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ON THE BURGE CORRESPONDENCE BETWEEN PARTITIONS AND BINARY WORDS

GEORGE E. ANDREWS* AND
D. M. BRESSOUD**

Dedicated to the memory of Ernst Straus

In 1981 Burge [3] published a very elegant solution to a problem posed by the first author, ([1], p. 156, Question 1) and described in detail below, that of finding a purely constructive bijection between two sets of partitions, each of which arises in generalization of the Rogers-Ramanujan identities. The key idea in Burge's proof is to establish separate correspondences between each of the sets of partitions and a certain set of binary words, the number being partitioned corresponding to the major index of the word.

This turned out to be a fruitful approach to partition identities, and in a subsequent article [4] Burge supplied a devastatingly simple proof of the principal result in the second author's memoir [2]. The purpose of this paper is to further strengthen the case for viewing partitions as binary words by demonstrating how this viewpoint leads to simple proofs of partition identities which on one side count partitions in which the parts are restricted to certain congruence classes. More than this, our new point of view actually leads to new such identities. As an example, we shall sketch the proof of the following generalization of the Göllnitz-Gordon identities.

THEOREM 1. *Let δ, i, k be integers satisfying $\delta = 0$ or 1, $1 \leqq 2i - \delta < 2k$, $2i - \delta \not\equiv 2 \pmod 4$. Given a partition of n, f_j denotes the number of times the part j appears in the partition. The following two sets of partitions are equinumerous:*

(1) *partitions of n in which no part is congruent to* $2 \pmod 4$ *or congruent to* $0, \pm (2i - \delta) \pmod{4k}$.

(2) *partitions of n in which $f_1 \leqq 2i - 1 - \delta$, $f_j + f_{j+1} \leqq 2k - 1 - \delta$ for all j, f_{2j} is always even, and if $\delta = 0$ and $f_j + f_{j+1} \geqq 2k - 2$ then*

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$$(j-1)f_j + jf_{j+1} \equiv 2(k+i+\sum_{t<j} f_t) \pmod 4).$$

1. The original problem. Given a parition π of n we shall use $\lambda_1 \geqq \lambda_2 \geqq \ldots$ to denote the parts in π and $\lambda'_1 \geqq \lambda'_2 \geqq \ldots$ to denote the parts in π', the partition conjugate to π. Equivalently, λ'_j is the number of parts in π which are greater than or equal to j and thus f_j, the number of times j appears in the parition, is equal to $\lambda'_j - \lambda'_{j+1}$. We shall need some elementary results on generating functions for certain partitions. Let $(a; q)_n$ denote the "rising q-factorial".

$$(a; q)_n = \prod(1 - \alpha q^i),\ 0 \leqq i \leqq n-1.$$

Then $1/(q; q)_\infty$ is the generating function for all partitions, $1/(q; q)_N$ is the generating function for partitions into parts less than or equal to N or, equivalently, is the generating function for partitions into at most N parts, and the Gaussian polynomial

$$\begin{bmatrix} N+M \\ N \ ;\ q \end{bmatrix} = \frac{(q; q)_{N+M}}{(q; q)_N (q; q)_M}$$

is the generating function for partitions into at most M parts all less than or equal to N ([**1**], p. 33).

The original problem solved by Burge involves the following theorem.

THEOREM 2. *Let i, k, n be fixed positive integers such that $i \leqq k$, then the following integers are equal:*

(1) *the cardinality of the set of partitions of n for which no part is congruent to* $0, \pm i \pmod{2k+1}$.

(2) *the cardinality of the set of partitions of n for which $f_1 \leqq i-1$ and $f_j + f_{j+1} \leqq k-1$ for all j.*

(3) *the cardinality of the set of partitions of n for which if $\lambda_j \geqq j$ then* $-i+2 \leqq \lambda_j - \lambda'_j \leqq 2k-i-1$.

