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A NOTE ON ORDER CONVERGENCE IN COMPLETE LATTICES

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ABSTRACT. Order convergence is studied in a complete lattice L_1 which is the image of another complete lattice L under a complete homomorphism. The goal is to relate order convergence in L to that in L_1. For instance, we show that order convergence in L_1 is pretopological if it is pretopological in L, while topological order convergence is in general not preserved under complete images. We conclude with some applications and examples.

1. Complete homomorphisms and order convergence. Throughout this note, let L and L_1 be complete lattices and suppose that $\varphi: L \to L_1$ is a surjective, complete homomorphism (i.e., $\varphi(L) = L_1$ and φ preserves arbitrary sups and infs). The lower adjoint $\varphi_*: L_1 \to L$ and the upper adjoint $\varphi^*: L_1 \to L$ of φ are given by

$$\varphi_*(y) = \inf\{x \in L: \varphi(x) = y\},$$
$$\varphi^*(y) = \sup\{x \in L: \varphi(x) = y\}$$

$(y \in L_1)$.

The following facts are well-known (see, e.g., [3] or [6] for a general theory of adjoint pairs):

$$\text{(i) } \varphi_*\varphi(x) \leqq x \text{ and } \varphi^*\varphi(x) \geqq x, \qquad x \in L,$$
$$\text{(ii) } \varphi\varphi_*(y) = y \text{ and } \varphi\varphi^*(y) = y, \qquad y \in L_1,$$
$$\text{(iii) } \varphi_* \text{ preserves sups, and } \varphi^* \text{ preserves infs.}$$

We set $[a) = \{x \in L: a \leqq x\}$, $(b] = \{x \in L: x \leqq b\}$, and $[a, b] = [a) \cap (b]$, for all $a, b \in L$. Sets of this form are called intervals. For $X \subseteq L$, $\downarrow X = \bigcup\{(x]: x \in L\}$ denotes the lower set generated by X.

A (set-theoretical) filter $\mathfrak{F}$ on L order converges to a point $x \in L$, written $\mathfrak{F} \to_L x$, if

$$x = \inf\{\sup F: F \in \mathfrak{F}\} = \sup\{\inf F: F \in \mathfrak{F}\}.$$

By the order convergence $\mathbf{O}(L)$ of L we mean the set of all pairs $(\mathfrak{F}, x)$

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with $\mathfrak{F} \to_L x$. In the order topology on L, closed sets are precisely those which contain all of their order-convergent limit points. It may also be characterized as the finest topology $\mathfrak{T}$ on L such that order convergence implies convergence with respect to $\mathfrak{T}$. If not stated otherwise, all topological notions refer to the order topology. Note that a set $U \subseteq L$ is open iff $x \in U$ and $\mathfrak{F} \to_L x$ implies $U \in \mathfrak{F}$. Another characterization of open sets has been given in **[5]**.

PROPOSITION 1. *A subset U of L is open if and only if for any up-directed set Y and any down-directed set Z with* $\sup Y = \inf Z \in U$, *there are elements $a \in Y$ and $b \in Z$ such that the interval $[a, b]$ is contained in U.*

Passing to complements, we see that any closed set is closed under up-directed sups and down-directed infs. For each $x \in L$, $\mathfrak{V}(x)$ denotes the intersection of all filters order converging to x; we refer to $\mathfrak{V}(x)$ as the pre-neighborhood filter of x. The neighborhood filter of x is denoted by $\mathfrak{U}(x)$. The order convergence of L is said to be pretopological (resp., topological) if each of the filters $\mathfrak{V}(x)$ (resp., $\mathfrak{U}(x)$) order converges to x. It is easy to see that order convergence is topological iff it agrees with convergence in the order topology, and in this case $\mathfrak{V}(x) = \mathfrak{U}(x)$ for all $x \in L$. Conversely, if this equality holds and order convergence is pretopological then it is already topological. References **[4, 5, 9, 10, 13]** give further background information on order convergence in lattices via filters, including various conditions under which order convergence is pretopological or topological.

For $x \in L$, let $I(x)$ denote the intersection of all ideals $I \subseteq L$ with $\sup I = x$, and $D(x)$ the intersection of all dual ideals $D \subseteq L$ with $\inf D = x$. The following results have been shown in **[9]**.

