ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 15, Number 1, Winter 1985

A CHARACTERIZATION OF THOSE SPACES HAVING ZERO-DIMENSIONAL REMAINDERS

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ABSTRACT. A 0-space is a completely regular Hausdorff space possessing a compactification with zero-dimensional remainder. It is well known that any rimcompact space is a 0-space, while the converse is not true. In this paper a proximal characterization of 0-spaces is presented. Those open sets U of βX for which $U \cap (\beta X \backslash X)$ is clopen in $\beta X \backslash X$ are characterized. This characterization is then utilized to define a relation α on $\mathscr{P}(X)$. It is shown that α is a proximity on X if and only if X is a 0-space. The definition of the relation α is motivated by the presentation of a proximal characterization of almost rimcompact spaces—a class of spaces intermediate between the classes of rimcompact spaces and 0-spaces.

1. Introduction and known results. The characterization of those completely regular Hausdorff spaces possessing a compactification with zero-dimensional remainder has been considered by various researchers (see for example [5], [6] and [9]). Such a compactification will be called 0-dimensional at infinity (denoted by O.I.); a 0-space is any space possessing a O.I. compactification. Recall that a space is rimcompact if it has a basis of open sets with compact boundaries ([5]). Each rimcompact space X possesses a compactification which has a basis of open sets whose boundaries are contained in X ([7], [9]). Hence a rimcompact space is a 0-space; the converse is not true ([9]). In [2] we introduced a natural generalization of rimcompactness called almost rimcompactness and obtained the following characterization, which we consider in this paper as a definition. A space X is almost rimcompact if and only if X possesses a compactification KX in which each point of $KX \backslash X$ has a basis (in KX) of open sets whose boundaries are contained in X. If KX is such a compactification of X, we say that $KX \backslash X$ is relatively 0-dimensionally embedded in KX. Hence each almost rimcompact space is a 0-space; in the same paper we show that the converse is not true. For the internal definition and a thorough discussion of almost rimcompactness, see [2] and [3].

Received by the editors on November 10, 1983.
This research was partially supported by a scholarship from the Natural Sciences and Engineering Research Council of Canada.
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In this paper we give an internal characterization of the class of 0-spaces. In §2 we characterize those open sets U of βX for which $U \cap (\beta X \backslash X)$ is clopen in $\beta X \backslash X$ by writing $U \cap X$ as the union of a family of open sets of X with special properties. In §3 we present a proximal characterization of both almost rimcompact spaces and 0-spaces. In §4 we briefly discuss a special class of 0-spaces called full 0-spaces.

In the remainder of this section, we present our notation and terminology and some known results. All spaces are assumed to be completely regular and Hausdorff. The notions used from set theory are standard. A map is a continuous surjection. A function f: $X \rightarrow Y$ is closed if whenever F is a closed subset of X, then $f[F]$ is a closed subset of Y.

The family $\mathscr{K}(X)$ of (equivalence classes of) compactifications of X is partially ordered in the usual way: $JX \leqq KX$ if there is a map f: $KX \rightarrow JX$ such that $f(x) = x$ for each $x \in X$; KX is equivalent to JX if f is a homeomorphism. For background information on compactifications the reader is referred to **[1]** or **[4]**. The maximum element of $\mathscr{K}(X)$, the Stone-Čech compactification of X, is denoted by βX. In the sequel, if $KX \in \mathscr{K}(X)$, the natural map from βX into KX is denoted by Kf.

If $KX \in \mathscr{K}(X)$, we often call $KX \backslash X$ the remainder of KX. For any space X, the residue of X (denoted by $R(X)$) is the set of points at which X is not locally compact. If $KX \in \mathscr{K}(X)$, then $Cl_{KX}(KX \backslash X) = R(X) \cup (KX \backslash X)$.

Our standard reference on proximities is **[8]**. In the sequel, any proximity considered on a space X is assumed to be compatible with the topology of X. Two proximites δ and α on X are equivalent if for $A, B \subseteq X$, $A\delta B$ if and only if $A\alpha B$. There is a $1-1$ correspondence between (equivalence classes of) proximities on a space X and (equivalence classes of) compactifications of X. That is, if δ is any proximity on X, then there is a unique compactification δX of X satisfying (for A, $B \subseteq X$) $A\delta B$ if and only if $Cl_{\delta X}A \cap Cl_{\delta X}B \neq \varnothing$. Conversely, if $KX \in \mathscr{K}(X)$, and δ is defined (for A, $B \subseteq X$) by $A\delta B$ if and only if $Cl_{KX}A \cap Cl_{KX}B \neq \varnothing$, then δ is a proximity on X and $\delta X = KX$.

If U is an open subset of X, and $\delta X \in \mathscr{K}(X)$, then $Ex_{\delta X}U$ is defined to be $\delta X \backslash Cl_{\delta X}(X \backslash U)$. The set $Ex_{\delta X}U$ is often called the extension of U in δX. It is an easy exercise to verify (i), (ii), (iii) and (iv) of the following proposition. Statement (v) is implicit in the proof of Lemma 2 of **[9]**, and (vi) follows from (v).

PROPOSITION 1.1. *Let* $\delta X \in \mathscr{K}(X)$.

(i) *If* W *is open in* δX, *then* $W \subset Ex_{\delta X}(W \cap X)$.

(ii) *If* U *and* V *are open in* X, *then* $Ex_{\delta X}(U \cap V) = (Ex_{\delta X}U) \cap (Ex_{\delta X}V)$.

(iii) *If* U *is open in* X, *then* $(Ex_{\delta X}U) \cap X = U$, *hence* $Cl_{\delta X}U = Cl_{\delta X}Ex_{\delta X}U$.

(iv) *If F is closed in X, U is open in X, and $F \cap U = \emptyset$, then $Cl_{\delta X}F \cap Ex_{\delta X}U = \emptyset$.*

(v) *If U and V are open in X, then $Ex_{\delta X}(U \cup V) \backslash (Ex_{\delta X}U \cup Ex_{\delta X}V) \subset Cl_{\delta X}U \cap Cl_{\delta X}V$.*

(vi) *If U and V are open in X, and $Cl_{\delta X}U \cap Cl_{\delta X}V = \emptyset$, then $Ex_{\delta X}(U \cup V) = Ex_{\delta X}U \cup Ex_{\delta X}V$.*

If U is any open subset of X, then it follows from 1.1 (i) that $Ex_{\delta X}\, U$ is the largest open subset of δX whose intersection with X is the set U. The collection $\{Ex_{\delta X}U\colon U \text{ is an open subset of } X\}$ of open sets of δX is easily seen to be a basis for the topology of δX.