(4) *the coefficient of q^n in the power series expansion of*

$$\sum_{n_1 \geqq n_2 \geqq \cdots \geqq n_{k-1} \geqq 0} \frac{q^{n_1^2 + n_2^2 + \cdots + n_{k-1}^2 + n_i + \cdots + n_{k-1}}}{(q; q)_{n_1 - n_2} (q; q)_{n_2 - n_3} \cdots (q; q)_{n_{k-1}}}.$$

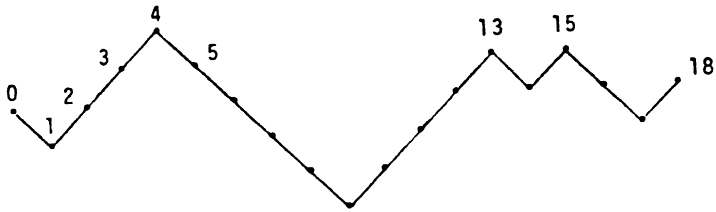
The equality of (1) and (2) is Gordon's generalization of the Rogers-Ramanujan identities (the case $k = 2$) [**5**]. The equality of (1) and (4) was proved by the first author ([**1**], §7.3) who also proved the equality of (1) and (3) where he asked for a direct bijective proof of the equality of (2) and (3). This is the problem Burge solved. We introduce a fifth integer:

(5) the cardinality of the set of binary words of major index n with "peak conditions" (k, i) where these terms will be defined below. Burge [**3, 4**] has supplied simple bijective proofs of the equality of (5) with each of

(2), (3) and (4). In this paper, we shall provide a simple though non-bijective proof of the equality of (5) and (1).

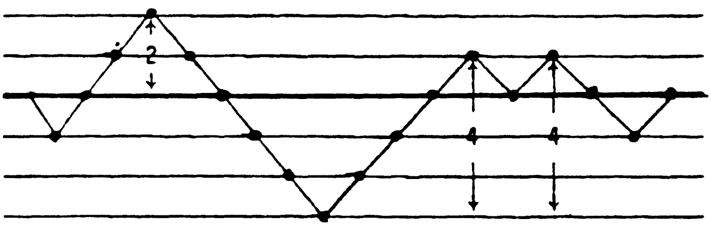
2. Binary words and a generating function. We shall work with finite words built from a two-letter alphabet, $\{\alpha, \beta\}$. For our purposes, two words ω and ω' are indistinguishable if one can be obtained from the other by appending or deleting α's at the end of the word. Therefore, we shall restrict ourselves to words ending in β and the null word. It is useful to represent a binary word as a lattice path, read from left to right, where an α denotes a descent and a β denotes as ascent. Thus the following word and path correspond:

$$\alpha\ \beta\ \beta\ \beta\ \alpha\ \alpha\ \alpha\ \alpha\ \alpha\ \beta\ \beta\ \beta\ \beta\ \alpha\ \beta\ \alpha\ \alpha\ \beta$$



The *major index* of the word ω, denoted MAJω, is the sum of all positions at which a peak (*a* β followed by an α or following *a* β at the end of the word) occurs. For the above example, the major index of ω equals $4 + 13 + 15 + 18 = 50$.

Peak conditions (k, i), $1 \leqq i \leqq k$, are restrictions on the height of the peaks: no peak is more than $k - 1$ above any valley (an α followed by *a* β) to its left and no peak is more than $i - 1$ above sea level (starting level). As we see below, our example satisfies peak conditions (5, 3):



Let $\mathscr{A}_{k,i}$ be the set of words ending in β with peak condition (k, i) and let $A_{k,i}$ be the generating function for this set with elements weighted by their major index,

$$A_{k,i} = \sum_{\omega \in \mathscr{A}_{k,i}} q^{\mathrm{MAJ}\omega} = \sum_{n \geqq 0} a(k, i; n) q^n$$

where $a(k, i; n)$ is the number of words in $\mathscr{A}_{k,i}$ of major index n. To prove that (1) and (5) of Theorem 2 are equal, it is sufficient to show that

$$A_{k,i} = \prod \frac{1}{(1 - q^n)}, \ n \geqq 1 \text{ and } n \not\equiv 0, \pm i \pmod{2k + 1}.$$

We shall find the generating function $A_{k,i}$ by using the number of peaks in a word to find a set of recursion relations which will uniquely determine $A_{k,i}$.

