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DIFFERENTIAL EQUATIONS ON LIE GROUPS AND TORI THE WAVE EQUATIONS AND HUYGENS' PRINCIPLE

H. D. FEGAN

1. Introduction. The purpose of this note is to describe one way in which a differential equation on a compact Lie group can be transferred to another equation on the maximal torus. This procedure is then illustrated by applying it to the case of the wave equation. From this we obtain a well known result about the non-existence of Huygens' principle on a compact Lie group. More interestingly we obtain the existence of Huygens' principle for a shifted wave equation.

Let G be a compact Lie group. Let ρ be half the sum of the positive roots for the Lie algebra t of some maximal torus $T \subseteq G$. (See Section 2). Let $\|\rho\|$ denote the norm of ρ with respect to an appropriate norm on t^*. (The norm $\| \quad \|$ is given by the killing form if G is semisimple. Again we refer to Section 2 for details.) The result on the existence of Huygen's principle is as follows.

THEOREM 1.1. *Let G be odd dimensional. Then the shifted wave equation*

$$(\Delta + \|\rho\|^2)u = \partial^2 u/\partial t^2 \tag{1.1}$$

satisfies Huygens' principle.

We explain below precisely what Huygen's Principle involves.

Notice that since $\|\rho\|^2$ is a constant, see Section 2, the Laplacian is shifted by a constant for Huygens' principle to hold. This result is particularly interesting in the light of the results in [1] and [2]. In [1] the asymptotic expansion for the trace of the heat equation is given. If we take the heat equation as $\Delta u + \partial u/\partial t = 0$ the expansion is, for a suitable constant c.

$$Z(t) \approx ct^{-k/2}(1 + \|\rho\|^2 + \cdots) \text{ as } t \to 0. \tag{1.2}$$

On the other hand if we use the shifted Laplacian and take the heat equation as $(\Delta + \|\rho\|^2)u + \partial u/\partial t = 0$, the expansion becomes

$$Z(t) \approx ct^{-k/2} \text{ as } t \to 0. \tag{1.3}$$

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This is of course a much simpler expression.

In [2] we are concerned with $H_a(x, t)$, a special solution which gives rise to Macdonald's identities. If we use the heat equation $\Delta u + \partial u/\partial t = 0$ then $H_a(x, t)$ has an infinite product expansion. In particular we are interested in $H_a(1, t)$ where 1 is the identity element of G. When we use the shifted Laplacian it happens that

$$H_a(1, t) = \eta(t)^{k/2} \tag{1.4}$$

where η is the Dedekind eta function. Since $\eta(t)$ is a modular form this is more interesting than the previous product. Both these effects, simplifying the asymptotic expansion and obtaining a modular form, are likely to result from an effect involving the curvature of G.

The passage between an equation on the group and one on the maximal torus is given as Lemma 2.1. This is useful in studying differential equations on Lie groups, having been implicitly used in [2] for example. Stating the result explicitly, as in Lemma 2.1, shows more clearly the relationship between the group and its maximal torus.

The wave problem consists of the wave equation and some initial data. The wave equation is

$$\Delta u = \frac{\partial^2 u}{\partial t^2} \tag{1.5}$$

and the initial data has two parts

$$u(x, 0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x, 0) = g(x). \tag{1.6}$$

Standard techniques involving Fourier transforms show that there are two distributions P_1 and P_2 on $G \times R$ such that

$$u = P_1 * f + P_2 * g. \tag{1.7}$$

Here $*$ represents convolution on the space G. The distributions P_1 and P_2 are called propagators. These are solutions of the wave equation with initial data

$$P_1(x, 0) = \delta_{x=0}, \qquad \frac{\partial P_1}{\partial t}(x, 0) = 0 \tag{1.8}$$

and

$$P_2(x, 0) = 0, \qquad \frac{\partial P_2}{\partial t}(x, 0) = \delta_{x=0} \tag{1.9}$$

where x is the variable on G and $\delta_{x=0}$ is the delta distribution concentrated at the identity element of G. Since the wave equation (1.5) is invariant under both right and left translations by elements of G, and since the initial data (1.8) and (1.9) is invariant under conjugation by G, the propagators P_1 and P_2 will be invariant under conjugation by G. In particular, they will be determined by the solution of the wave equation for conjugation-invariant initial data. Similar remarks apply to the shifted wave equation (1.1).

