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EXTENDED ARTIN-SCHREIER THEORY OF FIELDS

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Dedicated to the memory of Gus Efroymson

Introduction. This survey is concerned with recent developments in the Artin-Schreier theory of fields. The basic notion of an ordering of a field has been extended to the more general notion of an ordering of higher level. This extension has opened the way to a natural, far-ranging extension of the ordinary Artin-Schreier theory. Between 1924 and 1927, the foundation of the Artin-Schreier theory was laid by two papers of E. Artin ([**1**], [**2**]), by two joint papers of E. Artin and O. Schreier ([**3**], [**4**]) and by R. Baer's contribution [**5**]. As an introduction to this survey we recall some main features of these papers and the investigations they inspired.

It is thown that a field K can be ordered if and only if it is formally real which, by definition, means that -1 is not a sum of squares in K. Moreover, an element of K is proved to be a sum of squares if and only if it is contained in all orderings of K.

The maximal real agebraic extensions R of K (the real closures of K) are shown to admit the unique ordering R^2. Via $R \mapsto R^2 \cap K$ ($R^2 \cap K$ is an ordering in K), their conjugacy classes over K correspond bijectively to the orderings of K.

Starting from these results Artin was able to solve Hilbert's 17^{th} problem in the affirmative. Artin's proof related, for the first time, the theory of real fields with real algebraic geometry. This becomes especially clear in S. Lang's version of this proof [**35**] which in turn gave rise to the Real Nullstellensatz by Dubois [**27**] and Risler [**40**]. An up-to-date account of this relationship can be found in several papers contained in [**23**].

Real closed fields are characterized by the property that their algebraic closure is a finite, nontrivial extension or, equivalently, by the property that their absolute Galois group is a nontrivial finite group, in fact a cyclic group of order 2. From this point of view, the theory of real closed fields contributes to the question of characterizing which profinite groups may occur as the absolute Galois group of a field. It is this problem which is of interest for this paper.

Real closed fields were also important in the development of model theory. In 1948 A. Tarski published his famous Tarski Principle for these fields **[45]** and later A. Robinson proved the model completeness of the elementary theory of real closed fields. One can see from the introduction of the paper [3] of Artin and Schreier that at least Tarski's result on completeness fits nicely with the motivation of the founder of the theory.

Today it appears quite natural to treat orderings of a field by using valuation rings. It is interesting to learn that Artin and Schreier in [3] and, more systematically, Baer in [5], had already proceeded in this manner. They all were led to valuation rings by investigating Archimedes' axiom in arbitrary ordered fields. Besides model theory there is another area of current research that is not touched upon in the fundamental papers of Artin, Schreier and Baer. In 1966 A. Pfister **[38]** revealed the importance of the various orderings of a field K for the quadratic forms over K. He was dealing with the signatures $\mathrm{sgn}_P(\rho)$ of a quadratic form ρ with respect to the various orders P of K and was able to derive deep results on the Witt ring of K.

Our list of some of the main features of the ordinary Artin-Schreier theory is now complete. Each of the items will be discussed again in the extended theory. The extension is based on the replacement of orderings by orderings of higher level or by signatures of higher level. By definition, a signature χ of K is a character $\chi: K^* \to \mathbf{Q}/Z$ with an additively closed kernel, and the orderings of higher level are just the kernels of those signatures. The signatures $\chi: K^* \to (1/2n)\mathbf{Z}/\mathbf{Z}$ of level n are related to the sums of $2n$-th powers in K in the same way that the ordinary orderings are related to sums of squares. In extending Artin's result we find more generally that an element of K^* is a sum of $2n$-th powers if and only if it is contained in the intersection of all orderings of level n.

The plan of the survey is the following. In §1 we present the Kadison-Dubois representation theorem for Archimedean partially ordered rings. This theorem serves as the fundamental device in studying Archimedean properties of fields and leads to the definition and study of the real holomorphy ring. Signatures and orderings of higher levels are defined and investigated in §2. Their application to questions on sums of $2n$-th powers is discussed. In the next section we shall be concerned with higher reduced Witt rings by which Pfister's approach is extended to signatures of higher level. The last two sections are devoted to real closures of signatures of higher level (§4), and to the theory of these fields (§5), including a proof of the Nullstellensatz over them.

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1. The representation theorem of Kadison-Dubois and the real holomorphy ring. The importance of the Archimedean property of an ordering was noticed as early as in the papers of Artin-Schreier [3] and Baer [5]. It is the distinction between Archimedean and non-Archimedean orderings or partial orders that we adopt as one of the leading principles. The Kadison-Dubois representation theorem treats a rather general Archimedean situation. In the field case, the deviation from the Archimedean axiom is related to the appearance of nontrivial valuation rings with a formally real residue field. The real holomorphy ring $H(K)$ of a formally real field K comprises both aspects. It consists of those elements of the field which are finite with respect to every ordering of K. The valuation rings of K with a formally real residue field are just the localizations of $H(K)$. Hence, in the study of the various orderings of a field, the holomorphy ring is to be regarded as one of the basic global objects.