If $B \subset X$, the boundary of B in X, denoted by $bd_X B$, is defined to be the set $Cl_X B \cap Cl_X(X \backslash B)$. A compactification δX of X is a perfect compactification of X if for each open subset U of X, $Cl_{\delta X}(bd_X U) = bd_{\delta X}(Ex_{\delta X}U)$. According to the corollary to Lemma 1 of [9], βX is a perfect compactification of X.

The equivalence of (i), (ii), and (iii) of the following proposition appears in Theorems 1 and 2 of [9].

PROPOSITION 1.2. *Let $\delta X \in \mathscr{K}(X)$. The following are equivalent.*

(i) *δX is a perfect compactification of X.*

(ii) *If U and V are disjoint open sets of X, then $Ex_{\delta X}(U \cup V) = Ex_{\delta X}U \cup Ex_{\delta X}V$.*

(iii) For each $p \in \delta X$, $(\delta f)^{\leftarrow}(p)$ is a connected subset of βX.

The connected component of $x \in X$ is the union of all connected subspaces of X containing x. The quasi-component of $x \in X$ is the intersection of all closed-and-open (denoted clopen) subsets of X containing x. A space X is zero-dimensional (denoted 0-dimensional) if X has a basis of clopen sets.

For a detailed discussion of the disconnectedness of remainders of compactifications see [2]. Any 0-space X has a maximum O.I. compactification (which we denote by $F_{\circ}X$) which is also a minimum perfect compactification of X([6]). For each $p \in F_{\circ}X \backslash X$, $(F_{\circ}f)^{\leftarrow}(p)$ is the connected compact quasicomponent in $\beta X \backslash X$ of each element of $(F_{\circ}f)^{\leftarrow}(p)$.

Following the terminology of [7] and [9], we say that an open set U of X is π-open in X if $bd_X U$ is compact. The intersection and union of finitely many π-open sets are π-open, as is the complement of the closure of a π-open set. Also, if W is open in KX, and $bd_{KX}W \subseteq X$, then $W \cap X$ is π-open in X.

DEFINITIONS 1.3. (i) If $F_1, F_2 \subset X$, then F_1 and F_2 are π-separated in X if there is a π-open set U of X such that $F_1 \subset U$, and $Cl_X U \cap F_2 = \emptyset$. We say that F_1 is π-contained in $X \backslash F_2$ if F_1 and F_2 are π-separated.

(ii) If F is closed in X, U is open in X, and $F \subseteq U$, then F is nearly

π-contained in U if there is a compact subset K of F so that whenever F' is a closed subset of F, and $F' \cap K = \emptyset$, F' is π-contained in U.

2. Clopen subsets of remainders. We need some tools for studying clopen sets in remainders of compactifications. These are developed in 2.1–2.5 inclusive.

DEFINITIONS 2.1. (i) Let X be a space. An open set U of $KX \in \mathscr{K}(X)$ is clopen at infinity in KX (denoted by KX–$C.I.$) if $U \cap (KX\backslash X)$ is clopen in $KX\backslash X$. The set U is a full KX–$C.I.$ set if U is KX–$C.I.$, and $U = Ex_{KX}(U \cap X)$. Often a βX–$C.I.$ (respectively, full βX–$C.I.$) set will simply be called a $C.I.$ (respectively, full $C.I.$) set.

(ii) X is a full 0-space if X is a 0-space, and if for each $p \in \beta X\backslash X$, the connected component of p in $\beta X\backslash X$ has a basis in βX of full $C.I.$ sets.

(iii) If $\mathscr{E}$ is a family of open sets of X, and D is open in X, then D is small with respect to $\mathscr{E}$ if for each $E \in \mathscr{E}$, $Cl_X(D \cap E)$ is compact.

(iv) A family $\mathscr{E}$ of open sets of X is clopenly extendible (denoted $C.E.$) if there is a compact subset K of X so that if U is open in X, and $K \subset U$, there is $E \in \mathscr{E}$, and D small with respect to $\mathscr{E}$ such that $X = U \cup E \cup D$. A family $\mathscr{E}$ is a full $C.E.$ family if $\mathscr{E}$ is $C.E.$, and $Ex_{\beta X}(\bigcup\{E\colon E \in \mathscr{E}\}) = \bigcup\{Ex_{\beta X}E\colon E \in \mathscr{E}\}$.

If $bd_{KX}W \subseteq X$, then W is clearly a full KX–$C.I.$ subset of KX. The following shows that if W is any KX–$C.I.$ open set, then the sets W and $Ex_{KX}(W \cap X)$ can only differ in the locally compact part of $KX\backslash X$.

PROPOSITION 2.2. *If $KX \in \mathscr{K}(X)$, and if U is a KX–$C.I.$ set, then* $Ex_{KX}(U \cap X) \cap Cl_{KX}R(X) = U \cap Cl_{KX}R(X)$.

PROOF. Let U be a KX–$C.I.$ open set, and suppose that $p \in [Ex_{KX}(U \cap X) \cap Cl_{KX}R(X)]\backslash U$. As $p \in (KX\backslash X)\backslash U$, which is clopen in $KX\backslash X$, there is an open subset W of KX such that $p \in W \subset Ex_{KX}(U \cap X)$ and $W \cap (KX\backslash X) \cap U = \emptyset$. As $p \in Cl_{KX}R(X)$, there is $x \in W \cap R(X)$. Now $W \cap R(X) \subset Ex_{KX}(U \cap X) \cap X = U \cap X$, so $x \in W \cap U$, which is an open set of KX. Also, $x \in R(X)$, so $W \cap U \cap (KX\backslash X) \neq \emptyset$, which is a contradition to our choice of W. Then $Ex_{KX}(U \cap X) \cap Cl_{KX}R(X) \subset U \cap Cl_{KX}R(X)$. Since the reverse inclusion is always true, the result is proved.

We need to extend some results concerning open sets and perfect compactifications.

LEMMA 2.3. *Let $KX \in \mathscr{K}(X)$. If K is a compact subset of X, and if U is open in X, then $[Ex_{KX}(U\backslash K)] \cap (KX\backslash X) = (Ex_{KX}U) \cap (KX\backslash X)$. Hence if V is open in X, and $Cl_X(U \cap V)$ is compact, then $(Ex_{KX}U) \cap (Cl_{KX}V) \subset X$.*

PROOF. Since $Ex_{KX}(U\backslash K) \cap (KX\backslash X) = Ex_{KX}(U \cap (X\backslash K)) \cap (KX\backslash X)$

$$= Ex_{KX}U \cap Ex_{KX}(X\backslash K) \cap (KX\backslash X)$$
$$= Ex_{KX}U \cap (KX\backslash K) \cap (KX\backslash X)$$
$$= Ex_{KX}U \cap (KX\backslash X),$$

the first statement is true.

