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ON THE IMPRIMITIVITY THEOREM FOR ALGEBRAIC GROUPS

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Let G be an affine algebraic group, defined over the algebraically closed field k, and let A be a commutative k-algebra which is a rational G-module such that G acts on A as k-algebra automorphisms. An $A.G$-module M is an A module and a rational G-module such that $g(am) = g(a)g(m)$ for $g \in G$, $a \in A$ and $m \in M$, and a morphism of $A.G$-modules is a G-linear A-homomorphism. The $A.G$-modules, and their morphisms, form an abelian category which we denote $\mathrm{Mod}(A.G)$.

In [7, Theorem 3.1, p. 42] Parshall and Scott prove that if H is an affine algebraic subgroup of G such that the homogeneous space G/H is affine, then $\mathrm{Mod}(k[G/H]. G)$ is equivalent to $\mathrm{Mod}(H)$, the category of rational H-modules. The proof uses their earlier theorem [2, Theorem 4.3, p. 9] that G/H being affine implies that the induction functor from $\mathrm{Mod}(H)$ to $\mathrm{Mod}(G)$ is exact. See also [9].

The point of the present note is to observe that a slightly more general version of the above category equivalence can be derived directly from the fundamental (and easily proven) algebraic fact that if A is a simple $A.G$-module, then $A.G$-modules are all A-flat, due to I. Dorai swamy [3, Cor. 2.3, p. 792]. In the version presented here, it is the inverse of the induction functor that is easier to consider. The theorem is preceded by some standard observations on Hopf algebras and followed by some applications. The notation already introduced is retained throughout.

To define the functor, we assume there is a k-algebra homomorphism $\alpha: A \to k$. Let Y be the affine scheme represented by A, on which the group scheme G represented by $k[G]$ acts. Then the functor which assigns to each commutative k-algebra C the stabilizer in $G(C)$ of the α of $Y(C)$ is also an affine group scheme: the fibre product $Gx_Y\{e\}$ where the right map $\{e\} \to Y$ is $e \to \alpha$ and the left map $G \to Y$ is $g \to g\alpha$. It follows that the algebra $B = k[G] \otimes_Y k$ representing $Gx_y\{e\}$ is a Hopf algebra and that $k[G] \to B$ is a Hopf algebra morphism whose kernel J is a Hopf ideal. The range of the functor will be the category of B comodules.

We need to recall how G-modules can be regarded as $k[G]$-comodules. If M is a rational G-module, the map γ_M: $M \to M \otimes k[G]$, defined by

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$\gamma(m) = \sum m_i \otimes t_i$ if $gm = \sum f_i(g)m_i$ for all g in G, makes M a $k[G]$-comodule in the sense of [8, p. 30]. It is easy to check that if M is an $A.G$-module, then $\gamma(am) = \gamma_A(a)\gamma(m)$ for all $a \in A$. Also, we will need the map $\beta: A \to k[G]$ defined by $\beta = (\alpha \otimes 1)\gamma_A$. Finally, for an $A.G$-module M we have the tensor identity

$$(*) \qquad \begin{aligned} &(M \otimes_k k[G] \otimes_A k = (M \otimes_A k) \otimes_k (k[G] \otimes_A k) \\ &(m \otimes f \otimes y) \to (m \otimes 1) \otimes (f \otimes y) \end{aligned}$$

(Here $M \otimes k[G]$ is an A-module via γ_A and $k[G]$ an A-module via β.) To verify the identity, we observe that if $\gamma_A(a) = \sum a_i \otimes h_i$, then for $m \in M$ and $f \in k[G]$, $\sum(a_i m \otimes_A 1) \otimes_k (h_i f \otimes_A 1) = (m \otimes 1) \otimes (\sum \alpha(a_i)h_i) f \otimes 1 = (m \otimes 1) \otimes (\beta(a) f \otimes 1) = (m \otimes 1) \otimes (f \otimes \alpha)(a)$, which justifies the definition of the map in $(*)$.

Now, if M is an $A.G$-module, the map $\bar{\gamma}_M: M \otimes_A k \to (M \otimes_A k) \otimes_k B$ given by following $\gamma_M \otimes_A k$ with the identity $(*)$ is easily seen to give $M \otimes_A k$ the structure of a B-comodule, using the commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & M \otimes k[G] \\ \downarrow & & \downarrow \\ M \otimes_A k & \to & (M \otimes_A k) \otimes_k B. \end{array}$$

The same diagram also is used to see that if $f: M \to N$ is a morphism of $A.G$-modules, then $f \otimes_A k$ is a morphism of B-comodules. Thus $\otimes_A k$ provides a functor from the category of $A.G$-modules to the category of B-comodules.