The representation theorem is concerned with Archimedean partially ordered rings. Let R denote a ring with 1, neither associativity nor commutativity being assumed. An infinite preprime P (in the sense of Harrison [29]) is a subset of R subject to the conditions

$$(*) \qquad 0, 1 \in P; \ -1 \notin P; \ P + P \subset P \text{ and } PP \subset P.$$

Note that $(*)$ implies $\mathrm{char}(R) = 0$, hence $\mathbf{Z} \subset R$. If additionally

$$(**) \qquad (\forall a \in R)\, (\exists n \in \mathbf{N}) : n - a \in P$$

holds then P is called an Archimedean infinite preprime. Basic examples are provided by compact Hausdorff spaces X:

$$R_0 = C(X, \mathbf{R}),\ P_0 = C^+(X, \mathbf{R}) = \{f \in C(X, \mathbf{R}) \mid (\forall x \in X) : f(x) \geqq 0\}.$$

The Kadison-Dubois representation theorem relates an arbitrary ring with an Archimedean infinite preprime P to these examples $(C(X, \mathbf{R}), C^+(X, \mathbf{R}))$. Given (R, P) as above set

$$X = \{\phi \in \mathrm{Hom}(R, \mathbf{R}) \mid \phi(P) \subset \mathbf{R}^2\}.$$

By $\mathrm{Hom}(R, \mathbf{R})$ we mean the set of unitary ring homomorphisms. On X we impose the weak topology with regard to all the evaluation functions $\hat{a} : X \to \mathbf{R}$, $\phi \mapsto \phi(a)$. We get the representation

$$\Phi : \begin{cases} R \to C(X, \mathbf{R}) \\ a \to \quad \hat{a} \end{cases}.$$

1.1. Theorem i) *X is a non-empty compact Hausdorff space,*

ii) $\Phi^{-1}(C^+(X, R)) = \{a \in R \mid (\forall n \in \mathbf{N})\, (\exists m \in \mathbf{N}) : m(1 + na) \in P\}$,

iii) $\ker \Phi = \{a \in R \mid (\forall n \in \mathbf{N})\, (\exists m \in \mathbf{N}) : m(1 \pm na) \in P\}$,

iv) $\mathbf{Q} \cdot \Phi(R)$ *is dense* in $C(X, \mathbf{R})$,

v) $(\forall a \in R) : \begin{cases} a \in R^* \Leftrightarrow \Phi(a) \in C(X, \mathbf{R})^* \\ a \in R^* \cap P \Leftrightarrow \Phi(a) \in C(X, \mathbf{R})^* \cap C^+(X, \mathbf{R}). \end{cases}$

Here R^* denotes the group of units of R, whenever R is a ring.

This theorem and its proof have a long history. The first special case was treated by H. Stone **[32]** in 1942, his result was extended by Kadison **[32]** in 1952, later by D. Dubois **[26]** in 1968 and by the author **[8]** in 1979. Today an easy proof by N. Schwartz and the author is available **[15]**. In **[44]** R. G. Swan studied projective modules over certain rings by relating them to vector bundles over associated compact Hausdorff spaces. These rings had to satisfy certain axioms and it is interesting to note that in the situation above the image of R under Φ meets these requirements.

We next turn to the application of 1.1 to infinite preprimes of fields. such a preprime T of K is called a preordering if $T^* := T\backslash\{0\}$ is a subgroup of K^*, and a torsion preordering if additionally K^*/T^* is a torsion subgroup. Let T be a torsion preordering of K. In general, T need not be an Archimedean preprime. Following the approach of Artin-Schreier and Baer we therefore introduce the ring $A(T)$ of the "finite" elements.

$$A(T) = \{a \in K \mid (\exists n \in \mathbf{N}) \colon n \pm a \in T\}$$

$A(T)$ is readily checked to be a subring of K. As an application of 1.1, in **[8]** the following fundamental result is proved.

THEOREM 1.2. *Given a torsion preordering T of the field K, the following statements hold*:

i) *$A(T)$ is a Prüfer ring with quotient field K.*

ii) *as a ring, $A(T)$ is generated by the elements $1/(1 + t)$, $t \in T$;*

iii) *every valuation overring of $A(T)$ has a formally real residue field.*

Following one of the various definitions we call an integral domain a Prüfer ring if every localization is a valuation ring.

In view of statement iii) we obtain the important corollary that a field is formally real if and only if it admits a torsion preordering. Statement iii) immediately leads us to consider the intersection of all valuation rings of K with a formally real residue field, $H(K) = \bigcap V$, where V ranges over all valuation rings of K with formally real residue field. $H(K)$ is called the real holomorphy ring of K. It includes the two sorts of information we are interested in, the Archimedean property and the deviation from it measured by valuation rings. This is the essence of the next statement. Set

$$\sum_1^k K^n := \{\sum_1^k x_i^n \mid x_1, \ldots, x_k \in K\}, \sum K^n = \bigcup_{k=1}^{\infty} \sum_1^k K^n.$$

THEOREM 1.3. i) $H(K) = A(\sum K^2) = \{a \in K \mid (\exists n \in \mathbf{N}) \colon n \pm a \in \sum K^2\}$.

ii) *$H(K)$ is a Prüfer ring with quotient field K.*

iii) *A valuation ring V of K has a formally real residue field if and only if $H(K) \subset V$.*

In order to derive 1.3 from 1.2 one only has to observe that a valuation V of K has a formally real residue field if and only if $(1 + \sum_1^r x_i^2)^{-1} \in V$ holds for every $x_1, \ldots, x_r \in K$, $r \in \mathbf{N}$.