PROPOSITION 2. *The order convergence of L is pretopological if and only if* $x = \sup I(x) = \inf D(x)$, *for all $x \in L$. In this case, the intervals $[a, b]$ with $a \in I(x)$ and $b \in D(x)$ form a base for the pre-neighborhood filter $\mathfrak{V}(x)$.*

Hence $\mathbf{O}(L)$ is topological iff it is pretopological and the interior of the intervals $[a, b]$ with $a \in I(x)$ and $b \in D(x)$ always contains the point x.

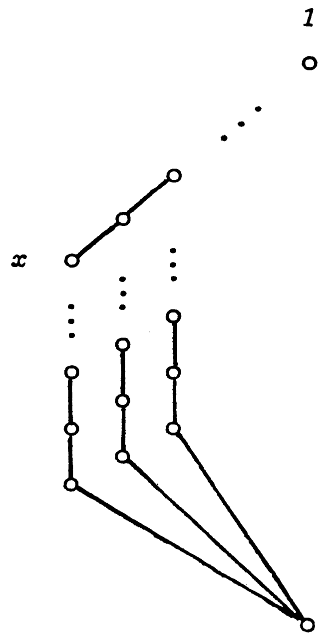
The existence of a complete lattice with pretopological but not topological order convergence has already been mentioned in **[9]**. However, the given example has to be modified slightly to ensure pretopological order convergence also at the greatest element.

EXAMPLE 1. The following subset of the real unit square

$$L = \{(x, y) \in [0, 1]^2 : x + y \leqq 1 \text{ or } x = y = 1\},$$

partially ordered componentwise, is a complete lattice with pretopological order convergence (as can easily be checked with the help of Proposition 2). But the pre-neighborhood filters of the points (0, 1) and (1, 0) differ from the topological neighborhood filters, whence $\mathbf{O}(L)$ cannot be topological.

EXAMPLE 2. Another complete lattice with pretopological but not topological order convergence is this.



Here the interval $[x)$ has empty interior, although $x \in I(1)$.

Recall that L is meet-continuous if the identity $x \wedge \sup I = \sup(x \wedge I)$ holds for every $x \in L$ and every ideal I of L. L is bicontinuous if L and its dual are meet-continuous. As shown in **[4]**, $\mathbf{O}(L)$ is in fact topological if $\mathbf{O}(L)$ is pretopological and L is bicontinuous. Further, L and its dual are continuous in the sense of Scott (see **[6]**) iff L is bicontinuous and $\mathbf{O}(L)$ is (pre)topological. These lattices have been called order-topological in **[4]**.

PROPOSITION 3. *A map between complete lattices is isotone and continuous if and only if it preserves up-directed sups and down-directed infs. In particular, the continuous lattice homomorphisms are precisely the complete homomorphisms.*

PROOF. Let ϕ: $L \to L_1$ be isotone and continuous, and consider an up-directed subset X of L. Setting $y = \sup \phi(X)$, we know that the inverse image $A = \psi^{-1}((y])$ is closed, and $\psi(x) \leqq y$ for all $x \in X$, i.e., $X \subseteq A$. Hence, by Proposition 1, $\sup X \in A$, and this means $\psi(\sup X) \leqq y = \sup \psi(X)$. On the other hand, we have $\sup \psi(X) \leqq \psi(\sup X)$ because ψ is isotone. Thus ψ preserves up-directed sups, and by duality, it also preserves down-directed infs. Conversely, if this is true then ψ is certainly isotone. Further, let $V \subseteq L_1$ be open. We have to show that $U = \psi^{-1}(V)$ is open. In view of Proposition 1, we consider an up-directed set $Y \subseteq L$ and a down-directed set $Z \subseteq L$ with $x = \sup Y = \inf Z \in U$. Then $\psi(Y)$ is up-directed, $\psi(Z)$ is down directed, and $\psi(x) = \sup \psi(Y) = \inf \psi(Z)$. Hence we find $a \in Y$ and $b \in Z$ such that $[\psi(a), \psi(b)]$ is contained in V. But then $[a, b] \subseteq \psi^{-1}([\psi(a), \psi(b)]) \subseteq \psi^{-1}(V) = U$. This proves U open.