In the proof of the theorem we will need to use some facts about Hopf algebras and comodules. We will state these for the Hopf algebra B with comultiplication Δ and counit ε, and a B-comodule M with coaction γ, but they are completely general and apply, for example, to $k[G]$ and its comodules:

First, the k-space $\mathrm{Comod}_B(M, B)$ of B-comodule morphisms from M to $B(\gamma_B = \Delta)$ is A-isomorphic to the k-linear dual M^* of M via the maps $T \to \varepsilon T$ and $f \to (f \otimes 1)\gamma$. This is actually a natural equivalence of functors $\mathrm{Comod}_B(\_\_, B) \to (\_\_)^*$. Second, if M_t denotes the underlying k-space of M with the trivial B coaction ${}_tM \to {}_tM \otimes B$ by $m \to m \otimes 1$, then $\gamma: M \to M_t \otimes B$ is a morphism of B-comodules. (${}_tM \otimes B$ is a comodule with coaction $1 \otimes \Delta$.) Since γ is a monomorphism, this shows that M is a subcomodule of a direct sum of copies of B, as B-comodule.

We can now state and prove the main result.

THEOREM. *Let G be an affine algebraic group over k, A an affine k-algebra and rational G-module with G acting as algebra automorphisms, and suppose A has no non-trivial G-stable ideals. Let $\alpha: A \to k$ be a k-algebra homomorphism and let B denote the Hopf algebra $K[G] \otimes_A k$. Then the functor*

$\_\_ \otimes_A k$ is an equivalence between the category of $A.G$-modules and the category of B-comdules.

PROOF. We establish first that $\_\_\otimes_A k$ is exact by showing that every $A.G$-module is A-flat. When G is connected this follows from [**3**, Cor. 2.3, p. 792]. In general, we let $P_1, \ldots, P_k$ denote the distinct minimal primes of A, observe that $P_1 \cap \cdots \cap P_k$ is a G-stable ideal, hence zero, so that A is reduced. The set X of primes Q in Spec(A) where A_Q is regular is open [**6**, Theorem 74, p. 248] and non-empty, since for example A_Q is a field for $Q = P_1$. Then $Y = \mathrm{Spec}(A) - X$ is closed and G-stable and hence corresponds to a G-stable radical ideal, which must be zero. Thus Y is empty, and it follows from [**5**, Theorem 168, p. 119] that $A = A_1 \times \cdots \times A_k$ where each A_i is a (regular) domain. Let e_i be the minimal idempotent of A corresponding to A_i, and let $H_i \subseteq G$ be the stabilizer of e_i. Then H_i acts rationally on A_i, and if I were a non-trivial H_i-stable ideal of A_i then $\sum g_{ij} I$ (where $\{g_{ij}\}$ is a set of left coset representatives of H_i in G_i) would be a non-trivial G-stable ideal of A. So each A_i has no non-trivial H_i-stable ideals. An $A.G$-module M then is a product $M = M_1 \times \cdots \times M_k$ ($M_i = e_i M$) where each M_i is an $H_i A_i$-module. So by [**2**, Prop. 2.2, p. 790] finitely generated $A.G$-modules are A-projective and all $A.G$-modules are A-flat.

Next, we show that every B-comodule is of the form $M \otimes_A k$ for a suitable $A.G$-module M. Applying the first of the two remarks immediately preceding the theorem to the $k[G]$-comodule $k[G]$ and the B-comodule B, we have the following chain of identities: $\mathrm{Hom}_{A.G}(k[G], k[G]) = \mathrm{Hom}_A(k[G], k) = \mathrm{Hom}_k(B, k) = \mathrm{Comod}_B(B, B)$, the map from the firxt to the last sending T to $T \otimes_A k$. Now let V be a B-comodule. From the second of the two remarks, we know there is an exact sequence of B-comodules $0 \to V \to B^{(\lambda)} \to B^{(\mu)}$ where $B^{(\lambda)} = {}_tV \otimes B$ and $B^{(\mu)} = {}_t(({}_tV \otimes B)/\gamma(V)) \otimes B$ are direct sums of copies of B as B-comodules. We can find an $A.G$-morphism T: $k[G]^{(\lambda)} \to k[G]^{(\mu)}$ such that $T \otimes_A k$: $B^{(\lambda)} \to B^{(\mu)}$ is the map of this exact sequence. If $M = \mathrm{Ker}(T)$, then by exactness we have $M \otimes_A k = V$.