By the procedure described before 1.1 we obtain a topological space M attached to $H = H(K)$, since H can be expressed as $A(\sum K^2)$. It turns out that $M = \mathrm{Hom}(H, \mathbf{R})$, since $H \cap \sum K^2 = \sum H^2$. But by 1.3, ii) and iii), every $\phi \in M$ extends uniquely to a real place $\lambda: K \to \mathbf{R} \cup \infty$ and every real place can be obtained in this way. For more details see, e.g., **[10]**. Moreover, M is seen to coincide with the subspace of closed points in the real spectrum of H **[10]**.

2. Signatures of fields. Set $\mu = \{z \in \mathbf{C} \mid z^n = 1 \text{ for some } n\}$. By definition, a signature χ of K is a character $\chi: K^* \to \mu$ with an additively closed kernel. This notion was introduced in **[14]** by J. Harman, A. Rosenberg and the author. It extends the notion of an ordering P in a character theoretical setting. Note that an ordering P can be equivalently replaced by the signature sgn_P where $\mathrm{sgn}_P(a) = 1$ or -1 if $a \in P^*$ or $a \in -P^*$ respectively. Obviously, we get P back via $P^* = \ker \mathrm{sgn}_P$. In general, the kernels of arbitrary signature, the so called orderings of higher level **[6]**, **[9]**, **[14]**, play an important role too. But in certain circumstances it is easier to start with signatures than with orderings of higher level.

We first extend Artin's characterization of sums of squares in a field.

THEOREM 2.1. *Let T be a torsion preordering of K. Then*

$$T^* = \bigcap \ker \chi,$$

where χ ranges over the set X_T of all signatures of K with $T^* \subset \ker \chi$.

This theorem generalizes Artin's result for the case $T = \sum K^2$, since the kernel of a signature $\chi \in X_{\Sigma K^2}$ necessarily has index 2 in K^* and is therefore of the type sgn_P, for an ordering P.

The other basic discovery of Artin-Schreier and Baer, namely that an ordering gives rise to a valuation ring, also extends to this more general situation.

For any signature χ of K we set

$$A(\chi) := A(\ker \chi \cup \{0\}),$$

$$I(\chi) := \{a \in K \mid (\forall n \in \mathbf{N}) : \frac{1}{n} \pm a \in \ker \chi\}.$$

As proved in **[8, (3.4)]** or **[14, (2.7)]**, we get the following result.

THEOREM 2.2.

i) *$A(\chi)$ is a valuation ring with the maximal ideal $I(\chi)$. The residue field $A(\chi)/I(\chi)$ is formally real,*

ii) *χ induces the signature of an Archimedean ordering P of $A(\chi)/I(\chi)$ via $\varepsilon + I(\chi) \mapsto \chi(\varepsilon)$, $\varepsilon \in A(\chi)^*$.*

We are now going to specialize the previous results. The ordinary Artin-Schreier theory provided basic tools for the study of sums of squares in fields. In an analogous manner, one can exploit the extended Artin-Schreier theory for the investigation of sums of $2n$-th powers. To this end we deal with the torsion preordering $\sum K^{2n}$. (Arbitrary torsion preorderings have not yet found any applications.) A detailed account can be found in **[9]**, **[11]**.

THEOREM 2.3. *For a field K the following statements are equivalent.*

i) $-1 \in \sum K^2$, *i.e., K is not formally real.*

ii) $-1 \in \sum K^{2n}$ *for some* $n \in \mathbf{N}$.

iii) $-1 \in \sum K^{2n}$ *for all* $n \in \mathbf{N}$.

This was first proved by Joly **[31]**, but a proof also easily follows from 1.2 since $\sum K^{2n}$ is a torsion preordering if we assume $-1 \notin \sum K^{2n}$.

A famous result of Pfister **[37]** states that the "Stufe" s of a field is always a power of 2. The Stufe s equals s_1 in the sense of the following definition.

$$s_n = s_n(K) = \min\{k \in \mathbf{N} \mid -1 \in \sum_1^k K^{2n}\} \text{ or } s_n = \infty.$$

To date, there are no hints indicating which way Pfister's result might be extended. To stimulate interest I would like to pose the first problem.

PROBLEM 1. What are the natural numbers which occur as the higher Stufen s_n of fields?

A signature χ which is trivial on $\sum K^{2n}$ necessarily has a finite order $2m$ in the character group of K^*, where $m|n$. This order is even since $\chi(-1) = -1$. In this case, $s(\chi) = m$ is called the level of χ. If $P = (\ker \chi) \cup \{0\}$ where χ is a signature with $s(\chi) = m$ then P is called an ordering of level m. The orderings of level 1 are just the usual orderings of the Artin-Schreier theory. This terminology follows the convention of **[14]** and supersedes the previous notations. Without a reference to signatures, an ordering of level m can be characterized as any subset of K subject to

$$K^{2m} \subset P,\ P + P \subset P,\ P \cdot P \subset P,\ K^*/P^* \text{ cyclic of order } 2m.$$

Note that the first three conditions imply that $P^* = P\backslash\{0\}$ is a subgroup of K^*.

In terms of orderings of higher level, theorem 2.1 reads as follows.

THEOREM 2.4. $\sum K^{2n} = \bigcap P$, *where P ranges over all orderings of level m, for which $m|n$.*

In the proof of 2.1 one has as an intermediate result that $T^* = \bigcap V^*T^*$, where T is any torsion preordering and V ranges over all valuation rings with a formally real residue field. This fact and the statement v) in 1.1 result in surprising relations between sums of $2n$-th powers and the real holomorphy rings.