Suppose that $Cl_X(U \cap V)$ is compact. Since $[U\backslash Cl_X(U \cap V)] \cap V = \varnothing$, by 1.1 (iv), $Ex_{KX}(U\backslash Cl_X(U \cap V)) \cap Cl_{KX}V = \varnothing$. Then $Ex_{KX}U \cap (KX\backslash X) \cap Cl_{KX}V = Ex_{KX}(U\backslash Cl_X(U \cap V)) \cap (KX\backslash X) \cap Cl_{KX}V = \varnothing$.

If $\mathscr{E}$ is a family of open subsets of X, let $Ex_{KX}\mathscr{E} = \bigcup \{Ex_{KX}E: E \in \mathscr{E}\}$. The following is an immediate consequence of 2.3.

COROLLARY 2.4. *Let $KX \in \mathscr{K}(X)$. Suppose that $\mathscr{E}$ is a family of open sets of X, and that D is open in X. If D is small with respect to $\mathscr{E}$, then $Cl_{KX}D \cap Ex_{KX}\mathscr{E} \cap (KX\backslash X) = \varnothing$ and $Ex_{KX}D \cap (\bigcup\{Cl_{KX}E: E \in \mathscr{E}\}) \cap (KX\backslash X) = \varnothing$.*

As pointed out in 1.2, the equivalence of (i) and (ii) in the following theorem appears in Theorem 1 of **[9]**; we will need the equivalence of (i) and (iii).

THEOREM 2.5. *Let $KX \in \mathscr{K}(X)$, and let U, V be open in X. Then the following are equivalent.*

(i) *KX is a perfect compactification of X.*

(ii) *If $U \cap V = \varnothing$, then $Ex_{KX}(U \cup V) = Ex_{KX}U \cup Ex_{KX}V$.*

(iii) *If $Cl_X(U \cap V)$ is compact, then $Ex_{KX}(U \cup V) = Ex_{KX}U \cup Ex_{KX}V$.*

PROOF. (iii) implies (ii). This is obvious.

(ii) implies (iii). Since $[Ex_{KX}(U \cup V)] \cap X = U \cup V = (Ex_{KX}U \cup Ex_{KX}V) \cap X$, it is sufficient to show that $Ex_{KX}(U \cup V) \cap (KX\backslash X) = (Ex_{KX}U \cup Ex_{KX}V) \cap (KX\backslash X)$. If $Cl_X(U \cap V)$ is compact, then according to 2.3,

$$(Ex_{KX}U \cap (KX\backslash X)) \cup (Ex_{KX}V \cap (KX\backslash X))$$
$$= [Ex_{KX}(U\backslash Cl_X(U \cap V)) \cap (KX\backslash X)] \cup [Ex_{KX}(V\backslash Cl_X(U \cap V)) \cap (KX\backslash X)]$$

(as $U\backslash Cl_X(U \cap V)$ and $V\backslash Cl_X(U \cap V)$ are disjoint open sets of X),

$$= Ex_{KX}[(U\backslash Cl_X(U \cap V)) \cup (V\backslash Cl_X(U \cap V))] \cap (KX\backslash X)$$
$$= Ex_{KX}[(U \cup V)\backslash Cl_X(U \cap V)] \cap (KX\backslash X)$$
$$= Ex_{KX}(U \cup V) \cap (KX\backslash X),$$

where the last equality follows from 2.3. The theorem follows.

If $\mathscr{E} = \{E(\alpha): \alpha \in A\}$ is a collection of sets, then $\mathscr{E}^F$ will denote the collection of sets $\{\bigcup\{E(\alpha_i): 1 \leqq i \leqq n\}: \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a finite subset of $A\}$. The following series of results will establish a correspondence between *C.E.* (respectively, full *C.E.*) families and *C.I.* (respectively, full *C.I.*) subsets of compactifications.

THEOREM 2.6. *Let KX be a compactification of X. If U is a C.I. subset of KX, then there is a C.E. family $\mathscr{E}$ such that $Ex_{KX}\mathscr{E} = U$.*

PROOF. Since U is an open subset of KX, for each $p \in U$ we can choose an open set E_p of X such that $p \in Ex_{KX}E_p \subset Cl_{KX}E_p \subset U$. Let $\mathscr{E}_1 = \{E_p: p \in U\}$, and $\mathscr{E} = \mathscr{E}_1^F$. Note that if $E \in \mathscr{E}$ then $Cl_{KX}E \subset U$.

Clearly $Ex_{KX}\mathscr{E} = U$. In order to show that $\mathscr{E}$ is a *C.E.* family, we must construct a compact subset K of X so that if V is open in X, and $K \subset V$, there is $E \in \mathscr{E}$ and D small with respect to $\mathscr{E}$ such that $X = V \cup D \cup E$. First we construct a second family of open sets of X. Since $U \cap (KX\backslash X)$ is clopen in $KX\backslash X$, for each $p \in (KX\backslash X)\backslash U$, we can choose an open set D_p of X such that $p \in Ex_{KX}D_p$ while $(Cl_{KX}D_p) \cap U \subset X$. Let $\mathscr{D}_1 = \{D_p: p \in (KX\backslash X)\backslash U\}$, and $\mathscr{D} = \mathscr{D}_1^F$. Note that if $D_1 \in \mathscr{D}_1$ and $E_1 \in \mathscr{E}_1$, then $Cl_{KX}D_1 \cap Cl_{KX}E_1 \subset X$, hence $Cl_X(E_1 \cap D_1)$ is compact. It follows that if $D \in \mathscr{D}$ and $E \in \mathscr{E}$, then $Cl_X(D \cap E)$ is compact (being a finite union of compact sets). In other words, if $D \in \mathscr{D}$, then D is small with respect to $\mathscr{E}$.

Let $K = KX\backslash\bigcup\{Ex_{KX}A: A \in \mathscr{E} \cup \mathscr{D}\}$. Then K is a compact subset of X. Suppose that $K \subset V$, where V is open in X. Then the collection of sets $\{Ex_{KX}A: A \in \mathscr{E} \cup \mathscr{D}\} \cup \{Ex_{KX}V\}$ is an open cover of KX, so there is a finite subcollection whose union covers KX. Then X is covered by the union of a finite subcollection of $\mathscr{E} \cup \mathscr{D} \cup \{V\}$. Since $\mathscr{E}$ and $\mathscr{D}$ are closed under finite unions, there are sets $E \in \mathscr{E}$ and $D \in \mathscr{D}$ such that $X = V \cup E \cup D$. Since D is small with respect to $\mathscr{E}$, $\mathscr{E}$ is a *C.E.* family.