The preceding argument also shows that the functor is full, i.e., if W is another B-comodule and f: $V \to W$ is a comodule morphism, then if we construct the exact sequence $0 \to W \to B^{(\alpha)} \to B^{(\beta)}$ analogous to that for V we have maps f_1: $B^{(\lambda)} \to B^{(\alpha)}$ and f_2: $B^{(\mu)} \to B^{(\alpha)}$ extending f. (For example, f_1: ${}_tV \otimes B \to {}_tW \otimes B$ is given by $f_1 = f \otimes 1$.) Let S: $k[G]^{(\alpha)} \to k[G]^{(\beta)}$ be the $A.G$-morphism with $N = \mathrm{Ker}(S)$ such that $N \otimes_A k = W$, and choose $A.G$-morphisms F_1: $k[G]^{(\lambda)} \to k[G]^{(\alpha)}$ and F_2: $k[G]^{(\mu)} \to k[G]^{(\beta)}$ such that $F_i \otimes_A k = f_i$. Then $(F_2 T \otimes_A k = (SF_1) \otimes_A k$, and since $\mathrm{End}_{A.G}(k[G]) = \mathrm{Comod}_B(B, B)$, we conclude that $F_2 T = SF_1$. Hence there is an $A.G$-module morphism F: $M \to N$ such that $F \otimes_A k =$

f (namely, $F = F_1|M$). This proves that the functor $\_\otimes_A k$ is full.

To complete the proof of the theorem, we must show that the functor is faithful; that is, if $f, g: M \to N$ are $A.G$-morphisms with $f \otimes_A k = g \otimes_A k$, then $f = g$. We can replace f by $f - g$ and assume $f \otimes_A k = 0$, which means, by exactness, that $(M/\mathrm{Ker}(f))$, $\otimes_A k = 0$. If V is a finitely generated $A.G$-submodule of $M/\mathrm{Ker}(f)$ then V is A-projective (see above) and $V \otimes_A k = 0$ also. The set of primes P in Spec(A) where $V_P = 0$ is open and G-stable, so either all of Spec(A) or empty. If $Q = \mathrm{Ker}(\alpha)$, then $V_Q = 0$ since $V \otimes_A k = 0$ (using that V is A-projective) so $V_P = 0$ for all P and hence $V = 0$. Thus $M/\mathrm{Ker}(f)$, being a union of its finitely generated $A.G$-submodules, is zero, so $f = 0$.

We have now shown that $\_\otimes_A k$ is exact, full, and faithful, and that every B comodule is isomorphic to one of the form $V \otimes_A k$. It follows that $\_\otimes_A k$ is an equivalence of categories by [**1**, (1.2), p. 49], completing the proof.

A major application of the theorem is to the case where $A = k[G]^H$ where H is an affine algebraic subgroup of G such that the quotient G/H is an affine variety. In this case A has no non-trivial G-stable ideals, since the radical of such an ideal, which is necessarily G-stable also, corresponds to a G-stable subvariety of G/H, of which there are no proper such. In this case the inverse equivalence to $\_\otimes_A k$ is the induction functor of [**2**, p. 1–14]. This functor has several descriptions; the one adopted here [**2**, p. 3] is chosen for its convenience in the proof of the preceding assertion. Let X be a rational H-module, and define a right H-module structure on $k[G] \otimes X$ by $(f \otimes v)^h = f \cdot h \otimes h^{-1}v$. Then $(k[G] \otimes X)^H$ is a rational G-module, denoted $X|_H^G$, and called the G module induced from H. It has the property that, for all G-modules, Y, $\mathrm{Hom}_G(Y, X|_H^G) = \mathrm{Hom}_H(Y, X)$ [**2**, Prop. 1.4, p. 9]. It is, moreover, clear that, as a functor, $(\ \ )|_H^G$ is left exact, preserves arbitrary direct sums, and carries H-modules to $A.G$-modules. From these facts we will deduce, via the theorem, that $(\ \ )|_H^G$ is exact, a result first obtained in [**2**, Thm. 4.3, p. 9].

