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A NOTE ON FIXED-POINT CONTINUED FRACTIONS AND AITKEN'S Δ^2-METHOD

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ABSTRACT. Limit periodic continued fractions can be accelerated, and, in some instances, analytically extended by the use of certain modifying factors. This procedure is actually Aitken's Δ^2-method when applied to equivalent continued fractions/power series. Both acceleration and continuation results are given.

The continued fraction

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots, \qquad \text{with complex } \{a_n\} \text{ and } \{b_n\}, \tag{1}$$

is called limit periodic if $a_n \to a$ and $b_n \to b$.

In accordance with the following procedure, (1) may be conceptualized as a composition of linear fractional transformations:

Let $t_n(w) = a_n/(b_n + w)$ for $n = 1, 2, \ldots,$ and set $T_1(w) = t_1(w)$, $T_n(w) = T_{n-1}(t_n(w))$ for $n = 2, 3, \ldots$. Then $a_1/b_1 + \cdots + a_n/(b_n + w) = T_n(w)$, and, in particular, the nth approximant of (1) equals $T_n(0)$.

It is usually the case that each t_n has two distinct fixed points, α_n and β_n. When $|\alpha_n| < |\beta_n|$, α_n is called the attractive fixed point and β_n, the repulsive fixed point of t_n. If we assume (1) is limit periodic, then ordinarily $t_n(w) \to t(w) = a/(b + z)$ with $\alpha_n \to \alpha$, $\beta_n \to \beta$. Each t_n can be written $t_n(w) = \alpha_n\beta_n/[(\alpha_n + \beta_n) - w]$, so that (1) can be recast in fixed-point form

$$\frac{\alpha_1\beta_1}{\alpha_1 + \beta_1} - \frac{\alpha_2\beta_2}{\alpha_2 + \beta_2} - \cdots \tag{2}$$

Over the last ten years several papers have appeared describing and investigating a simple modification of (1) that frequently accelerates convergence and may analytically continue the function represented by (1) (if $a_n = a_n(z)$, $b_n = b_n(z)$) into a larger domain. See, e.g., [**2**], [**3**], [**7**], [**8**]. The modification requires the use of $T_n(\mu_n)$ in lieu of $T_n(0)$. μ_n customarily takes on fixed point values α, β, α_{n+1} or β_{n+1}. Motivation for this technique comes from the study of infinite iterations of a single linear

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fractional transformation, and is somewhat geometrical in nature. See, e.g., [5].

The purpose of the present paper is to investigate an extension of these geometrically motivated ideas into the realm of power series, particularly those series that are equivalent (see (3), below) to certain limit periodic continued fractions. In pursuing this course of action we will

1) derive Aitken's Δ^2-method [4] for power series in an unusual fashion,
2) provide interesting accelerative inequalities for this method, and
3) illustrate the use of the method to analytically extend a function interpreted either as a continued fraction or as a power series. Thus, we will relate two simple and important modification procedures for continued fractions and power series.

We begin with a description of Euler's equivalent continued fraction [6].

If $\rho_n \neq 0$ for $n \geqq 1$, then

$$\begin{aligned} &1 + \rho_1 z + \rho_1\rho_2 z^2 + \cdots + \rho_1\rho_2\cdots\rho_n z^n \\ (3)\qquad &= \frac{1}{1} - \frac{\rho_1 z}{1+\rho_1 z} - \frac{\rho_2 z}{1+\rho_2 z} - \cdots - \frac{\rho_n z}{1+\rho_n z}, \quad \text{for } n \geqq 1. \end{aligned}$$

The continued fraction in (3) is limit periodic if $\rho_n \to \rho$. Henceforth, we will assume this to be the case.