THEOREM 2.5. *Let K be a formally real field. Then the following statements hold for $H = H(K)$.*
i) $H = A(\sum K^{2n})$,
ii) $\sum K^{2n} = (H^* \cap \sum K^2)(\sum K^2)^n$.

PROOF.
i) Theorem 1.2, iii), says that $H \subset A(\sum K^{2n})$. By 1.2, ii), the latter ring is generated by the elements $1/1 + \sum_1^r x_i^{2n}$, $x_i \in K$, $r \in \mathbf{N}$. These elements themselves are contained in every valuation ring with a formally real residue field, hence $A(\sum K^{2n}) \subset H$.
ii) Because of $H \cap \sum K^2 = \sum H^2$, the statement i) just proved, and 1.1, v), we get $H^* \cap \sum K^2 \subset \sum K^{2n}$. Take $q \in (\sum K^2)^*$, then $q^n \in V^*K^{*2n}$ for every valuation ring V with a formally real residue field. Hence, $q^n \in \bigcap V^*K^{*2n} = \sum K^{2n}$. This all implies $\sum K^{2n} \supset (H^* \cap \sum K^2)(\sum K^2)^n$. To prove the opposite inclusion write

$$\sum x_i^{2n} = \frac{\sum x_i^{2n}}{(\sum x_i^2)^n} \cdot (\sum x_i^2)^n$$

and check that the first factor on the right hand side lies in $H^* \cap \sum K^2$.

A detailed study of the group $H^* \cap \sum K^2$ has led to results on the finiteness of the Waring numbers

$$g(K, n) = \min\{\ell \in \mathbf{N} \cup \{\infty\} \mid \sum K^n = \sum_1^\ell K^n\}$$

where K is any field, $n \in \mathbf{N}$. In [11], this number was denoted by $P_n(K)$ and called the n-th Pythagoras number. But, following the more classical notations $g(u)$ and $G(u)$ of Hardy-Littlewood, I would like to make this change.

THEOREM 2.6. ([11]) *Given a field K the following statements are equivalent.*
i) $g(K, 2) < \infty$,
ii) $g(K, 2n) < \infty$ *for some n,*
iii) $g(K, 2n) < \infty$ *for all n.*

Moreover, there is a polynomial bound for $g(K, 2n)$ in terms of $g(K, 2)$.

Pfister has shown $g(\mathbf{R}(X_1, \ldots, X_k), 2) \leqq 2^k$ [**39**]. Hence we see $g(\mathbf{R}(X_1, \ldots, X_k), 2n) < \infty$ for all k, n. The precise values are unknown. The study of the numbers $g(K, 2)$ is in general extremely difficult, and the numbers $g(K, 2n)$ present even more difficulties. Moreover it seems that studying the $g(K, 2n)$, $n > 1$, one requires new ideas and methods beyond those already known in quadratic form theory. This is the reason for the second problem.

PROBLEM 2. Find general results on the $g(K, 2n)$, either sharp bounds or exact values.

If one tries to extend the above applications of signatures and the real holomorphy rings, one is immediately led to the question of which subsets T of K^* can be represented as an intersection of kernels of signatures. For T to be represented in this way there are two obvious necessary conditions: T is a preordering and $H^* \cap \sum K^2 \subset T$. The latter condition follows since $H^* \cap \sum K^2 = \bigcap \ker \chi$ where χ ranges over the whole set of signatures ([**14**, (2.10)], [**11**, (1.1)]). In general, these two conditions are not sufficient as examples show [**18**], but cf 2.1. For further applications it is therefore interesting to deal with the following problem.

PROBLEM 3. Characterize the intersections of kernels of signatures and understand their meaning.

3. The reduced Wittrings of higher level. Let $\mathrm{sgn}(K)$ denote the set of all signatures of K and $K^{\wedge *}$ the compact character group of the discrete group K^*. We give $\mathrm{sgn}(K)$ the subspace topology. In general, $\mathrm{sgn}(K)$ is not closed in $K^{\wedge *}$. Its closure $\overline{\mathrm{sgn}(K)}$ has been determined in [**18**].

THEOREM 3.1. *A character χ: $K^* \to S^1$ lies in $\overline{\mathrm{sgn}(K)}$ if and only if it satisfies*:

i) *$A(\chi)$ is a valuation ring of K,*

ii) *$I(\chi)$ is the maximal ideal of $A(\chi)$,*

iii) *χ induces via $\varepsilon + I(\chi) \mapsto \chi(\varepsilon)$, $\varepsilon \in A(\chi)$, the signature sgn_P of an Archimedean ordering P of $A(\chi)/I(\chi)$.*

Here, of course, $A(\chi)$ and $I(\chi)$ are defined in the same manner as in §2.