COROLLARY 1. *Let G be an affine algebraic group over k, H an affine algebraic subgroup and assume the quotient G/H is an affine variety. Then the induction functor from rational H-modules to rational G-modules is exact.*

PROOF. We let $A = k[G]^H = k[G/H]$ as above. Since $Gx_{(G/H)}\{e\} = H$, we have, in the notation on theorem, that $B = k[G] \otimes_A k = k[H]$. Also, we have $k[H]|_H^G = k[G]$; for $k[G] \otimes k[H] = k[G \times H]$, and the above right H-action becomes $f^h(g, x) = f(hg, xh^{-1})$, so $k[G \times H]^H = k[G]$. If V is an H-module, it was shown in the proof of the theorem that there exists an exact sequence $0 \to V \to k[H]^{(\lambda)} \to k[H]^{(\mu)}$ of H-modules. Applying the induction functor, which is left exact and preserves direct sums, we obtain an exact sequence $0 \to V|_H^G \to k[G]^{(\lambda)} \to k[G]^{(\mu)}$. Now applying $\_\otimes_A k$ as in the proof of the theorem we obtain $(V|_H^G) \otimes_A k = V$. It is now easy to see that induction preserves epimorphisms, so is exact; if $V \to W$ is an H-module surjection and C the cokernel of $V|_H^G \to W|_H^G$, then $C \otimes_A k = 0$. By faithfulness $C = 0$ and the induced map is a surjection.

As a second application of the theorem, we consider infinitesimal subgroups in positive characteristic [**4**, (1.4), p. 271]. Assume that k has characteristic $p > 0$ and let σ: $k[G] \to k[G]$ be $\sigma(f) = f^p$. For $n = 1, 2, 3, \ldots$ let $A_n = \sigma^n(k[G])$. Then A_n is a G-stable subalgebra of $k[G]$ and has no non-trivial G-stable ideals. Let α: $A_n \to k$ be given by evaluation at e, and consider $B_n = k[G] \otimes_{A_n} k$; if $I = \{f \in k[G] \mid f(e) = 0\}$, then $\mathrm{Ker}(\alpha) = \sigma^n(I)$ so that $B_n = k[G]/\sigma^n(I)k[G]$. This is the Hopf algebra of the infinitesimal group scheme G_n which is the kernel of the n-th-power of the Frobenius on G [**4**, (1.4), p. 271]. Rational G_n-modules are, by definition, B_n-comodules. Hence we conclude with the following corollary.

COROLLARY 2. *Let G be an affine algebraic group over k, and assume k has positive characteristic p. Let A_n be the subalgebra of $k[G]$ consisting of all p^n-powers and let G_n be the finite group scheme which is the kernel of the n-th-power of the Frobenius on G. Then $\_\otimes_{A_n} k$ is an equivalence between the category of $A_n.G$-modules and the category of rational G_n-modules.*

If $n \geqq m$, we have a surjection $B_n \to B_m$, and if V is a B_n-comodule with coaction $V \to V \otimes B_n$, composition with this surjection gives a coaction $V \to V \otimes B_m$ making V a B_m-comodule. This operation is a functor from G_n-modules to G_m-modules, called restriction. It is interesting to interpret this functor in the light of corollary 2; here we map $A_n.G$-modules to $A_m.G$-modules, the operation sending the $A_n.G$-module M to $A_m \otimes_{A_n} M$, which is an $A_m.G$-module. (To see that this does coincide with the above restriction functor, we observe that $(A_m \otimes_{A_n} M) \otimes_{A_m} k = M \otimes_{A_m} k$.) Similarly, we can restrict from G-modules to G_n-modules. Here the operation sends M to $A_n \otimes_k M$.

Finally, we note that the converse of the theorem is also valid. That is, if for every choice of α the functor $\_\otimes_A k$ is an equivalence, then A has no non-trival G-stable ideals. For if I were a non-zero G-stable ideal, then choose α so that $\mathrm{Ker}(\alpha)$ contains I. The sequence $0 \to I \to A \to A/I \to 0$ of G-modules must remain exact after tensoring with k via α, so $I \otimes_A k = 0$, and hence $I = 0$ since $\_\otimes_A k$ is an equivalence.

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