It is a straightforward computation to verify that if $KX = \beta X$, and if U is a full *C.I.* subset of βX, then $\mathscr{E}$ as defined in the proof of 2.6 is a full *C.E.* family. We observe that in the proof of 2.6, the only conditions that $\mathscr{E}$ is required to satisfy are that (i) for each $E \in \mathscr{E}$, $Cl_{KX}E \subset U$, and (ii) $Ex_{KX}\mathscr{E} = U$. Therefore, we could have chosen $\mathscr{E}$ to be $\{V: V$ is open in X and $Cl_{KX}V \subset U\}$.

THEOREM 2.7. *Let KX be a perfect compactification of X, and let $\mathscr{E}$ be a C.E. family of open sets of X. Suppose that $p \in (KX\backslash X)\backslash Ex_{KX}\mathscr{E}$. Then*

(i) *There is a set D small with respect to $\mathscr{E}$ such that $p \in Ex_{KX} D$*, hence,

(ii) $(Ex_{KX}\mathscr{E}) \cap (KX\backslash X) = \bigcup\{Cl_{KX}E: E \in \mathscr{E}\} \cap (KX\backslash X)$, *and*

(iii) $Ex_{KX}\mathscr{E}$ *is KX–C.I.*

PROOF. (i). Let K be a compact subset of X which witnesses the fact that $\mathscr{E}$ is a *C.E.* family, and let $p \in (KX\backslash X)\backslash Ex_{KX}\mathscr{E}$. Since $p \notin Cl_{KX}K = K$, there is an open set U of X such that $K \subset U$, while $p \notin Cl_{KX}U$. Choose

D to be small with respect to $\mathscr{E}$, and choose $E \in \mathscr{E}$, such that $X = U \cup E \cup D$. Now $X \backslash Cl_X U \subset D \cup E$, so $p \in KX \backslash Cl_{KX} U = Ex_{KX}(X \backslash Cl_X U) \subset Ex_{KX}(E \cup D) = Ex_{KX}E \cup Ex_{KX}D$, where the last equality follows from 2.5. Since $p \notin Ex_{KX}E$, it follows that $p \in Ex_{KX}D$.

(ii) and (iii). Suppose that $p \in (KX \backslash X) \backslash Ex_{KX}\mathscr{E}$. According to (i) and 2.4, there is an open set D of X such that $p \in Ex_{KX}D$, and $Ex_{KX}D \cap (\bigcup\{Cl_{KX}E\colon\ E \in \mathscr{E}\}) \subset X$. Then $Ex_{KX}\mathscr{E} \cap (KX \backslash X) = (\bigcup\{Cl_{KX}E\colon\ E \in \mathscr{E}\}) \cap (KX \backslash X)$. Thus, $p \notin Cl_{KX \backslash X}[Ex_{KX}\mathscr{E}) \cap (KX \backslash X)]$, so $[Ex_{KX}\mathscr{E}) \cap (KX \backslash X)$ is clopen in $KX \backslash X$.

It follows easily from the above that if $\mathscr{E}$ is a full *C.E.* family, then $Ex_{\beta X}\mathscr{E}$ is a full *C.I.* subset of βX.

When we defined a *C.E.* family $\mathscr{E}$, we did not specify that $\mathscr{E}$ is to be closed under finite unions, although the *C.E.* family $\mathscr{E}$ constructed in the proof of 2.6 is closed under finite unions. The following result shows that it is not necessary to specify this property in the definition of a *C.E.* family.

THEOREM 2.8. *Let KX be a perfect compactification of X, and let $\mathscr{E}$ be a C.E. family of open sets of X. Then*

(i) *$\mathscr{E}^F$ is a C.E. family.*

(ii) $Ex_{KX}\mathscr{E} = Ex_{KX}(\mathscr{E}^F)$.

(iii) *If B is a closed subset of X, then $Cl_{KX}B \subset Ex_{KX}\mathscr{E}$ if and only if there is $E \in \mathscr{E}^F$ such that $B \subset E$.*

PROOF. (i). Note that if D is small with respect to $\mathscr{E}$, then D is small with respect to $\mathscr{E}^F$. It is then clear that if $\mathscr{E}$ is a *C.E.* family, $\mathscr{E}^F$ is also.

(ii). If U and V are any open subsets of a space X, and if δX is any compactification of X, then an easy computation shows that $(Ex_{\delta X}U) \cup (Ex_{\delta X}V) \subset Ex_{\delta X}(U \cup V) \subset Cl_{\delta X}(U \cup V) = Cl_{\delta X}U \cup Cl_{\delta X}V$. Then it follows that $(Ex_{KX}\mathscr{E}) \cap KX \backslash X) \subseteq (Ex_{KX}\mathscr{E}^F) \cap (KX \backslash X) \subseteq [\bigcup\{Cl_{KX}E\colon\ E \in \mathscr{E}^F\}] \cap (KX \backslash X) = [\bigcup\{Cl_{KX}E\colon\ E \in \mathscr{E}\}] \cap (KX \backslash X) = (Ex_{KX}\mathscr{E}) \cap (KX \backslash X)$, where the last equality is (ii) of 2.7. Hence $Ex_{KX}\mathscr{E}^F = Ex_{KX}\mathscr{E}$. Clearly $(Ex_{KX}\mathscr{E}) \cap X = (Ex_{KX}\mathscr{E}^F) \cap X$.

(iii). Note that $Ex_{\delta X}U \cup Ex_{\delta X}V \subset Ex_{\delta X}(U \cup V)$, for any compactification δX of X, and open sets U, V of X. Hence if $Cl_{KX}B \subset Ex_{KX}\mathscr{E}$, by compactness there is a set $E \in \mathscr{E}^F$ such that $Cl_{KX}B \subset Ex_{KX}E$; that is, $B \subset E$. On the other hand, if $B \subset E$, where $E \in \mathscr{E}^F$, then $Cl_{KX}B = (Cl_{KX}B \cap (KX \backslash X)) \cup B \subseteq (Cl_{KX}E \cap (KX \backslash X)) \cup E \subseteq Ex_{KX}\mathscr{E}^F = Ex_{KX}\mathscr{E}$, where the last inclusion and the equality follow from 2.7 (ii), and (ii) of the present result respectively.