The continued fraction in (3) involves (2), with $\rho_n z$ and 1 being the fixed points of t_n, $n \geqq 1$.

$$T_n(w) = 1 + \rho_1 z + \rho_1\rho_2 z^2 + \cdots + \rho_1\rho_2\cdots\rho_{n-1}z^{n-1} + \frac{\rho_1\rho_2\cdots\rho_n z^n}{1-w}.$$

If $|z| < 1/|\rho_n|$, then $\alpha_n = \rho_n z$ and $\beta_n \equiv 1$. If $|z| > 1/|\rho_n|$, then $\alpha_n \equiv 1$ and $\beta_n = \rho_n z$. It has been discovered, with reference to (2), that convergence is accelerated if one employs $T_n(\alpha)$ or $T_n(\alpha_{n+1})$ and the function represented by (2) (if $\alpha_n = \alpha_n(z)$, $\beta_n = \beta_n(z)$) may possibly be analytically continued if one uses $T_n(\beta)$ or $T_n(\beta_{n+1})$, [7], [8]. Thus, the modification $T_n(\rho_{n+1}z)$ may serve both purposes in (3). We find that

$$T_n(\rho_{n+1}z) = 1 + \rho_1 z + \rho_1\rho_2 z^2 + \cdots + \rho_1\rho_2\cdots\rho_{n-1}z^{n-1} + \frac{\rho_1\rho_2\cdots\rho_n z^n}{1-\rho_{n+1}z},$$

in terms of the series in (3). This is precisely the expression given by Aitken's Δ^2-method applied to $x_n(z) = 1 + \rho_1 z + \cdots + \rho_1\cdots\rho_n z^n$.

The following heuristic exposition, which treats (3) as a series, puts this situation in perspective. Paraphrasing Henrici's motivational discussion of Aitken's method [4], we define an iterative procedure, dependent upon an appropriately chosen function f, by setting $x_1 = f(x_0)$,

$\ldots, x_n = f(x_{n-1}) = f \circ f \circ \cdots \circ f(x_0) \equiv F_n(x_0)$, where $x_n \to s$. The assumption upon which the method is structured is

$$\frac{F_{n+1}(x_0) - s}{F_n(x_0) - s} \simeq \text{constant}. \tag{4}$$

One then infers the familiar form $x_n' = x_n - (x_{n+1} - x_n)^2/(x_{n+2} - 2x_{n+1} + x_n)$, which can be written

$$x_n' = x_n + (x_{n+1} - x_n)\Big/\left(1 - \frac{x_{n+2} - x_{n+1}}{x_{n+1} - x_n}\right). \tag{5}$$

Generally, x_n' is a much better approximation of s than x_n.

If $f(w) = 1 + \rho z w$, then $F_n(1) = 1 + \rho z + \rho^2 z^2 + \cdots + \rho^n z^n$. f has two fixed points, $\alpha = 1/(1 - \rho z)$ and $\beta = \infty$. It is easy to show that

$$F_n(w) - \alpha = K^n(w - \alpha), \text{ where } K = \rho z, \tag{6}$$

so that $F_{n+1}(\alpha) \equiv \alpha$ and $\lim_n F_n(w) = \alpha$ for $w \neq \beta$ if $|z| < 1/|\rho|$. We find that (4) is satisfied, since (6) implies $(F_{n+1}(w) - \alpha)/(F_n(w) - \alpha) = K$. Hence, $x_n = F_n(1) = 1 + \rho z + \cdots + \rho^n z^n$, so that $x_{n+\kappa} - x_{n+\kappa-1} = F_{n+\kappa}(1) - F_{n+\kappa-1}(1) = \rho^{n+\kappa} z^{n+\kappa}$, and (5) becomes

$$x_n' = 1 + \rho z + \cdots + \rho^n z^n + \frac{\rho^{n+1} z^{n+1}}{1 - \rho z} \equiv \frac{1}{1 - \rho z}.$$

Thus, $F_{n+1}(\alpha) = x_n' \equiv \alpha$. If $|z| > 1/|\rho|$, then $\lim x_n$ does not exist, but $F_{n+1}(\alpha)$ provides an analytic continuation of $1 + \rho z + \rho^2 z^2 + \cdots$.

In the more general limit periodic setting, we have $f_n(w) = 1 + \rho_n z w$, with fixed points $1/(1 - \rho_n z)$ and ∞. $x_{n+\kappa} - x_{n+\kappa-1} = \rho_1 \rho_2 \cdots \rho_{n+\kappa} z^{n+\kappa}$, and $x_n' = F_{n+1}(\alpha_{n+2}) = 1 + \rho_1 z + \cdots + \rho_1 \cdots \rho_n z^n + (\rho_1 \cdots \rho_{n+1} z^{n+1})/(1 - \rho_{n+2} z)$.