In some sense, this is a satisfying theorem because it shows that, even in the most general context, the two basic features of the Artin-Schreier theory are preserved (the Archimedean property and the occurence of valuation rings). But on the other hand, it seems to suggest that one should deal with all characters $\chi \in \overline{\mathrm{sgn}(K)}$ and not just with signatures χ: $K^* \to \mu$ as defined here. For future investigations this might become necessary. Actually however, there is no need to be concerned with the "most general signatures" $\chi \in \overline{\mathrm{sgn}(K)}$. To see this, first set $X_T = \{\chi \in \mathrm{sgn}(K) \mid \chi(T^*) = 1\}$ where T is any preordering of K. Denoting the closure by $\bar{X}_T$ we get

$$\bigcap_{\chi \in X_T} \ker \chi = \bigcap_{\chi \in \bar{X}_T} \ker \chi.$$

Hence, in this regard, there is no need at all to turn to $\bar{X}_T$. In the case of a torsion preordering T we even get that X_T is closed, since K^*/T'^* is closed in $K^{\wedge *}$. Thus we have

Proposition 3.2. *If T is a torsion preordering, then X_T is a compact Hausdorff space.*

If $T = \sum K^2$ then X_T is just the well known Harrison space $X(K)$ of all the orderings of K, see, e.g., **[10]**. The theory of reduced quadratic forms (**[12]**, **[13]**, **[21]**, **[36]**) is concerned with the subring $W_{\mathrm{red}}(K) \subset C(X(K), \mathbf{Z})$ which is generated by the evaluation functions $\hat{a}$: $X(K) \to \mathbf{Z}$, $P \mapsto \mathrm{sgn}_P(a)$. This definition is easily extended to our general situation. Instead of Z we have to deal with $Z[\mu] =: \sigma$ which is, by the classical result of Dedekind-Weber, the ring of integers of the maximal abelian extension of $\mathbf{Q}$. Each $a \in K^*$ induces the function $\hat{a}$: $X_T \to \sigma$, $\chi \mapsto \chi(a)$. Hence, we define W_T to be the subring of $C(X_T, \sigma)$ which is generated by these functions a, $\hat{a} \in K^*$. Here σ is regarded as a discrete ring, and T is assumed to be a torsion preordering.

These rings W_T are the reduced Witt rings of higher level. A forthcoming paper of A. Rosenberg and the author is devoted to them. But that paper treats only the case that K^*/T^* has a finite exponent $2n$. A sample of the results will be described in the sequel.

We fix a preordering T subject to $K^{2n} \subset T$. A "form" over T of dimension k is a k-tuple $\rho = (a_1, \ldots, a_k)$, $a_i \in K^*$. A form ρ induces the function $\hat{\rho} = \sum \hat{a}_i \in W_T$. Two forms ρ and τ are called isometric ($\rho \simeq \tau$) if $\dim \rho = \dim \tau$ and $\hat{\rho} = \hat{\tau}$. In the case of quadratic forms one has the two fundamental relations (Satz 7 of Witt **[46]**)

$$\begin{aligned} &(x^2 a) \simeq (a) && \text{for } a, x \in K^*, \\ &(a, b) \simeq (a + b, (a + b)ab) && \text{for } a, b, a + b \in K^*. \end{aligned}$$

In our context the following statements are valid.

Proposition 3.3.
i) $(ta) \simeq (a)$ *for $a \in K^*$, $t \in T$,*
ii) $(a, b) \simeq (a + b, a^{2n}b + ab^{2n})$ *for $a, b, a + b \in K^*$.*

The proof relies heavily on the fundamental fact that for every signature χ, $A(\chi)$ is a valuation ring and that χ induces the signature of an ordering on the residue field. Witt's Satz 7 states that it is possible to pass from one form to an isometric one by a finite number of applications of the basic relations. This is true in our situations as well. To have a convenient notion at hand we say that a form ρ over T is chain-equivalent to the form τ over T if τ can be obtained by a finite number of the substitutions as given in i) and ii) of 3.3.

THEOREM 3.4. *Two forms over T are isometric if and only if they are chain-equivalent.*

This is one of the basic theorems which enables a detailed study of the higher reduced Witt rings. Some of its many consequences follow.

COROLLARY 3.5. *If $\rho = (a_1, \ldots, a_k)$ and $\tau = (b_1, \ldots, b_k)$ are isometric then their value sets coincide:*

$$D_T(\rho) := \sum_1^k Ta_i = \sum_1^k Tb_i =: D_T(\tau).$$

COROLLARY 3.6. *The kernel of the natural epimorphism $Z[K^*/T^*] \to W_T$ induced by $a \mapsto \hat{a}$ is generated by the elements*

$$\bar{1} + \overline{(-1)}, \bar{a} + \bar{b} - \overline{(a + b)} - \overline{(a^{2n}b + ab^{2n})}$$

where $a, b, a + b \in K^*$, $\bar{a} = aT^*$ etc.

Every reader who is familiar with the theory of reduced quadratic forms ([12], [13], [21], [36]) will easily recognize the analogy. This similarity remains valid for nearly all aspects of the theory. For instance, the prime spectrum of W_T and its relation to orderings of higher level, the characterization of fans (see [8] for this notion) by the fact that W_T is a group ring over an integral domain, and, finally, the representation theorem of [13] can all be carried over to our situation.

Putting all these results together one is led to restart the program of abstract Witt rings [34], [36] and spaces of orderings [36] by considering appropriate factor rings $Z[G]/J$ where G might be any torsion group. It is interesting to note that the pair $(K^*/T^*, X_T)$ represents a space of orderings of higher level [36] if one makes the obvious changes in Marshall's definition. Therefore the following problem is posed.

PROBLEM 4. Extend the theory of abstract Witt rings and of spaces of orderings in order to incorporate the rings W_T defined above.