In the following results, we will assume without loss of generality that any *C.E.* family is closed under finite unions.

The correspondence between $C.I.$ open sets and $C.E.$ families developed in 2.7 has an interesting form in the special situations discussed below.

PROPOSITION 2.9. *Let U be an open subset of X. Then*
(i) *$\{U\}$ is a C.E. family if and only if $bd_X U$ is compact.*
(ii) *$Ex_{\beta X} U$ is C.I. in βX if and only if $\{V$: $Cl_X V$ is completely separated from $X \backslash U\}$ is a C.E. family.*

PROOF. (i) Suppose that $\{U\}$ is a $C.E.$ family. Then by 2.7 (ii) $Cl_{\beta X} U \cap (\beta X \backslash X) = Ex_{\beta X} U \cap (\beta X \backslash X)$. That is, $bd_{\beta X} Ex_{\beta X} U = Cl_{\beta X} bd_X U \subset X$.
Conversely suppose that $bd_X U$ is compact and let $K = bd_X U$. If $K \subset V$, where V is open in X, then $X = U \cup V \cup (X \backslash Cl_X U)$. Since $Cl_X(U \cap (X \backslash Cl_X U)) = \emptyset$, $\{U\}$ is a $C.E.$ family.
(ii) Suppose that $\mathscr{U}' = \{V$: $Cl_X V$ is completely separated from $X \backslash U\}$ is a $C.E.$ family. Then $Ex_{\beta X}\mathscr{U}'$ is a $C.I.$ set of βX and equals $Ex_{\beta X} U$.
On the other hand, suppose that $Ex_{\beta X} U$ is a $C.I.$ subset of βX. According to the remark following 2.6, the family $\{V$: $Cl_X V$ is completely separated from $X \backslash U\}$ is a $C.I.$ family.

3. A proximal characterization of almost rimcompact spaces and of 0-spaces. If X is almost rimcompact, the connected components of $\beta X \backslash X$ have a particularly nice form. According to 2.14 of [2] the connected component in $\beta X \backslash X$ of $p \in \beta X \backslash X$ is the set $\bigcap \{Cl_{\beta X} U$: U is π-open in X, $p \in Ex_{\beta X} U\}$. Identifying the connected components of $\beta X \backslash X$ in this way allowed us to show directly that by collapsing these connected components, we obtain an upper semicontinuous decomposition of βX with certain special properties. The connected components of $\beta X \backslash X$ are not as easily identified for an arbitrary 0-space X. Rather than working with this decomposition, we will characterize 0-spaces in terms of proximity theory. We would like to motivate this characterization by first considering almost rimcompact spaces from the viewpoint of proximities.

Recall that for a rimcompact space X, the proximity δ associated with $F_\circ X$ is defined as follows: for $A, B \subset X$, $A \delta B$ if and only if $Cl_X A$ and $Cl_X B$ are π-separated in X (see [5]). If X is any space, define γ to be a relation on $\mathscr{P}(X)$ as follows: for $A, B \subset X$, $A \gamma B$ if and only if $Cl_X A$ is nearly π-contained in $X \backslash Cl_X B$. For the rest of this section, γ will be defined as above.

If δ is as in the previous paragraph, then δ is clearly symmetric, while it is not clear that γ is symmetric. It is not necessary to build symmetry into the definition of γ. Recall that if $KX \in \mathscr{K}(X)$, and ρ is the relation on $\mathscr{P}(X)$ defined by (for $A, B \subset X$) $A \rho B$ if and only if $Cl_{KX} A \cap Cl_{KX} B \neq \emptyset$, then ρ is a proximity on X. We apply this fact to prove that if X is almost rimcompact, then γ is a proximity on X and therefore is symmetric (and satisfies the remaining defining properties of a proximity).

THEOREM 3.1. *For any space X, the following are equivalent.*

(i) *X is almost rimcompact.*

(ii) *γ is a proximity on X.*

If γ is a proximity on X, then $\gamma X = F_\circ X$.

PROOF. (i) implies (ii). If X is almost rimcompact, then X is a 0-space and $F_\circ X \backslash X$ is relatively 0-dimensionally embedded in $F_\circ X$. We will show both that γ is a proximity on X and that $\gamma X = F_\circ X$ by showing that if F_1, F_2 are subsets of X, then $Cl_{F_\circ X}F_1 \cap Cl_{F_\circ X}F_2 = \emptyset$ if and only if $F_1 \bar{\gamma} F_2$.

Suppose that $Cl_{F_\circ X}F_1 \cap Cl_{F_\circ X}F_2 = \emptyset$. For each $p \in Cl_{F_\circ X}F_1 \backslash Cl_X F_1$, choose an π-open subset $U(p)$ of X such that $p \in Ex_{F_\circ X}U(p)$, and $Cl_{F_\circ X}U(p) \cap Cl_{F_\circ X}F_2 = \emptyset$. Let $K = Cl_{F_\circ X}F_1 \backslash \bigcup \{Ex_{F_\circ X}U(p)\colon p \in Cl_{F_\circ X}F_1 \backslash X\}$. Then K is a compact subset of $Cl_X F_1$. Suppose that F_1' is a closed subset of $Cl_X F_1$ and that $F_1' \cap K = \emptyset$. Then $Cl_{F_\circ X}F_1' \subset \bigcup\{Ex_{F_\circ X}U(p)\colon p \in Cl_{F_\circ X}F_1 \backslash X\}$. By compactness there is a finite set $\{p_1, p_2, \ldots, p_n\} \subset Cl_{F_\circ X}F_1 \backslash X$ such that $Cl_{F_\circ X}F_1' \subset \bigcup\{Ex_{F_\circ X}U(p_i)\colon 1 \leqq i \leqq n\}$. Then $F_1' \subset \bigcup\{U(p_i)\colon 1 \leqq i \leqq n\}$, which is a π-open subset of X whose closure has empty intersection with F_2. In other words, F_1' and F_2 are π-separated, so $F_1 \bar{\gamma} F_2$.

Conversely, suppose that $F_1 \bar{\gamma} F_2$, and let K be a compact subset of $Cl_X F_1$ witnessing this fact. Let $p \in Cl_{F_\circ X}F_1 \backslash Cl_X F_1$. There is a closed subset F_p of $Cl_X F_1$ such that $p \in Cl_{F_\circ X}F_p$, and $(Cl_{F_\circ X}F_p) \cap K = \emptyset$. Thus $p \in Cl_{F_\circ X}F_p$, and by our choice of K, F_p is π-separated from F_2. Since $F_\circ X$ is a perfect compactification of X, an easy computation shows that $Cl_{F_\circ X}F_p \cap Cl_{F_\circ X}F_2 = \emptyset$. Then $p \notin Cl_{F_\circ X}F_2$, and as p was arbitrarily chosen in $Cl_{F_\circ X}F_1$, $Cl_{F_\circ X}F_1 \cap Cl_{F_\circ X}F_2 = \emptyset$.