At this point one wonders whether more general periodic and limit periodic continued fractions are related to Aitken's method in this way. If f is a linear fractional transformation having finite fixed points α, $\beta \neq \alpha$, then the counterpart of (6) is

$$\frac{F_n(w) - \alpha}{F_n(w) - \beta} = K^n \frac{w - \alpha}{w - \beta},$$

where $|K| < 1$, if f is non-elliptic [1].

This equation allows the formulation of (4) as

$$\frac{F_{n+1}(w) - \alpha}{F_n(w) - \alpha} = \frac{f(F_n(w)) - \alpha}{F_n(w) - \alpha} = \frac{K(\alpha - \beta)}{F_n(w)(1 - K) + K\alpha - \beta} \simeq K,$$

for large n, since $F_n(w) \to \alpha$ if $w \neq \beta$. The periodic continued fraction $-[-a/b]^\infty$ is generated by setting $f(w) = a/(b - w)$. If one looks at $-[-2/3]^\infty$ one finds $\alpha = 1$, $\beta = 2$ and $F_n(\alpha) \equiv \alpha$ for $n \geqq 1$; If $x_1 = f(0)$ and $x_n = f(x_{n-1})$, $n \geqq 1$, we find, from (5), that $|x_2' - \alpha| \simeq .3$.

Consequently, Aitken's method is not identical to the fixed point modification method in more general, non-equivalent fraction structures, although the two may be related in a less obvious manner. It also appears that Aitkin's method is somewhat less effective than the fixed point approach for limit periodic fractions that are nearly periodic.

In the discussion that follows we will develop an interesting inequality supporting the acceleration of convergence of (3), one that is very similar to a result presented by Thron and Waadeland in connection with limit periodic continued fractions. Also, we will show how, under certain circumstances, the use of the repulsive fixed point $\beta_{n+1}(z) = \rho_{n+1}z$ analytically extends the domain of convergence of the continued fraction/power series in (3).

1. Acceleration of Convergence. The principal acceleration result in the study of limit periodic continued fractions is due to Thron and Waadeland. It is abbreviated and paraphrased here, since our concern is with the general form of the theorem in terms of fixed points.

THEOREM 2.1 [7]. *Suppose* $\alpha_n \to \alpha \neq 0$, $\beta_n = -1 - \alpha_n$ *in* (2), *where* $|\alpha_n| < |\beta_n|$, $|\alpha| < |\beta|$ *and* $|\alpha - \alpha_n|$ *is "small". Then*

$$|T_n(\alpha) - T| < G(\alpha) \max_{m \geqq n} |\alpha_n - \alpha| \cdot |T_n(0) - T|,$$

where

$$T = \lim_n T_n(0).$$

REMARK. $G(\alpha)$ can be quite large for values of α such that $\alpha(\alpha + 1)$ is near the ray $z = -1/4 - p, p \in [0, \infty)$.

The author considered the case $\alpha = 0$ in [2]. In abbreviated form, we have

THEOREM 3 [2]. *Suppose* $\alpha_n \to \alpha = 0$, $\beta_n = -1 - \alpha_n$ *in* (2). *where* $|\alpha_n| < |\beta_n|, |\alpha| < |\beta|$, *and* $\max_{m \geqq n}|\alpha_m - \alpha_{m+1}| \leqq |\alpha_{n+1}|$. *Then* $|T_n(\alpha_{n+1}) - T| < H \cdot \max_{m \geqq n} |\alpha_m| \cdot |T_n(0) - T|$.

Let us now turn our attention to (3) and develop a theorem similar to the two above, but in terms of $\max_{m \geqq n}|\alpha_m - \alpha_{m+1}|$ in both cases.

Clearly, $\lim_n T_n(\rho_{n+1}z) = \lim_n T_n(0) = T$ on $\Omega_1 = \{z: |z| \leqq r/|\rho| < 1/|\rho|\}$ if $\rho \neq 0$, or $\Omega_2 =$ a bounded compact subset of the z-plane, if $\rho = 0$. Let $\delta_n = \max_{m \geqq n}|\rho_{m+2} - \rho_{m+1}|$.