Let $\mathscr{A}_{k,i}(N) = \{\omega \in \mathscr{A}_{k,i} \mid \omega$ has exactly N peaks$\}$ and $\mathscr{B}_{k,i}(N) = \{\omega \in \mathscr{A}_{k,i} \mid$ if $\omega = \beta\omega'$ then ω has N peaks, if $\omega = \alpha\omega'$ then ω has $N - 1$ peaks$\}$. The respective generating functions are

$$A_{k,i}(N) = \sum q^{\mathrm{MAJ}\omega}, \ \omega \in \mathscr{A}_{k,i}(N),$$
$$B_{k,i}(N) = \sum q^{\mathrm{MAJ}\omega}, \ \omega \in \mathscr{B}_{k,i}(N).$$

If $\omega \in \mathscr{A}_{k,i}(N)$ and $i < k$ and if $\omega = \alpha\omega'$, then removing this initial α raises the entire lattice path by one and so we have that $\omega' \in \mathscr{A}_{k,i+1}(N)$. The position of each peak has been decreased by one, and so $\mathrm{MAJ}\omega = \mathrm{MAJ}\omega' + N$. If, on the other hand, $\omega = \beta\omega'$, then removing the β lowers the lattice path by one and will remove one peak if $\omega' = \alpha\omega''$ and leave the number of peaks fixed if $\omega' = \beta\omega''$. Therefore we see that in this case, $\omega' \in \mathscr{B}_{k,i-1}(N)$. Again, $\mathrm{MAJ}\omega = \mathrm{MAJ}\omega' + N$. This gives us the first piece of our recursion:

$$(2.1) \qquad 1 \leqq i < k\colon A_{k,i}(N) = q^N A_{k,i+1}(N) + q^N B_{k,i-1}(N).$$

We note here that no lattice path can stay below its starting level, and so

$$(2.2) \qquad B_{k,0}(N) = 0.$$

If $\omega \in \mathscr{A}_{k,k}(N)$ and $\omega = \alpha^j\beta\omega'$, $j \geqq 0$, then $\beta\omega'$ is also in $\mathscr{A}_{k,k}(N)$ and so $\omega' \in \mathscr{B}_{k,k-1}(N)$. Further, $\mathrm{MAJ}\omega = (j + 1)N + \mathrm{MAJ}\omega'$ and so

$$(2.3) \qquad A_{k,k}(N) = \sum_{j\geqq 0} q^{(j+1)N} B_{k,k-1}(N) = \frac{q^N}{1 - q^N} B_{k,k-1}(N).$$

Similarly, if $\omega \in \mathscr{B}_{k,i}(N)$, $i < k$, and if $\omega = \alpha\omega'$, then ω has $N - 1$ peaks and $\omega' \in \mathscr{A}_{k,i+1}(N - 1)$, $\mathrm{MAJ}\omega = N - 1 + \mathrm{MAJ}\omega'$. If $\omega = \beta\omega'$, then ω has N peaks and $\omega' \in \mathscr{B}_{k,i-1}(N)$. $\mathrm{MAJ}\omega = N + \mathrm{MAJ}\omega'$. We thus get

$$(2.4) \qquad 1 \leqq i < k\colon B_{k,i}(N) = q^{N-1}A_{k,i+1}(N - 1) + q^N B_{k,i-1}(N).$$

Finally, there is just one word with no peaks, the null word,

$$(2.5) \qquad A_{k,i}(0) = 1.$$

We note that equations (2.1)–(2.5) uniquely define $A_{k,i}(N)$ for if for given k and N, $A_{k,i}(N)$ is known for all i, $1 \leqq i \leqq k$, then (2.2) and

(2.4) uniquely determine $B_{k,1}(N+1)$ and thence by induction on i, $B_{k,i}(N+1)$ for all $i < k$. Equation (2.3) then determines $A_{k,k}(N+1)$ and by a descending induction on i, (2.1) determines $A_{k,i}(N+1)$ for all i. Since $A_{k,i}(0)$ is known for all i by (2.5). $A_{k,i}(N)$ is uniquely determined for all k, i, and N, $N \geqq 0$, $1 \leqq i \leqq k$.