(ii) implies (i). Suppose that γ is a proximity on X. We will show that the proximal compactification γX associated with γ has relatively 0-dimensionally embedded remainder, and therefore that X is almost rimcompact.

Note that if U is a π-open subset of X, and if A, B are closed subsets of X contained in U, $X \backslash Cl_X U$ respectively, then A and B are π-separated in X, hence $A \bar{\gamma} B$. That is, $Cl_{\gamma X}A \cap Cl_{\gamma X}B = \emptyset$.

We now claim that if U is a π-open subset of X, then $bd_X U = bd_{\gamma X}Ex_{\gamma X}U$. For suppose that $p \in bd_{\gamma X}Ex_{\gamma X}U \backslash bd_X U$. Then $p \in Cl_{\gamma X}Ex_{\gamma X}U \cap Cl_{\gamma X}(X \backslash U)$. As U is π-open in X, $bd_X U$ is closed in γX. Hence we can choose an open subset W of X such that $p \in Ex_{\gamma X}W$, and $Cl_{\gamma X}W \cap bd_X U = \emptyset$. Since $p \in Cl_{\gamma X}U \cap Ex_{\gamma X}W$, $p \in Cl_{\gamma X}(W \cap U)$. Similarly, $p \in Cl_{\gamma X}(W \cap (X \backslash U)) = Cl_{\gamma X}(W \cap (X \backslash Cl_X U))$, since $W \cap bd_X U = \emptyset$. Hence $p \in Cl_{\gamma X}(W \cap U) \cap Cl_{\gamma X}(W \cap (X \backslash Cl_X U))$. However, $Cl_X(W \cap U) \subset Cl_X W \cap Cl_X U \subset (Cl_X W) \cap U$, while $Cl_X(W \cap (X \backslash Cl_X U)) \subset Cl_X W \cap Cl_X(X \backslash Cl_X U) \subset (Cl_X W) \cap (X \backslash Cl_X U)$. Then $Cl_X(W \cap U)$ and $Cl_X(W \cap (X \backslash Cl_X U))$ are π-separated in X, hence $Cl_{\gamma X}(W \cap U) \cap Cl_{\gamma X}(W \cap$

$(X \backslash Cl_X U)) = \emptyset$, which contradicts our choice of p. Therefore $bd_X U = bd_{\gamma X} Ex_{\gamma X} U$ and our claim is verified.

Suppose that T is a closed subset of γX, and that $p \in (\gamma X \backslash X) \backslash T$. Choose open sets U and V of X such that $p \in Ex_{\gamma X} U$, $T \subset Ex_{\gamma X} V$, and $Cl_{\gamma X} U \cap Cl_{\gamma X} V = \emptyset$. Then $Cl_X U \gamma Cl_X V$; let K be a compact subset of $Cl_X U$ witnessing this fact. Since $p \notin K$, there is a closed subset F of $Cl_X U$ such that $p \in Cl_{\gamma X} F$, and $F \cap K = \emptyset$. Then F is π-separated from $Cl_X V$. Choose W to be a π-open subset of X such that $F \subset W$, and $Cl_X W \cap Cl_X V = \emptyset$. Then $bd_{\gamma X} Ex_{\gamma X} W \subseteq X$, and $p \in Cl_{\gamma X} F \cap (\gamma X \backslash X) \subset Cl_{\gamma X} W \cap (\gamma X \backslash X) = Ex_{\gamma X} W \cap (\gamma X \backslash X)$, while $T \cap Ex_{\gamma X} W \subset (Cl_{\gamma X} V) \cap (Ex_{\gamma X} W) = \emptyset$. This shows that $\gamma X \backslash X$ is relatively 0-dimensionally embedded in γX, as required.

A proximity similar to γ will be defined using *C.E.* families instead of π-open sets. Just as in the case of almost rimcompact spaces, when considering 0-spaces we are only concerned with what happens "away from compact subsets" of X.

DEFINITIONS 3.2. (i) If $A, B \subset X$, A is *C.E.*-separated from B if there is a *C.E.* family $\mathscr{E}$ such that $A \subset E$ for some $E \in \mathscr{E}$, and $Cl_X(\bigcup \mathscr{E}) \cap Cl_X B = \emptyset$.

(ii). Let X be any space, and define α to be a relation on $\mathscr{P}(X)$ as follows: for $A, B \subset X$, $A \not\alpha B$ if and only if (i) $Cl_X A \cap Cl_X B = \emptyset$ and (ii) there is a compact subset K of $Cl_X A$, so that if A' is a closed subset of $Cl_X A$, and $A' \cap K = \emptyset$, then A' is *C.E.*-separated from B.

For the rest of this paper, α will be as defined above. We shall prove that X is a 0-space if and only if α is a proximity on X, in which case $\alpha X = F_\circ X$ (3.6). Unless specifically stated, in the following results α is not assumed to be a proximity on X.

LEMMA 3.3. *Suppose that KX is a perfect compactification of X, and that F_1, F_2 are closed subsets of X such that $F_1 \not\alpha F_2$. Then if $p \in Cl_{KX} F_1 \backslash F_1$, there is a KX–C.I. subset U_p such that $p \in U_p$ and $Cl_X(U_p \cap X) \cap F_2 = \emptyset$, hence $U_p \cap Cl_{KX} F_2 = \emptyset$.*

PROOF. Suppose that $F_1 \not\alpha F_2$; let K be a compact subset of F_1 witnessing this fact. If $p \in Cl_{KX} F_1 \backslash F_1$, then $p \notin K$, so there is a closed subset F_1' of F_1 such that $p \in Cl_{KX} F_1'$, and $F_1' \cap K = \emptyset$. Thus $p \in Cl_{KX} F_1'$ and F_1' is *C.E.*-separated from F_2. Let $\mathscr{E}$ be a *C.E* family such that $(Cl_X(\bigcup \mathscr{E})) \cap F_2 = \emptyset$, and $F_1' \subset E$, for some $E \in \mathscr{E}$. Since KX is a perfect compactification of X, by 2.7 (iii), $Ex_{KX} \mathscr{E}$ is *C.I.* in KX. Also, $p \in Cl_{KX} F_1' \subset Ex_{KX} \mathscr{E}$ by 2.8 (iii), while $Cl_X(\bigcup \mathscr{E}) \cap F_2 = \emptyset$, hence $Ex_{KX} \mathscr{E} \cap Cl_{KX} F_2 = \emptyset$.