THEOREM 1. *If* (i) $\rho \neq 0$ *and* $\delta_n \leqq ((1 - r)^3 d - \sigma_1)/r(1 + r)$, *where* $\sigma_1 > 0$ *and* $|\rho_n| \geqq d$ *if* $n \geqq 1$, *or*

(ii) $\rho = 0$, $|\rho_n|(\downarrow)$ and $\delta_n \leqq (|\rho_n|(1 - r)^3 - \sigma_2)/(1 + r)$, *where* $\sigma_2 > 0$, *then*

$$|T_n(\rho_{n+1}z) - T| \leqq M(1 + r) \cdot \delta_n \cdot |T_n(0) - T|,$$

where $M = r/\sigma_1$ *if* $\rho \neq 0$ *or* $M = 1/\sigma_2$ *if* $\rho = 0$.

The inequality is valid on $\Lambda_n = \{z\colon |z| \leqq r/p_n\}$, where $p_n = \sup_{m\geqq n}|\rho_m|$.

PROOF. Consider the quotient

$$\frac{T - T_n(\rho_{n+1}z)}{T - T_n(0)} = \frac{(\rho_{n+1}z + \rho_{n+1}\rho_{n+2}z^2 + \cdots) - \rho_{n+1}z/(1-\rho_{n+1}z)}{\rho_{n+1}z + \rho_{n+1}\rho_{n+2}z^2 + \cdots}$$

$$(7) \qquad = \frac{G^{(n)}(z) - \rho_{n+1}/(1-\rho_{n+1}z)}{G^{(n)}(z)}, \qquad \text{where, as } n \to \infty,$$

$$G^{(n)}(z) = \rho_{n+1}(1 + \rho_{n+2}z + \rho_{n+2}\rho_{n+3}z^2 + \cdots) \to \rho/(1-\rho z) \text{ on } \Omega_1 \text{ (or } \Omega_2).$$

Let $\qquad G^{(n)}(z) = D_n(z) + \rho_{n+1}/(1 - \rho_{n+1}z).$

Now, $\qquad G^{(m)}(z) = \rho_{m+1}(1 + zG^{(m+1)}(z)),\ m \geqq n.$

Hence, $\qquad D_m(z) = \rho_{m+1}(1 + zG^{(m+1)}(z)) - \rho_{m+1}/(1 - \rho_{m+1}z)$

$$= \rho_{m+1}(z)\, D_{m+1}(z) + \rho_{m+1}z \cdot \frac{\rho_{m+2} - \rho_{m+1}}{(1-\rho_{m+1}z)(1-\rho_{m+2}z)},$$

so that

$$|D_m(z)| \leqq |\rho_{m+1}z| \cdot |D_{m+1}(z)| + |\rho_{m+1}z| \cdot |\rho_{m+2} - \rho_{m+1}|/(1-r)^2 \text{ on } \Lambda_n.$$

Assume, for the moment, that $|D_{m+1}(z)| \leqq R_n$ on Λ_n.

Case (i). For $\rho \neq 0$, $|D_m(z)| \leqq rR_n + r\delta_n/(1 - r)^2$. Thus, $|D_m(z)| \leqq R_n$ if $R_n = r\delta_n/(1 - r)^3$. Since $\lim_n D_n(z) = 0$ on Ω_1 there exists $\kappa > 0$ such that, for fixed m, n ($m \geqq n$), $|D_{m+\kappa}(z)| \leqq R_n$. It then follows that $|D_m(z)| \leqq R_n$ for $m \geqq n$ on Ω_1. From (7),

$$\left|\frac{T - T_n(\rho_{n+1})z)}{T - T_n(0)}\right| \leqq \left[\left|\frac{\rho_{n+1}/(1-\rho_{n+1}z)}{D_n(z)}\right| - 1\right]^{-1} \leqq \frac{r}{\sigma_1} \cdot \delta_n \cdot (1 + r).$$

Case (ii). For $\rho = 0$ and $|\rho_n|\,(\downarrow)$, $|D_m(z)| \leqq rR_n + |\rho_{n+1}|\delta_n/(|\rho_n| \cdot (1 - r)^2)$. Thus, $|D_m(z)| \leqq R_n$ if $R_n = |\rho_{n+1}|\, \delta_n/(|\rho_n| \cdot (1 - r)^3)$. The $\lim_n D_n(z) = 0$ on Ω_2 and, as before, $|D_m(z)| \leqq R_n$ for $m \geqq n$ on Ω_2. From (7), as in case (i),

$$\left|\frac{T - T_n(\rho_{n+1}z)}{T - T_n(0)}\right| \leqq \frac{1}{\sigma_2} \cdot \delta_n \cdot (1 + r).$$

This completes the proof of Theorem 1.