Of course, a continuous map between complete lattices need not be isotone. For example, every map between two finite lattices is continuous because the order topologies are discrete.

As an application of Proposition 3, we note that L is meet-continuous iff each local meet-function $\rho_a\colon x \to a \wedge x$ is continuous.

2. Which convergence properties are preserved by complete homomorphisms? The order convergence on L is a limitierung (that is, $\mathfrak{F}_1 \to_L x$ and $\mathfrak{F}_2 \to_L x$ always implies $\mathfrak{F}_1 \cap \mathfrak{F}_2 \to_L x$) iff for all ideals I_1, I_2 of L with $\sup I_1 = \sup I_2 = x$, we have $\sup (I_1 \cap I_2) = x$, and dually (see [5]). This criterion is used to prove the next proposition.

PROPOSITION 4. *If the order convergence on L is a limitierung then so is the order convergence on* L_1.

PROOF. Assume that $\mathbf{O}(L)$ is a limitierung, and let $\sup J_1 = \sup J_2 = y$ for ideals J_1, J_2 of L_1. Then $I_1 = \downarrow\varphi_*(J_1)$ and $I_2 = \downarrow\varphi_*(J_2)$ are ideals of L with $\varphi_*(y) = \sup I_1 = \sup I_2$, and it follows that $\sup(J_1 \cap J_2) = \sup \varphi(I_1 \cap I_2) = \varphi(\sup(I_1 \cap I_2)) = \varphi\varphi_*(y) = y$.

PROPOSITION 5. *If L has pretopological order convergence then so has* L_1.

The proof is essentially the same as for Proposition 4. Alternatively, one may apply the following lemma in connection with Proposition 2:

LEMMA 6. *For all* $y \in L_1$, $I(y) = \varphi(I(\varphi_*(y)))$ *and* $D(y) = \varphi(D(\varphi^*(y)))$.

PROOF. In order to prove the inclusion $I(y) \subseteq \varphi(I(\varphi_*(y)))$, consider an element $z \in I(y)$ and an ideal I of L with $\sup I = \varphi_*(y)$. As φ preserves sups, we have $y = \varphi\varphi_*(y) = \sup \varphi(I)$, and $\varphi(I)$ is easily seen to be an ideal of L_1. Thus $z \in \varphi(I)$ and so $\varphi_*(z) \in I$. This shows $\varphi_*(z) \in I(\varphi_*(y))$, whence $z = \varphi\varphi_*(z) \in \varphi(I)\varphi_*(y)))$. Conversely, let $x \in I(\varphi_*(y))$ and $\sup J = y$ for some ideal $J \subseteq L_1$. Then $\varphi_*(y) = \sup \varphi_*(J)$, hence there exists $z \in J$ with $x \leqq \varphi_*(z)$, and therefore $\varphi(x) \leqq z$, i.e. $\varphi(x) \in J$.

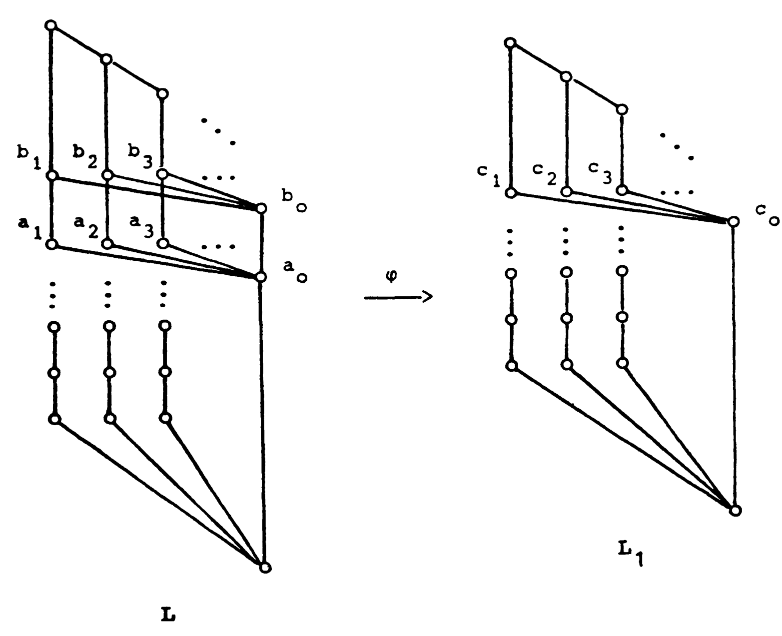
Notice that the order-topological lattices are precisely the images of bicompactly generated lattices under complete homomorphisms (cf. [**6**; I-4. 16]). In a bicompactly generated lattice L, the neighborhood filter $\mathfrak{U}(x)$ has a base consisting of all intervals $[a, b]$ which contain x, where a is compact and b is cocompact (see [**5**] and [**10**]). Hence from Lemma 6, we infer the following fact:

PROPOSITION 7. *If L is bicompactly generated then L_1 has topological order convergence, and a base for the neighborhood filter of $y \in L_1$ is formed by the intervals $[\varphi(a), \varphi(b)]$ with compact $a \leqq \varphi_*(y)$ and cocompact $b \geqq \varphi_*(y)$.*

In case that L is bicontinuous, Proposition 5 states that a complete image of an order-topological lattice is again order-topological (cf. [**6**; I-2.14]).

Since a smooth lattice-theoretical characterization of topological order convergence seems to be impossible, one might expect that this property is not preserved under complete homomorphisms. Indeed, a closer examination leads to the following counterexample.

EXAMPLE 3. Consider the complete lattices L, L_1 and the complete homomorphism φ as defined by the diagram below (where $\varphi(a_i) = \varphi(b_i) = c_i$).



It is a routine matter to verify that $\mathbf{O}(L)$ is topological. However, $\mathbf{O}(L_1)$ is not topological, since c_0 is not an element of the interior of $[c_0)$ but $c_0 \in I(c_0)$.

3. Concluding remarks. The preceding considerations apply to the following situation: Let L_0 be a subset of a complete lattice L which contains the least element of L, and assume that

(C_1) For any subset A of L_0, the sup of A in L belongs to L_0.

(C_2) The map $\psi\colon\ L \to L_0,\ \ x \to x_0 = \sup\{y \in L_0\colon\ y \leqq x\}$ preserves arbitrary sups.

Condition (C_1) means that L_0 is a complete lattice in its own right and arbitrary sups in L_0 coincide with those formed in L. Condition (C_2) ensures that ψ is a complete homomorphism from L onto L_0. The lower adjoint ψ_* is simply the inclusion map from L_0 into L, while the upper adjoint has to be defined according to §1. On the other hand, every complete image L_1 of L is isomorphic to a substructure L_0 of this kind. Indeed, if $\varphi\colon L \to L_1$ is a surjective, complete homomorphism then the lower adjoint φ_* induces an isomorphism between L_1 and a subset L_0 of L satisfying (C_1) and (C_2). The corresponding map ψ is given by $\psi(x) = \varphi_*\varphi(x)$ $(x \in L)$.

In a complete Boolean lattice L, the conditions (C_1) and (C_2) are very strong. In fact, the only subsets L_0 of L satisfying (C_1) and (C_2) are the principal ideals. More generally, we can show the following result for a surjective, complete homomorphism $\varphi\colon L \to L_1$.

PROPOSITION 8. *If L is modular and complemented then the adjoints φ_* and φ^* are complete homomorphisms satisfying*

$$\varphi_*(y) = \varphi^*(y) \wedge \varphi_*(1),$$
$$\varphi^*(y) = \varphi_*(y) \vee \varphi^*(0)$$

$(y \in L_1)$.

Furthermore, φ_* induces an isomorphism between L_1 and the principal ideal $(\varphi_*(1)]$, while φ^* induces an isomorphism between L_1 and the principal dual ideal $[\varphi^*(0))$.