According to general principles, see [3], the distributions P_1 and P_2 (for either of equations (1.1) or (1.5)) have their singular support on the light cone in $G \times \mathbf{R}$. To define the light cone we introduce a multiple valued function, ℓ, on G. If $x \in G$ then $\ell(x)$ is the set of lengths of geodesics joining the identity element of G to x. This is multiple valued since there are closed geodesics on G. However, it is discretely valued. The light cone is then

$$((x, t) \colon t \in \ell(x)). \tag{1.10}$$

Huygens' principle is that the propagators are supported entirely on the light cone. This condition may be formulated in terms of individual solutions $u(x, t)$ as follows:

$$\operatorname{supp} u(x, t) \subseteq \{y \colon (y^{-1} \operatorname{supp} u(x, 0), t) \cap \ell(x) \neq \phi\} \tag{1.11}$$

where supp f denotes the support of f. The reader should see [4] for more details and the relationship with the Heisenberg group.

The author would like to thank all the many mathematicians with whom he has talked about this material, and the referee for his helpful suggestions. They are far too many to thank individually.

2. Passage between the torus and the group. The aim of this section is to establish a technical lemma. Before we state this lemma we introduce our notation, see [2]. Let G be a compact connected Lie group with simply connected commutator subgroup. Let T be a maximal torus in G. Let g and t be the Lie algebras of G and T. Let $B(,)$ be a positive definite conjugation invariant bilinear form on g such that the restriction of B to the commutator subalgebra of g is the negative Killing form. Let Δ_G be the Laplacian on G associated to the bi-invariant metric defined by B, and let Δ_T be the analogous Laplacian on T. We shall choose the signs of Δ_G and Δ_T so that they are positive operators. Choose a set of positive roots of $\alpha \colon t \to \mathbf{R}$ for the adjoint action of t on g. For $x \in T$ define

$$j(x) = \prod_{\alpha} (2i \sin \pi\alpha(x)). \tag{2.1}$$

This function j is the denominator of the Weyl character formula. Let $\rho = 1/2 \sum \alpha$ be half the sum of the positive roots. Let $\|\rho\|$ be the norm of ρ with respect to the inner product defined on t^*, the dual of t, by B.

Let P_t be a differential operator in the variable t and let $Q(z)$ be a polynomial in the indeterminate z. We can now give our technical lemma.

LEMMA 2.1. *Let $u(g, t)$, $g \in G$, $t \in \mathbf{R}$, be a central function on G for each t and for $x \in T$ let $v(x, t) = j(x)u(x, t)$. If v satisfies $Q(\Delta_T)v = P_t v$ then u satisfies $Q(\Delta_G + \|\rho\|^2)u = P_t u$. Equivalently, if u satisfies $Q(\Delta_G)u = P_t u$ then v satisfies $Q(\Delta_T - \|\rho\|^2)v = P_t v$.*

PROOF. This follows immediately by a simple calculation using the result

$$j(x)(\Delta_G f)(x) = (\Delta_T - \|\rho\|^2)(j(f|T)(x). \tag{2.2}$$

This result is due to Harish-Chandra and is given in **[4]**. Notice that the sign of $\|\rho\|^2$ is different from that in **[4]**. This is due to the choice of sign for the Laplacians so that they have positive eigenvalues.