In the case of exponent 2, i.e., $K^2 \subset T$, it is well known [12] that W_T is a factor ring of the ordinary Witt ring $W(K)$ of K. It seems to be a major problem to figure out the right definition for a non-reduced Witt ring of higher level. Various suggestions have already been made by Kleinstein-Rosenberg [33], by Carlsson [22] and, in the case of rings, by

Craven [24], [25]. Based on 3.6 we propose a further definition for the Witt ring $W_n(K)$ of level n.

$$W_n(K) = Z[K^*/K^{*2n}]/(\bar{1} + \overline{(-1)}, \bar{a} + \bar{b} - \overline{a+b} - \overline{a^{2n}b + ab^{2n}})$$

where $a, b, a + b \in K^*$, $\bar{a} = aK^{*2n}$ etc.

Which definition is to be chosen is not yet clear. So far all of the proposals lead to infinite series of higher Witt rings, even to projective systems. What is the importance of these series for the field K? This is not yet know.

PROBLEM 5. Find the right definition for the Witt rings of higher level, relate them to the theory of forms of higher degree (if there is any relationship) and study the whole system of Witt rings and reduced Witt rings of the various levels.

4. Real closures. In this section we are concerned with the extensions of a given ordering P of level n to algebraic extensions L of the base field K. Define an ordering P_L of L to be a faithful extension of P if $s(P_L) = s(P) = n$ and $P_L \cap K = P$. In this case, we write $(L, P_L) \mid (K, P)$. A real closure $(R, \tilde{P})$ is by definition any maximal algebraic extension $(R, \tilde{P}) \mid (K, P)$. Real closures exist by Zorn's lemma. It is often advantageous to switch over to signatures. This is possible because of the following easily proved fact. Given a signature χ with $P^* = \ker \chi$ then for every faithful extension $(L, P_L) \mid (K, P)$ there is a signature χ_L of L such that $\chi_L(L^*) = \chi(K^*)$, $\ker \chi_L = P_L^*$ and $\chi_L|K^* = \chi_K$.

Now let $(R, \tilde{P})$ be a real closure of (K, P) and choose χ_P, $\tilde{\chi}$ with $\ker\chi_P = P^*$, $\ker \tilde{\chi} = \tilde{P}^*$. We are going to consider the valuation ring $A(\tilde{\chi})$ and its additive value group Γ. In [9] the next result has been established.

THEOREM 4.1.

i) *$A(\tilde{\chi})$ is a Henselian valuation ring with a residue field which is real closed in the ordinary sense of Artin-Schreier.*

ii) *If p denotes a prime then we have*:

$$\Gamma = p\Gamma \qquad \text{if } p \nmid n,$$
$$[\Gamma : p\Gamma] = p \qquad \text{if } p \mid n.$$

Note that in the case of an ordering ($n = 1$) the value group Γ is divisible. This implies that R^2 is the only ordering of an higher level m, $m \in \mathbf{N}$. In the case of $n > 1$ there is, on the other hand, a great variety of orderings and signatures of higher levels. Set $\mu(2m) = \{\zeta \in \mathbf{C} \mid \zeta^{2m} = 1\}$ and choose a signature χ_0 attached to an ordering (of level 1) of R. According to [14], signatures of a level $s|m$ can be constructed by $\chi = \chi_0 \circ (\eta \circ \upsilon)$ where η: $\Gamma \to \mu(2m)$ is any character and υ: $K^* \to \Gamma$ is the Krull valuation corresponding to $A(\tilde{\chi})$. In fact we even have

PROPOSITION 4.2.

i) *Every signature of R of level s, $s \in \mathbf{N}$, can be constructed as given above.*

ii) *If χ is any signature of R then $s(\chi) = \prod_{p|n} p^{\alpha_p}$, $\alpha_p \in \mathbf{N} \cup \{0\}$.*

iii) *R is real closed with respect to a signature χ (or* ker $\chi \cup \{0\}$*) if and only $s(\chi) = \prod_{p|n} p^{\alpha_p}$, where $\alpha_p > 0$ for all p.*

If $n > 1$ there are in fact infinitely many signatures of R as seen from 4.1, ii). Let us recall the situation we started from. A signature χ_p of K was chosen and then extended to the signature $\tilde{\chi}$ of R. Obviously, R is real closed with respect to $\tilde{\chi}$. But now many new signatures appear, including some for which R is also real closed. Therefore it is a basic task to organize this huge set of signatures. This can be done by appealing to the notion of a chain of signatures, first introduced by J. Harman **[28]**. After further investigations of this concept, especially by R. Brown **[19]**, **[20]**, it was N. Schwartz **[42]** who quite recently came up with a new, surprisingly easy and effective definition of a chain of signatures. We follow his approach.

Let F be any formally real field, set $\hat{\mathbf{Z}} = \lim_{\leftarrow n} \mathbf{Z}/n\mathbf{Z} = \prod \mathbf{Z}_p$, where $\mathbf{Z}_p$ is the ring of p-adic integers, and where $\hat{\mathbf{Z}}$, $\mathbf{Z}_p$ are considered with their natural compact topologies. By Schwartz's definition, a chain of signatures is a homomorphism $\kappa: K^* \to \mathbf{Z}^* \times \hat{\mathbf{Z}}$ such that the kernel of κ, denoted by T, satisfies: $a \notin -T$ implies $1 + a \in T \cup Ta$. (In the terminology of **[8]**, T is a fan.) A chain κ gives rise to a certain sequence of signatures in the following way. Set $\kappa(x) = (\kappa_0(x), \kappa_1(x))$ and let $\pi_t: \hat{\mathbf{Z}} \to \mu(t)$ be the homomorphism which is induced by $1 \mapsto \exp(2\pi i/t)$. Then define $\chi_t = \kappa_0 \cdot (\pi_t \circ \kappa_1)$, $t \in \mathbf{N}$. It turns out that $\{\chi_t\}_{t \in N}$ is a sequence of signatures which satisfies

$$(*) \qquad \chi_1^2 = 1,\ (\chi_{ts}\chi_1^{-1})^s = \chi_t\chi_1^{-1} \text{ for any } t, s \in \mathbf{N}.$$