The following is an immediate consequence of 3.3.

COROLLARY 3.4. *Suppose that KX is a perfect compactification of X,* *and that F_1, F_2 are closed subsets of X. If $F_1 \not\alpha F_2$, then $Cl_{KX}F_1 \cap Cl_{KX}F_2 = \emptyset$.*

LEMMA 3.5. *Suppose that α is a proximity on X, and that αX is a perfect compactification of X. Then $\alpha X \backslash X$ is 0-dimensional, hence X is a 0-space and $\alpha X = F_\circ X$.*

PROOF. Suppose that T is a closed subset of $\alpha X \backslash X$, and that $p \in (\alpha X \backslash X) \backslash T$. We must find a clopen subset U of $\alpha X \backslash X$ such that $p \in U$, while $U \cap T = \emptyset$. Now $p \notin Cl_{\alpha X}T$, so there exist open sets V, W of X such that $p \in Ex_{\alpha X}U$, $Cl_{\alpha X}T \subset Ex_{\alpha X}W$, and $Cl_{\alpha X}V \cap Cl_{\alpha X}W = \emptyset$. Hence $Cl_X V \not\alpha\, Cl_X W$. If αX is a perfect compactification of X, then according to 3.3 there is an αX–*C.I.* open set U_p such that $p \in U_p$, while $U_p \cap Cl_{\alpha X}W = \emptyset$. Then $U_p \cap (\alpha X \backslash X)$ is a clopen subset of $\alpha X \backslash X$ having the desired properties.

THEOREM 3.6. *If X is any space, then the following are equivalent.*
(i) *X is a 0-space.*
(ii) *α is a proximity on X.*
Furthermore, if α is a proximity on X, then $\alpha X = F_\circ X$.

PROOF. (i) implies (ii). Suppose that X is a 0-space. We will prove that α is a proximity on X, and that $\alpha X = F_\circ X$ by showing that if F_1, F_2 are closed subsets of X, then $Cl_{F_\circ X}F_1 \cap Cl_{F_\circ X}F_2 = \emptyset$ if and only if $F_1 \not\alpha F_2$.

Suppose that $F_1 \not\alpha F_2$. Since $F_\circ X$ is a perfect compactification, according to 3.4, $Cl_{F_\circ X}F_1 \cap Cl_{F_\circ X}F_2 = \emptyset$.

On the other hand, suppose that $Cl_{F_\circ X}F_1 \cap Cl_{F_\circ X} F_2 = \emptyset$. Since $F_\circ X \backslash X$ is 0-dimensional, for each $p \in (Cl_{F_\circ X}F_1)\backslash X$, there is an $F_\circ X$–*C.I.* open set $U(p)$ such that $p \in U(p)$ while $Cl_X(U(p) \cap X) \cap F_2 = \emptyset$. Let $K = Cl_{F_\circ X}F_1 \backslash \bigcup\{U(p) : p \in Cl_{F_\circ X}F_1 \backslash X\}$. Then K is a compact subset of F_1. If F_1' is a closed subset of F_1 such that $F_1' \cap K = \emptyset$, then $Cl_{F_\circ X}F_1' \subset \bigcup\{U(p) : p \in Cl_{F_\circ X}F_1 \backslash X\}$. By compactness, there is a finite subset $\{p_1, p_2, \ldots, p_n\} \subset Cl_{F_\circ X}F_1 \backslash X$ such that $Cl_{F_\circ X}F_1' \subset \bigcup\{U(p_i) : 1 \leqq i \leqq n\}$. Now $\bigcup\{U(p_i) : 1 \leqq i \leqq n\}$ is a *C.I.* open set of $F_\circ X$, so by 2.6, there is a *C.E.* family $\mathscr{E}$ of open sets of X such that $Ex_{F_\circ X}\mathscr{E} = \bigcup\{U(p_i) : 1 \leqq i \leqq n\}$. Now $Cl_{F_\circ X}F_1' \subset Ex_{F_\circ X}\mathscr{E}$, so by 2.8 (iii), there is $E \in \mathscr{E}$ such that $F_1' \subset E$. Also, since $Cl_X(\bigcup\{U(p_i) \cap X : 1 \leqq i \leqq n\}) \cap F_2 = \emptyset$, $Cl_X(\bigcup \mathscr{E}) \cap F_2 = \emptyset$. In other words, F_1' is *C.E.* separated from F_2; that is, $F_1 \not\alpha F_2$.
(ii) implies (i). Suppose that α is a proximity on X. According to 3.5, to show that X is a 0-space it suffices to prove that αX is a perfect compactification of X.

First, suppose that V_1 and V_2 are disjoint *C.I.* subsets of βX. If $y_i \in V_i \cap (\beta X \backslash X)$ $(i = 1, 2)$, we claim that $(\alpha f)(y_1) \neq (\alpha f)(y_2)$. To see this, note that there are closed subsets F_i of X such that $y_i \in Cl_{\beta X}F_i \subset V_i$ $(i = 1, 2)$. By 2.6 there exists a *C.E.* family $\mathscr{E}$ such that $Ex_{\beta X}\mathscr{E} = V_1$.

Since $Cl_{\beta X}F_1 \subset Ex_{\beta X}\mathscr{E}$, by 2.8 (iii), $F_1 \subset E$, for some $E \in \mathscr{E}$. Also, $Cl_X(\bigcup\mathscr{E}) \cap F_2 \subset (Cl_{\beta X}V_1) \cap V_2 = \varnothing$, so F_1 is $C.E.$ separated from F_2; that is $F_1 \not\alpha F_2$. Then $Cl_{\alpha X}F_1 \cap Cl_{\alpha X}F_2 = \varnothing$. Since $(\alpha f)(y_i) \in Cl_{\alpha X}F_i$, $(\alpha f)(y_1) \neq (\alpha f)(y_2)$, and our claim is verified.

