one person who probably would have recognized these q-Hermite polynomials as orthogonal polynomials was Stieltjes, and he unfortunately died shortly after the appearance of [**23**]. The next paper of Rogers [**24**] introduced a more general set of polynomials, q-extensions of ultraspherical polynomials. They are also orthogonal, and their weight function is the type of function Ramanujan would have liked. It is

$$w(x) = \prod_{k=0}^{\infty} \frac{1-2(2x^2-1)q^k+q^{2k}}{1-2(2x^2-1)\beta q^k+\beta^2 q^{2k}} \quad \frac{1}{\sqrt{1-x^2}}$$

when $-1 < \beta,\ q < 1$. The integral of $w(x)$ is an extension of the symmetric beta integral

$$\int_{-1}^{1} (1 - x^2)^a \, dx.$$

The orthogonality of the continuous q-ultraspherical polynomials of Rogers can be used in a direct way to solve the connection coefficient problem for these polynomials, and so leads to a well motivated derivation of the Rogers-Ramanujan identities. See **[3]** and **[8]**. In addition to his interest in the R.-R. identities, Ramanujan found two extensions of the beta integral on $[0, \infty)$. In **[18]** he stated

$$\int_0^\infty t^{x-1} \frac{(-tq^{x+y}; q)_\infty}{(-t; q)_\infty} dt = \frac{\pi(q^{1-x}; q)_\infty (q^{x+y}; q)_\infty}{\sin \pi x (q; q)_\infty (q^y; q)_\infty}.$$

In **[19**, Chapter XVI, Entry 17, p. 196**]** he stated

$$\sum_{-\infty}^{\infty} \frac{(bq^n; q)_\infty}{(aq^n; q)_\infty} t^n = \frac{(at; q)_\infty (q/at; q)_\infty (q; q)_\infty (b/a; q)_\infty}{(t; q)_\infty (b/at; q)_\infty (a; q)_\infty (q/a; q)_\infty}.$$

Both of these extend

$$\int_0^\infty t^{x-1}(1 + t)^{-x-y} dt = \Gamma(x)\Gamma(y)/\Gamma(x + y).$$

See **[1]** for simple proofs. It is clear from these remarks that Ramanujan would have liked the integral of $w(x)$. There is a more general integral that gives the orthogonality for a more general set of orthogonal polynomials, see **[5]**. I have finally figured out a proof of the type that Ramanujan would have worked out very rapidly **[2]**.

There is one aspect of Ramanujan's work that shows in this chapter and it can be considered a strength or a weakness depending on how one looks at it. This is Ramanujan's preference for specific results over general ones. He was more like Euler than Gauss in this respect. Consider the continued fractions in Entries 20 and 22. Gauss distilled the essence of this by finding all the three term contiguous relations, while Euler and Ramanujan were content to work out specific examples, but did not treat the subject exhaustively. I know this is an oversimplification for both Gauss and Euler, since Gauss treated a number of isolated examples that others built into big theories, and Euler often got to the heart of a problem rather than just starting it, but I think it is an apt description of Ramanujan. He loved specific results, and judging by the material in the "Lost Notebook", when he had little time left he spent it trying to find completely new results rather than trying to more fully understand those he had discovered previously. I think this is a strength, but others might disagree. We need both types, and Ramanujan's imagination was so powerful we can only be glad that he opened up as many topics for us as he did.

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