Conversely, it can be proved that every sequence $\{\chi_t\}_{t \in \mathbf{N}}$ of signatures which satisfies $(*)$ is induced by a chain κ. Here, $\kappa_0 = \chi_1$ and $\kappa_1 = \lim_{\leftarrow} \chi_t\chi_1^{-1}$, after identifying $\lim_{\leftarrow} \mu(t)$ with $\hat{\mathbf{Z}}$ in the natural way.

Given a signature χ of level $s(\chi) = n$, we say that a chain κ passes through χ if $\chi = \chi_{2n}$ in the construction as above. In general, the image $\kappa(K^*)$, for κ a chain of signatures, is not closed in $\mathbf{Z}^* \times \hat{\mathbf{Z}}$. Since the closed subgroups of $\mathbf{Z}_p$ are known, we get for the closure,

$$\overline{\kappa(K^*)} = \mathbf{Z}^* \times \prod_p p^{\alpha_p} \mathbf{Z}_p, \qquad \alpha_p \in \mathbf{N} \cup \{0, \infty\}.$$

Here, we set $p^\infty \mathbf{Z}_p = 0$.

PROPOSITION 4.2. *Given any signature χ of F there is a chain of signatures κ passing through χ and satisfying*

$$\overline{\kappa(K^*)} = \mathbf{Z}^* \times \prod_{p \mid s(\chi)} \mathbf{Z}_p.$$

Two chains κ_1 and κ_2 of F are called equivalent if there is an automorphism α of $\mathbf{Z}^* \times \hat{\mathbf{Z}}$ with $\alpha \circ \kappa_1 = \kappa_2$. See [**42**, Th. 30] for a characterization of equivalent chains.

Recall that $(R, \tilde{P})$ is a real closure of (K, P) where P has level n. In this situation we have

THEOREM 4.3. *Up to equivalence, R admits a unique κ satisfying*

$$\overline{\kappa(R^*)} = \mathbf{Z}^* \times \prod_{p \mid n} \mathbf{Z}_p.$$

This theorem directly generalizes, and in fact implies, the result that a real closure in the sense of Artin-Schreier has R^2 as the only ordering of a higher level m, $m \in \mathbf{N}$. For if R is such a field then $\mathbf{R}^{*2}$ is divisible which implies $\kappa(\mathbf{R}^{*2}) = 1$. Then $\kappa(R^*) = \mathbf{Z}^*$, as claimed.

Given a real closure $(R, \tilde{P})$, $\tilde{P}^* = \ker \tilde{\chi}$, we pick a chain κ through $\tilde{\chi}$ which satisfies $\overline{\kappa(R^*)} = \mathbf{Z}^* \times \prod_{p \mid n} \mathbf{Z}_p$. By restricting it to K we obtain a signature κ_0 passing through χ and satisfying $\overline{\kappa_0(K^*)} = \mathbf{Z}^* \times \prod_{p \mid n} \mathbf{Z}_p$. Obviously, equivalent chains induce equivalent ones on K. The following theorem generalizes the result of Artin-Schreier that two ordinary real closures are isomorphic over K if and only if they induce the same ordering.

THEOREM 4.4. *Given (K, P) with $s(P) = n$, choose a signature χ with* $\ker \chi = P^*$. *Let $(R_i, \tilde{P}_i)$, $i = 1, 2$ be two real closures of (K, P). Then the following statements hold.*

i) *The conjugacy classes over K of real closures $(R, \tilde{P})$ over (K, P) correspond bijectively with the equivalence classes of chains κ of K passing through χ and satisfying $\overline{\kappa(K^*)} = \mathbf{Z}^* \times \prod_{p \mid n} \mathbf{Z}_p$,*

ii) $R_1 \simeq_\kappa R_2 \Leftrightarrow R_1^{2t} \cap K = R_2^{2t} \cap K$ *for all* $t \in \mathbf{N}$.

As a consequence of 4.4 one can show that in general there are infinitely many conjugacy classes of real closures, see [**6**, Ch. IV], [**20**] or [**42**] for details.

The results above mentioned suggest that one should deal with fields and arbitrary chains κ on them. In particular, a real closure $(R, \tilde{\kappa})$ of (K, κ) can be defined as well. The study of arbitrary pairs (K, κ), κ a chain, was begun in [**28**], then extended in [**19**], [**20**] by R. Brown and finally reformulated by N. Schwartz in [**42**].

5. The theory of generalized real closed fields. In this section we are concerned with fields R which admit an ordering P of level n such that (R, P) does not have any proper algebraic extension $(L, P_L) | (R, P)$. These fields are called generalized real closed fields. In the case of an ordering

$P(s(p) = 1)$ these are just the real closed fields of Artin and Schreier. From [9, Bem. 3.8 (iv)] we get the following characterization.