Now suppose that αX is not a perfect compactification of X. According to 1.2, there is $p \in \alpha X \backslash X$ such that $(\alpha f)^{\leftarrow}(p)$ is not connected. Write $(\alpha f)^{\leftarrow}(p) = T_1 \cup T_2$, where T_1 and T_2 are disjoint closed subsets of $(\alpha f)^{\leftarrow}(p)$. Since $(\alpha f)^{\leftarrow}(p)$ is compact, T_1 and T_2 are disjoint compact subsets of βX, so there are open sets U_1 and U_2 such that $T_i \subset U_i$ $(i = 1, 2)$, and $Cl_{\beta X}U_1 \cap Cl_{\beta X}U_2 = \varnothing$. Since αf is a closed map, and $(\alpha f)^{\leftarrow}(p) \subset U_1 \cup U_2$, there are open sets W_1 and W_2 of X such that $p \in Ex_{\alpha X}W_1 \subset Cl_{\alpha X}W_1 \subset Ex_{\alpha X}W_2$, and $Cl_{\beta X}W_2 \subset (\alpha f)^{\leftarrow}[Cl_{\alpha X}W_2] \subset U_1 \cup U_2$. Now $Cl_{\alpha X}W_1 \cap Cl_{\alpha X}(X\backslash W_2) = \varnothing$; that is $Cl_X W_1 \not\alpha (X\backslash W_2)$. Also, $(\alpha f)^{\leftarrow}(p) \subseteq (\alpha f)^{\leftarrow}[Cl_{\alpha X}W_1] = Cl_{\beta X}W_1$.

In the following $i = 1, 2$. Choose $z_i \in (\alpha f)^{\leftarrow}(p) \cap U_i \cap Cl_{\beta X}W_1$. According to 3.3, there are βX–$C.I.$ sets S_i such that $z_i \in S_i$, and $S_i \cap Cl_{\beta X}(X\backslash W_2) = \varnothing$. Now $S_i \subset U_1 \cup U_2$. Let $S_i' = S_i \cap U_i$. Since $Cl_{\beta X}U_1 \cap Cl_{\beta X}U_2 = \varnothing$ and $S_i \cap (\beta X\backslash X)$ is clopen in $\beta X\backslash X$, $S_i' \cap (\beta X\backslash X)$ is clopen in $\beta X\backslash X$. In other words, S_i' is a $C.I.$ subset of βX. Also $z_i \in S_i'$, $i = 1, 2$, while $S_1' \cap S_2' \subset U_1 \cap U_2 = \varnothing$. It follows from our earlier claim that $(\alpha f)(z_1) \neq (\alpha f)(z_2)$, which contradicts the fact that $z_i \in (\alpha f)^{\leftarrow}(p)$. Thus $(\alpha f)^{\leftarrow}(p)$ is connected for each $p \in \alpha X\backslash X$, hence αX is a perfect compactification of X.

4. Full 0-spaces. The correspondence between full $C.I.$ sets and full $C.E.$ families that is outlined in the remarks following 2.6 and 2.7 allows us to characterize full 0-spaces.

DEFINITIONS 4.1. (i). If $A, B \subset X$, then A is fully $C.E.$ separated from B if there is a full $C.E.$ family $\mathscr{E}$ such that $Cl_X(\bigcup\mathscr{E}) \cap Cl_X B = \varnothing$, while $A \subset E$ for some $E \in \mathscr{E}$.

(ii). If X is any space, define α' to be a relation on $\mathscr{P}(X)$ as follows: for $A, B \subset X$, $A \not\alpha B$ if and only if there is a compact subset K of $Cl_X A$ so that if A' is a closed subset of $Cl_X A$, and $A' \cap K = \varnothing$, then A' is fully $C.E.$ separated from B.

Then results 3.3–3.6 hold, if in the statements and proofs of the results, "$C.E.$", "$C.I.$", "α", and "0-space" are replaced by "full $C.E.$", "full $C.I.$", "α'", and "full 0-space" respectively, leaving us with the following characterization of full 0-spaces.

THEOREM 4.2. *If X is any space, then the following are equivalent.*

(i) *X is a full 0-space.*

(ii) *α' is a proximity on X.*

If α' is a proximity on X, then $\alpha' X = F_0 X$.

Recall that a closed subset F of X is regular closed in X if $Cl_X \mathrm{int}_X F = F$. The following result is 2.4 of **[10]**.

LEMMA 4.3. *If A is a regular closed subset of X, B is closed in X, and* $Cl_{\beta X}A \backslash A \subset Cl_{\beta X}B \backslash B$, then $Cl_X(A \backslash B)$ is pseudocompact.

PROPOSITION 4.4. *Let U be open in X. If $Ex_{\beta X}U$ is C.I. in βX, and $p \in bd_{\beta X}(Ex_{\beta X}U) \cap (\beta X \backslash X)$, then there is a closed pseudocompact subset F of X such that $p \in Cl_{\beta X}F$.*

PROOF. By assumption $Ex_{\beta X}U \cap (\beta X \backslash X)$ is clopen in $\beta X \backslash X$. Note that $bd_{\beta X}(Ex_{\beta X}U) \backslash X = [Cl_{\beta X}U \backslash Ex_{\beta X}U] \backslash X$. If $p \in (Cl_{\beta X}U \backslash Ex_{\beta X}U) \backslash X$, there exists an open subset V of X such that $p \in Ex_{\beta X}V$, while $(Cl_{\beta X}V) \cap Ex_{\beta X}U \subset X$. Let $B = X \backslash U$. Then $Cl_{\beta X}V \cap (\beta X \backslash X) \subset \beta X \backslash Ex_{\beta X}U = Cl_{\beta X}B$. Since $Cl_X(V \backslash B)$ is regular closed, according to 4.3, $Cl_X[Cl_X(V \backslash B) \backslash B] = Cl_X(V \backslash B)$ is a pseudocompact subset of X. Now $Cl_X(V \backslash B) = Cl_X(V \cap U)$, and it is easily checked that $p \in Cl_{\beta X}(V \cap U)$. The proposition follows.

COROLLARY 4.5. *Suppose X is a space in which pseudocompact closed subsets are compact. If X is a full* 0*-space, then X is almost rimcompact.*

PROOF. Suppose that pseudocompact closed subsets of X are compact. It follows from 4.4 that if $Ex_{\beta X}U$ is any full *C.I.* subset of βX, then $bd_{\beta X}Ex_{\beta X}U \subset X$. This implies that any connected component of $\beta X \backslash X$ having a basis in βX of full *C.I.* sets has a basis of open sets whose boundaries are contained in X. In other words, if X is a full 0-space, then X is almost rimcompact.

COROLLARY 4.6. *If X is realcompact or metacompact, then X is a full* 0*-space if and only if X is almost rimcompact.*

In 3.9 of **[2]** we constructed a 0-space which was not almost rimcompact. The details of 3.9 indicate that this space is a full 0-space. We do not have an example of a 0-space which is not full—this question is left open to the reader.

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