We now pull the following functions out of the air:

$$A'_{k,i}(N) = \sum (-1)^n q^{(2k+1)\binom{n+1}{2}-in} \frac{q^{(N-n)(N+n)}}{(q;q)_{N-n}(q;q)_{N+n}}, \quad -N \leqq n \leqq N,$$

$$B'_{k,i}(N) = \sum (-1)^n q^{(2k+1)\binom{n+1}{2}-in} \frac{q^{(N-n-1)(N+n)}}{(q;q)_{N-n-1}(q;q)_{N+n}}, \quad -N \leqq n \leqq N-1,$$

and observe by simple algebraic manipulation that they satisfy:

$$(2.6) \qquad 1 \leqq i < k\colon A'_{k,i}(N) = q^N A'_{k,i+1}(N) + q^N B'_{k,i+1}(N)$$

$$(2.7) \qquad B'_{k,0}(N) = 0$$

$$(2.8) \qquad A'_{k,k}(N) = \frac{q^N}{1-q^N} B'_{k,k-1}(N)$$

$$(2.9) \qquad 1 \leqq i < k\colon B'_{k,i}(N) = q^{N-1} A'_{k,i+1}(N-1) + q^N B'_{k,i-1}(N)$$

$$(2.10) \qquad A'_{k,i}(0) = 1.$$

Therefore, $A_{k,i}(N) = A'_{k,i}(N)$ and so we have that

$$\begin{aligned} A_{k,i} &= \sum_{n\geqq 0} A_{k,i}(N) \\ (2.11) \qquad &= \sum_{N\geqq 0} \sum_{n=-N}^{N} (-1)^n q^{(2k+1)\binom{n+1}{2}-in} \frac{q^{(N-n)(N+n)}}{(q;q)_{N-n}(q;q)_{N+n}} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{(2k+1)\binom{n+1}{2}-in} \sum_{N\geqq |n|} \frac{q^{(N-n)(N+n)}}{(q;q)_{N-n}(q;q)_{N+n}}. \end{aligned}$$

The inner sum in the last line of (2.11) is the partition generating function, $1/(q;q)_\infty$. This can be seen by observing that for a fixed integer n, to each partition there exists a unique maximal integer $N \geqq |n|$ such that the partition has at least $N-n$ parts of size at least $N+n$. Subtracting $N+n$ from each of these $N-n$ parts leaves us with a partition into at most $N-n$ parts (generated by $1/(q;q)_{N-n}$). The remaining parts must have size less than or equal to $N+n$, else N could have been chosen larger, and so are generated by $1/(q;q)_{N+n}$. We thus see that

$$\begin{aligned} A_{k,i} &= \frac{1}{(q;q)_\infty} \sum (-1)^n q^{(2k+1)\binom{n+1}{2}-in}, \quad -\infty < n < \infty \\ (2.12) \qquad &= \frac{(q^i;q^{2k+1})_\infty (q^{2k+1-i};q^{2k+1})_\infty (q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty} \\ &= \prod \frac{1}{1-q^n}, \quad n \geqq 1 \text{ and } n \not\equiv 0, \pm i \pmod{2k+1}, \end{aligned}$$

which is what we set out to prove. The second line of (2.12) is the Jacobi Triple Product Identity ([1], p. 21).

3. Proving similar identities. The procedure presented in §2 is precisely the reverse of what one wants to do to discover new identities. One starts with a suitable specialization of the generating function for partitions into parts excluded from certain residue classes, as for example $A'_{k,i}(N)$, $B'_{k,i}(N)$. One finds the recursions they satisfy and then determines what words have generating functions satisfying these recursions. Following this procedure, we shall sketch proofs of the following three theorems. The first two are due to Burge [3, 4], the third is new. As before, all binary words are assumed to end in β.

THEOREM 3. *Let i, k, n be fixed positive integers, $i < k$, then the following integers are equal:*

(1) *the cardinality of the set of partitions of n for which no part is congruent to* $0, \pm i \pmod{2k}$.

(2) *the cardinality of the set of binary words with major index n, peak conditions (k, i) and such that any valley lying exactly $k - 1$ below a peak to its right is in a position with the same parity as $k + i$.*

THEOREM 4. *Let i, k, n be positive integers, $i \leqq k$, then the following integers are equal:*

(1) *the cardinality of the set of partitions of n for which no part is congruent to* $2 \pmod 4$ *or congruent to* $0, \pm (2i - 1) \pmod{4k}$.