2. Analytic Continuation. The simple modification we have discussed can be proven to analytically extend the limit periodic series $T = 1 + \rho_1 z + \rho_1\rho_2 z^2 + \cdots$ provided $\{\rho_n\}$ converges very rapidly to $\rho \neq 0$. Thus the equivalent limit periodic fraction is cxtended.

THEOREM 2. *Suppose there exists a sequence* $\{\mu_n\}$ *with* $\mu_1 = 0$ *and* $|\mu_n| \leqq (\varepsilon/|\rho|)^n$ *for* $0 \leqq \varepsilon < |\rho|$ *and* $n \geqq 1$ *and such that* $\rho_n = \rho(1 + \mu_{n+1})/(1 + \mu_n) \neq 0$ *for* $n \geqq 1$. *Then* $T_n(\rho_{n+1}z)$ *analytically continues* T *beyond* $|z| < 1/|\rho|$ *into any simply connected domain* D *contained in* $\Omega = \{z\colon |z| \leqq R\} \cap \{z\colon |z - 1/\rho| \geqq r\}$, *where* $R < \varepsilon^{-1}$ *and* $0 < \delta < r < 1/|\rho|$, *where* $|1/\rho_n - 1/\rho| \leqq \delta$ *for* $n \geqq 1$.

PROOF. The hypotheses imply $\Pi_1^n(\rho_j/\rho) = 1 + \mu_{n+1}$, which implies $|\rho^n z^n(\Pi_1^n(\rho_j/\rho) - 1)| = |\rho z|^n \cdot |\mu_n| \leqq \varepsilon^n |z|^n$. This, in turn, proves $|(1 + \rho_1 z + \rho_1\rho_2 z^2 + \cdots) - (1 + \rho z + \rho^2 z^2 + \cdots)| \leqq \sum |\rho^n z^n(\Pi_1^n(\rho_j/\rho) - 1)| \leqq \sum \varepsilon^n |z|^n \leqq 1/(1 - \varepsilon\,|z|)$, $|z| < 1/\varepsilon$. Therefore

$$\left|T_n(\rho_{n+1}z) - \frac{1}{1-\rho z}\right| = \left|1 + \rho_1 z + \cdots + \frac{\Pi_1^n \rho_j z^n}{1-\rho_{n+1}z} - \left(1 + \rho z + \cdots + \frac{\rho^n z^n}{1-\rho z}\right)\right|$$
$$\leqq \sum_1^{n-1} (\varepsilon|z|)^j + \left|\frac{\Pi_1^n \rho_j z^n}{1-\rho_{n+1}z} - \frac{\rho^n z^n}{1-\rho z}\right|,$$

where

$$\left|\frac{\Pi_1^n \rho_j z^n}{1-\rho_{n+1}z} - \frac{\rho^n z^n}{1-\rho z}\right|$$
$$\leqq \frac{\left|\rho^n z^n\left(\Pi_1^n\left(\frac{\rho_j}{\rho}\right)-1\right)\right| + |\rho z|^n\left[\left|\rho z\left(\Pi_1^n\left(\frac{\rho_j}{\rho}\right)-1\right)\right| + |\rho z - \rho_{n+1}z|\right]}{|1-\rho z| \cdot |1-\rho_{n+1}z|}.$$

It follows, from the hypotheses, that $|T_n(\rho_{n+1}z) - 1/(1 - \rho z)| < M$ for $|z| \leqq R$ and $|z - 1/\rho| \geqq r > \delta > 0$. Consequently, $\{T_n(\rho_{n+1}z)\}$ is uniformly bounded on $D \subset \Omega$. Now, $\{T_n(\rho_{n+1}z)\}$ converges to an analytic function T in $\{z\colon |z| < 1/|\rho|\} \cap D$. Thus, Stieltjes' theorem **[9]** implies $T_n(\rho_{n+1}z) \to T$ in D.

REMARK. If we had interpreted (3) as a continued fraction, Lemma 2.1 **[8]** could have been used to show the boundedness of $\{T_n(\rho_{n+1}z)\}$ under the hypotheses of Theorem 2.

Other results given in **[8]** could have been brought into play here, also.

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