THEOREM 5.1. *The following statements are equivalent.*

i) *R is real closed with respect to an ordering of level n.*

ii) *R admits a Henselian valuation ring with a residue field which is real closed in the ordinary sense, and with a value group Γ satisfying $\Gamma = p\Gamma$ if $p \nmid n$ and $[\Gamma: p\Gamma] = p$ if $p|n$.*

In the situation of this theorem, one can show that the value group of every Henselian valuation ring with a real closed residue field satisfies the conditions as given above. This follows from the fact that for $p \nmid n$, $R^{*2} = R^{*2p}$ and for $p|n$ we have $[R^{*2p}: R^{*2p}] = p$.

N. Schwartz [42, Th. 18] and J. Harman (unpublished) have extended 5.1 to a characterization of fields which are real closed with respect to a chain of signatures; instead of only finite sets there occur arbitrary sets S of primes satisfying $\Gamma = p\Gamma$ for $p \notin S$, $[\Gamma: p\Gamma] = p$ for $p \in S$.

If R is real closed with respect to P, $s(P) = n$, there are many Henselian valuation rings with the property of 5.1, ii). The ring $A(P)$ is one of them but there is another distinguished one, as first shown by B. Jacob [30]. Set

$$O_1 = \{x \in R | x \notin R^{*n} \cup -R^{*n},\ 1 + x \in R^{2n}\},$$

$$O_2 = \{x \in R | x \in R^{*n} \cup -R^{*n},\ xO_1 \subset O_1\}, \qquad O = O_1 \cup O_2.$$

PROPOSITION 5.2. *$A(P)$ is the smallest and O is the largest valuation ring satisfying the conditions of* 5.1, ii).

As stated in the introduction, the model theory of ordinary real closed field has been developed mainly by Tarski and A. Robinson. B. Jacob [30] extended their results to generalized real closed fields. In fact, his unpublished results even cover a bigger class, e.g., the class of chain closed fields (R, κ), see [42].

For the sequel fix a field R real closed with respect to P, $s(P) = n$. Let L denote the ordinary field language. For each $p|n$ we adjoin $D_p(x)$, $T_p(x)$, and we adjoin the predicate $O(x) \leftrightarrow x \in O$ where O is the valuation ring as above. The definitions of $D_p(x), T_p(x)$ can be found in [30]. Denote the extended field language by L_n. The proof of the following theorem is given in [30].

THEOREM 5.3. *The theory $Th(R)$ is decidable and is model-complete in the language L_n.*

It is well known that the model theory of the ordinary real closed fields provides a proof of the Dubois—Risler Nullstellensatz. Following this approach, by appealing to 5.3, B. Jacob and the author have derived the Nullstellensatz over arbitrary generalized real closed fields. Details will appear in [17].

Let R be as above, and let $\mathbf{A}$ be an ideal in $R[X_1, \ldots, X_k]$. Set $V_R(\mathbf{A}) = \{x \in R^k | F(x)=0$ for all $F \in \mathbf{A}\}$ and $I_R V_R(\mathbf{A}) = \{F \in R[X_1, \ldots, X_k] | F=0$ on $V_R(\mathbf{A})\}$. In [17] it will be proved that there exists a certain semi-ring $S \subseteq R[X_1, \ldots, X_k]$ generated by $R[X_1, \ldots, X_k]^{2n}$, P and certain distinguished polynomials such that the following theorem holds.

THEOREM 5.4. $\mathrm{rad}_S(\mathbf{A}) = I_R V_R(\mathbf{A})$ *where* $\mathrm{rad}_S(\mathbf{A}) = \{f \mid f^{2m} + s \in \mathbf{A}$ *for some* $m \in \mathbf{N}$, $s \in S\}$.

The generalized real closed fields R represent examples of fields with a known absolute Galois group G_R. In fact we have [9]

$$G_R \simeq \mathbf{Z}/2\mathbf{Z} \times \prod_{p|n} \mathbf{Z}_p,$$

where the first factor operates on the second one by taking the inverse. The basic motivation for the author's work [6] was to extend the Galois-theoretical characterization of ordinary real closed fields, as given by Artin—Schreier, by looking for a larger class of formally real fields with a "simple" absolute Galois group. In this context, hereditarily pythagorean fields play an important role. By definition, these are formally real fields K such that every formally real algebraic field extension L is pythagorean, i.e., $L^2 + L^2 = L^2$. Quite recently, Engler, Jansen and Viswanathan have supplemented the result of [7]. Putting various results together we can state.

THEOREM 5.5. *Let K be a formally real field. Then the following statements hold.*

i) *If* $1 < \# G_K < \infty$, *then K is real closed and* $G_{K(i)} = 1$.

ii) $G_{K(i)}$ *Abelian* $\Leftrightarrow$ *K is hereditarily pythagorean.*

iii) *If* $G_{K(i)}$ *is not Abelian then the 2-Sylow-subgroup of* G_K *contains a free pro- 2-group on an infinite number of generators.*

Generalized real closed fields R are hereditarily pythagorean but they are in general not characterized by the statement $G_{R(i)} = \prod_{p|n} \mathbf{Z}_p$. Theorem 5.5 shows distinctive "jumps" in the set of profinite groups which can occur as the absolute Galois groups of formally real fields. Which profinite groups actually occur is not known. Their determination seems to be an important task and is a contribution to the more general problem of determining these profinite groups which are Galois groups of arbitrary fields. In accordance to this we pose the final problem.

PROBLEM 6. Determine the profinite groups which occur as the absolute Galois groups of formally real fields.

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