(2) *the cardinality of the set of binary words with major index n, peak conditions $(2k - 1, 2i - 1)$ and every valley is in an even position.*

THEOREM 5. *Let i, k, n be positive integers, $i < k$ and i even, then the following integers are equal:*

(1) *the cardinality of the set of partitions of n for which no part is congruent to* $2 \pmod 4$ *or congruent to* $0, \pm 2i \pmod{4k}$.

(2) *the cardinality of the set of binary words with major index n, peak conditions $(2k, 2i)$, every valley is in an even position and any valley lying at least $2k - 2$ below a peak to its right has height (vertical displacement from sea level) congruent to* $2k + 2i \pmod 4$.

SKETCH OF PROOF OF THEOREM 3. We define the following functions,

$$C_{k,i}(N) = \sum (-1)^n q^{2k\binom{n+1}{2}-in} \frac{q^{(N-n)(N+n)}}{(q;q)_{N-n}(q;q)_{N+n}}, \quad -N \leqq n \leqq N,$$

$$D_{k,i}(N) = \sum (-1)^n q^{2k\binom{n+1}{2}-in} \frac{q^{(N-n-1)(N+n)}}{(q;q)_{N-n-1}(q;q)_{N-n}}, \quad -N \leqq n \leqq N-1.$$

These are chosen to be similar to $A_{k,i}(N)$ and to satisfy the property that

(3.1) $$\sum_{N\geqq 0} C_{k,i}(N) = \prod \frac{1}{1-q^n},\ N \geqq 1 \text{ and } n \not\equiv 0, \pm i \pmod{2k}.$$

These functions satisfy the following set of recursions:

(3.2) $$1 \leqq i < k:\ C_{k,i}(N) = q^N C_{k,i+1}(N) + q^N D_{k,i-1}(N).$$

(3.3) $$D_{k,0} = 0,$$

(3.4) $$C_{k,k}(N) = \frac{q^N}{1-q^{2N}} D_{k,k-1}(N) + \frac{q^{2N}}{1-q^{2N}} D_{k,k-2}(N),$$

(3.5) $$1 \leqq i < k:\ D_{k,i}(N) = q^{N-1} C_{k,i+1}(N-1) + q^N D_{k,i-1}(N),$$

(3.6) $$C_{k,i}(0) = 1.$$

By showing that the generating functions must satisfy the same set of recursions, it can be shown that $C_{k,i}(N)$ is the generating function for binary words with N peaks, peak conditions (k, i) and any valley lying exactly $k-1$ below a peak to its right is in a position with the same parity as $k+i$, while $D_{k,i}(N)$ is the generating function for binary words with $N-1$ peaks if the first letter is α, otherwise N peaks, and the rest of the conditions the same.

SKETCH OF PROOF OF THEOREM 4. Here we define the functions

$$E_{k,i}(N) = \sum (-1)^n q^{4k\binom{n+1}{2}-(2i-1)n} \frac{q^{(N-n)(N+n)}(-q;q^2)_N}{(q^2;q^2)_{N-n}(q^2;q^2)_{N+n}},$$

$$F_{k,i}(N) = \sum (-1)^n q^{4k\binom{n+1}{2}-2in} \frac{q^{(N-n-1)(N+n)}(-q;q^2)_N}{(q^2;q^2)_{N-n-1}(q^2;q^2)_{N+n}}.$$

It is an elementary consequence of Heine's identity ([1], p. 20) that

(3.7) $$\sum_{N=-n}^{n} \frac{q^{(N-n)(N+n)}(-q;q^2)_N}{(q^2;q^2)_{N-n}(q^2;q^2)_{N+n}} = \frac{(q^2;q^4)_\infty}{(q;q)_\infty},$$

and therefore that

(3.8) $$\sum_N E_{k,i}(N) = \prod \frac{1}{1-q^n},\ n \geqq 1,\ n \not\equiv 2 \pmod 4$$
$$n \not\equiv 0, \pm(2i-1) \pmod{4k}.$$

These functions satisfy the following set of recursions:

(3.9) $$1 \leqq i < k:\ E_{k,i}(N) = q^{2N} E_{k,i+1}(N) + q^N F_{k,i-1}(N)$$

(3.10) $$F_{k,0}(N) = 0$$

(3.11) $$E_{k,k}(N) = \frac{q^N}{(1-q^{2N})} F_{k,k-1}(N),$$

(3.12) $$1 \leqq i < k:\ F_{k,i}(N) = q^{N-1}(1+q^{2N-1}) E_{k,i+1}(N-1) + q^{2N} F_{k,i-1}(N)$$

(3.13) $$E_{k,i}(0) = 1.$$

The same set of recursions is satisfied by the generating functions for binary words with N peaks, peak conditions $(2k - 1, 2i - 1)$ and every valley is in an even position (corresponding to $E_{k,i}(N)$) and the generating function for binary words with $N - 1$ peaks if the first letter is α else N peaks, peak conditions $(2k - 1, 2i)$ and every valley is in an odd position (corresponding to $F_{k,i}(N)$).

SKETCH OF PROOF OF THEOREM 5. Here we define the functions

$$G_{k,i}(N) = \sum (-1)^n q^{4k\binom{n+1}{2}-2in} \frac{q^{(N-n)(N+n)}(-q;q^2)_N}{(q^2;q^2)_{N-n}(q^2;q^2)_{N+n}},$$

$$H_{k,i}(N) = \sum (-1)^n q^{4k\binom{n+1}{2}-(2i+1)n} \frac{q^{(N-n-1)(N+n)}(-q;q^2)_N}{(q^2;q^2)_{N-n-1}(q^2;q^2)_{N+n}}.$$

These satisfy the following set of recursions:

(3.14) $$1 \leqq i < k: G_{k,i}(N) = q^{2N}G_{k,i+1}(N) + q^N H_{k,i-1}(N),$$

(3.15) $$H_{k,0}(N) = q^{N-1}G_{k,1}(N-1),$$

(3.16) $$G_{k,k}(N) = \frac{q^N}{1-q^{4N}} H_{k,k-1}(N) + \frac{q^{3N}}{1-q^{4N}} H_{k,k-2}(N),$$

(3.17) $$i \leqq i < k: H_{k,i}(N) = q^{N-1}(1+q^{2N-1})G_{k,i+1}(N-1) + q^{2N}H_{k,i-1}(N),$$

(3.18) $$G_{k,i}(0) = 1.$$

By showing that they satisfy the same set of recursion, it can be shown that $G_{k,i}(N)$ is the generating function for binary words with N peaks, peak conditions $(2k, 2i)$ every valley is in an even position and any valley lying at least $2k - 2$ below a peak to its right has height congruent to $2k + 2i$ (mod 4), while $H_{k,i}(N)$ is the generating function for binary words with $N - 1$ peaks if the first letter is α else N peaks, peak conditions $(2k, 2i + 1)$, every valley is in an odd position and any valley lying at least $2k - 2$ below a peak to its right has height congruent to $2k + 2i + 1$ (mod 4).

4. Concluding remarks. Theorem 1 is a corollary to Theorems 4 and 5. If we take the Burge correspondence that was used to give the bijection between binary words with peak conditions (k, i) and partitions for which $f_1 \leqq i - 1$, $f_j + f_{j+1} \leqq k - 1$ [3] and apply it to the more restricted binary words of Theorems 4 and 5, we get precisely Theorem 1.

It is obvious that more can be done along these lines, but the only other examples which have been studied to date ha muchve more complicated restrictions on the lattice paths.

REFERENCES

1. G. E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, Mass., 1976.

2. D. M. Bressoud, *Analytic and combinatorial generalizations of the Rogers-Ramanujan identities*, Mem. Amer. Math. Soc. **227** (1980).

3. W. H. Burge, *A correspondence between partitions related to generalizations of the Rogers-Ramanujan identities*, Discrete Math **34** (1981) 9–15.

4. W. H. Burge, *A three-way correspondence between partitions*, Europ. J. Combinatorics **3** (1982), 195–213.

5. B. Gordon, *A combinatorial generalization of the Rogers-Ramanujan identities*, Amer. J. Math. **83** (1961), 393–399.

PENNSYLVANIA STATE UNIVERSITY