PROOF. We claim that $\varphi^*\varphi(x) = x \vee \varphi^*(0)$ $(x \in L)$. Let x' be a complement of x. Then $\varphi^*\varphi(x) \wedge x' \leqq \varphi^*\varphi(x) \wedge \varphi^*\varphi(x') = \varphi^*\varphi(x \wedge x') = \varphi^*(0)$, and applying modularity we get $\varphi^*\varphi(x) = \varphi^*\varphi(x) \wedge (x' \vee x) = (\varphi^*\varphi(x) \wedge x') \vee x \leqq \varphi^*(0) \vee x$ (since $x \leqq \varphi^*\varphi(x)$). The other inequality is obvious. Now, setting $x = \varphi^*(y)$, we obtain $y = \varphi(x)$ and $\varphi^*(y) = \varphi_*(y) \vee \varphi^*(0)$. As φ_* preserves sups, the same holds for φ^*. Finally, $\varphi^*(L_1) = \varphi^*\varphi(L) = L \vee \varphi^*(0) = [\varphi^*(0))$. The remaining assertions follow by duality.

We know that any complete homomorphism φ is continuous. A related question is: when is φ open, closed or at least a quotient map? These problems will be discussed in a forthcoming note, but a few simple remarks should be added already at this point.

Call a map f between topological spaces X and Y pseudo-open if, given $y \in f(x)$ and a neighborhood U of $f^{-1}(y)$, the image $f(U)$ is a neighborhood of y. Notice that every open and every surjective closed map is pseudo-open, and that pseudo-open continuous surjections are already quotient maps. Furthermore, if there exists a continuous $g: Y \to X$ with $fg = id_Y$ then it is easy to see that f is pseudo-open. In particular, if φ_* or φ^* is continuous then φ is pseudo-open. But in view of Proposition 3, the upper adjoint φ^* is continuous iff it preserves up-directed sups. (This is certainly the case if $\varphi^*(y) = \sup\ \varphi^*(I(y))$ for $y \in L_1$. In the presence of pretopological $\mathbf{O}(L_1)$, both conditions are equivalent.) Hence we have the following consequence of Proposition 8 which applies in particular to Boolean lattices.

COROLLARY 9. *If L is complemented and modular then φ is pseudo-open.*

We conclude with two examples.

EXAMPLE 4. Associated with any uniform space $(S, \mathfrak{U})$ are two complete lattices which are called, respectively, the scale P and the retracted scale P_0 of $(S, \mathfrak{U})$. These concepts were introduced by D. Bushaw in 1967 and have various applications as diverse as topological dynamics and generalized metrization (see [1] and [2]). The scale is a subcomplete lattice of an atomic Boolean algebra, while the retracted scale is a sublattice satisfying (C_1) and (C_2) (but not a subcomplete lattice) of the scale. However, P_0 is completely distributive, being the image of P under a complete homomorphism. Such lattices are order-topological and have compact T_2 order topologies (cf. [12]). Therefore every complete homomorphism defined on a completely distributive lattice is not only continuous, but also closed onto its image. In particular, this holds for the retraction from P onto P_0. Proposition 6 provides an explicit description of the neighborhood filters for the order topology of P_0 in terms of that for P. These results turn out to have interesting applications in the study of topological compactifications which will be published elsewhere.

EXAMPLE 5. Starting with an uncountable set S, let $\mathscr{C}(S)$ be the set of all convergence structures on S, and let $\mathscr{T}(S)$ be the set of all topologies on S. (A convergence structure is called a "localized convergence relation" in [5]). Every topology can be considered as a special case of a convergence structure, and so $\mathscr{T}(S)$ can be regarded as a subset of $\mathscr{C}(S)$; furthermore, the usual order relations on these sets are mutually compatible. Indeed, relative to their usual order relations, $\mathscr{C}(S)$ is a subcomplete lattice of an atomic Boolean algebra, and $\mathcal{T}(S)$ is a complete lattice which satisfies our condition (C_1) relative to $\mathcal{C}(S)$. If $q \in \mathcal{C}(S)$, then q_0 (as defined in (C_2)) is what is commonly called the topological modification of q. Since order convergence in $\mathcal{T}(S)$ is not pretopological (cf. **[10]** and **[11]**), it follows from our Proposition 7 that the pair $\mathcal{T}(S)$, $\mathcal{C}(S)$ fails to satisfy (C_2). In other words, the topological modification of the least upper bound of a set of convergence structures on S is not necessarily the least upper bound of their topological modifications.

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