The formula (2.2) was used in **[2]**. However, in the case of **[2]** we were concerned with the heat equation

$$\Delta_G u - \frac{1}{2\pi i}\frac{\partial u}{\partial t} = 0 \tag{2.3}$$

with complex valued t such that $Imt > 0$. In this case the shift in the Laplacian by $\|\rho\|^2$, when we passed to the torus, was moved to give a change in the variable t. The change appearing as multiplication of the solution by the factor $\exp(-2\pi\|\rho\|^2 t)$. In this paper our main application is to the wave equation and it is not as useful to move the shift to the time variable.

3. Huygens' principle on a compact Lie group. In this section we apply the technical result of section two to the wave equation. This gives clearly and easily both the non-existence of Huygens' principle for the wave equation, and the existence of Huygens' principle for the shifted Laplacian in the case of an odd number of space dimensions. The author finds this second result interesting since the same shift occurs in studying the heat equation, see **[1]** and **[2]**.

We start with the non existence of Huygens' principle for the standard wave equation. Consider the case of the wave equation on $\mathbf{R}'$ shifted by C^2. Thus the equation is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + C^2 u. \tag{3.1}$$

Notice that with our sign convention $\Delta u = -\partial^2 u/\partial x^2$. Take as initial data $u(x, 0) = 0$ and $\partial u/\partial t(x, 0) = \delta_{x=0}$, that is the solution u is the propagator P_2. Using the Riemann-Volterra method, see **[6]**, the solution for $t > 0$ is

$$u(x, t) = \begin{cases} \frac{1}{2} J_0{}^{(C\sqrt{x^2-t^2})} & |x| < t \\ 0 & |x| > t \end{cases} \tag{3.2}$$

Along the characteristics, that is $|x| = t$, the initial singularity is propagated. In (3.2) the function J_0 is a Bessel function and so u has support that is not on the wavefront. Thus the shifted wave equation does not satisfy Huygens' principle. Generalizing this leads to the following negative result.

PROPOSITION 3.1. *Let $u(x, t)$, $x \in G$ and $t \in \mathbf{R}$, satisfy $\Delta u + \partial^2 u/\partial t^2 = 0$ then u does not satisfy Huygens' principle.*

PROOF. Let $u(x, t)$ be conjugation invariant in x, and define v on T by $v(x, t) = j(x)u(x, t)$ as in Lemma 2.1. Then v satisfies $\Delta_T v - \|\rho\|^2 v + \partial^2 v/\partial t^2 = 0$ on the maximal torus T. As is well known, and illustrated by the above example, v does not satisfy Huygens' principle and so neither does u.

REMARK. This result with essentially the same proof can be found in **[4]**. There it is given for the same purpose as here, i.e., to illustrate the use of passing between the group and the torus. Since the torus is a flat quotient of $\mathbf{R}^n$ analysis on it is basically the same as analysis on $\mathbf{R}^n$.

The following result is proved in a similar way.

THEOREM 3.2. *Let G be an odd dimensional compact Lie group then Huygens' principle holds for the shifted wave equation.*

PROOF. Again let $u(x, t)$ be conjugation invariant in X and for $x \in T$ set $v(x, t) = j(x)u(x, t)$. Then, if u satisfies equation (1.1), v satisfies

$$\Delta_T v + \partial^2/\partial t^2 = 0. \tag{3.3}$$

Now if μ is the number of positive roots of G we have

$$\dim G = \dim T + 2\mu.$$

Thus $\dim T$ is odd and Δ_T satisfies Huygen's principle. Hence the individual solutions $v(x, t)$ satisfy the version of Huygen's principle (relative to T) given at the end of Section 1. Therefore so does $u(x, t)$. Letting the initial data $u(x, 0)$ and $\partial u(x, 0)/\partial t$ approach the initial data (1.8) or (1.9) for the propagators, we find Huygen's principle holds for equation (1.1).

As we noted in the introduction this result is interesting in view of the results in **[1]** and **[2]**. In the case of the heat equation when the Laplacian is shifted by $\|\rho\|^2$ the fundamental solution has a trivial asymptotic expansion. In the case of the special Macdonald's identities this shift give rise to a modular form.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF NEW MEXICO, ALBUQUERQUE, NM 87131