ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 3, Summer 1984

SELECTIONS OF MULTIFUNCTIONS OF TWO VARIABLES

BIAGIO RICCERI

0. Introduction. Let T, X, Y be three non-empty sets and F be a multifunction from $T \times X$ into Y. Then we may ask whether there exists a selection of F having a certain type of regularity with respect to the first variable and a certain other with respect to the second one. In practice, one provides the sets T, X, Y with some measurable or topological or even algebraic structures to consider types of regularity as measurability, continuity, belonging to some Baire class and so on. In particular, it is interesting to find Carathéodory's selections. Recently, in **[1]**, we have given a contribution to this problem.

The aim of the present paper is to show how the technique of proof used in **[1]** can be formalized in such a manner as to obtain, in a unified way, several results about our present and more general problems. Thus, as corollaries of a unique abstract theorem (Theorem 2.1), two specified versions of the results of **[1]** are obtained as well as improvements in several directions of some remarkable particular cases of the results contained in **[2]** and **[3]**. We want also to stress that some theorems we derive from Theorem 2.1 provide, as their consequences, new results on selections of class α on $T \times X$ ($0 \leqq \alpha < \omega_1$) of the given multifunction F, in the case where T, X, Y are topologized. In particular, if $\alpha = 0$, we don't assume that the product space $T \times X$ is normal.

The present paper has four sections. §1 is devoted to the notations used and to the definitions which are needed in order to state Theorem 2.1. This result is proved in §2. In §3 we explain the definitions put in §1. Finally, in §4 we present several consequences of Theorem 2.1 which differ from one another in some remarkable feature of topological nature. We conclude by establishing, as an application of a result previously obtained, an existence theorem on differential inclusions in Banach spaces which extends Theorem 2 of **[3]**.

1. Notations and definitions. For every set $A \neq \emptyset$, we denote by $\mathscr{P}(A)$ the family of all subsets of A and by 2^A the family $\mathscr{P}(A)\backslash\{\emptyset\}$. Given two sets A', $A'' \neq \emptyset$, we denote by $\mathscr{F}(A', A'')$ the set of all functions from A' into A''. Given two sets B, $C \neq \emptyset$ and given $F \in \mathscr{F}(B, 2^C)$ (resp.,

Received by the editors July 13, 1983.

$F \in \mathscr{F}(B, \mathscr{P}(C))$), we say that F is a multifunction (resp., generalized multifunction) from B into C and, for every $A \in 2^B$ and every $\Omega \in \mathscr{P}(C)$, we put $F(A) = \bigcup_{t \in A} F(t)$ and $F^-(\Omega) = \{t \in B: F(t) \cap \Omega \neq \emptyset\}$. Moreover, we put $\mathscr{S}(F) = \{f \in \mathscr{F}(B, C): f(t) \in F(t) \; \forall \, t \in B\}$. The elements of $\mathscr{S}(F)$ are called the selections of F.

Given a generalized metric space (Σ, d) (that is $d(x, y)$ can be $+\infty$ for some $x, y \in \Sigma$), for every $x \in \Sigma, \Omega, \Omega' \in 2^\Sigma$ and $r \in \mathbf{R}^+$, we put: $K_d(\Omega, r) = \{y \in \Sigma: \; \exists z \in \Omega: \; d(y, z) < r\}$; $\bar{K}_d(\Omega, r) = \{y \in \Sigma: \; \exists z \in \Omega: \; d(y, z) \leqq r\}$; $d(x, \Omega') = \inf\{(d(x, y): y \in \Omega'\}$.

Given a set $D \neq \emptyset$ and a generalized metric space (Σ', d'), we will always consider $\mathscr{F}(D, \Sigma')$ equipped with the usual generalized metric $\rho_{d'}(f, g) = \sup_{t \in D} d'(f(t), g(t))$ for every $f, g \in \mathscr{F}(D, \Sigma')$. Moreover, for every $\mathscr{E} \in 2^{\mathscr{P}(D)}$, we put

$$\mathscr{F}_{\mathscr{E}}(D, \Sigma') = \{f \in \mathscr{F}(D, \Sigma'): f^{-1}(K_{d'}(x, r)) \in \mathscr{E} \; \forall x \in f(D), \forall r \in R^+\};$$

$$\mathscr{F}_{\mathscr{E}}(D, 2^{\Sigma'}) = \{F \in \mathscr{F}(D, 2^{\Sigma'}): F^-(K_{d'}(x, r)) \in \mathscr{E} \; \forall x \in F(D), \forall r \in R^+\};$$

$$\bar{\mathscr{F}}_{\mathscr{E}}(D, 2^{\Sigma'}) = \{F \in \mathscr{F}(D, 2^{\Sigma'}): F^-(\bar{K}_{d'}(x, r)) \in \mathscr{E} \; \forall x \in F(D), \forall r \in R^+\}.$$

If D is a topological space and $\mathscr{E}$ is the family of all open subset of D, then $\mathscr{F}_{\mathscr{E}}(D, 2^{\Sigma'})$ (resp., $\mathscr{F}_{\mathscr{E}}(D, \Sigma')$) is the set of all lower semicontinuous (resp., continuous) multifunctions (resp., functions) from D into Σ'. If $\mathscr{E}$ is the family of all closed subset of D, then $\bar{\mathscr{F}}_{\mathscr{E}}(D, 2^{\Sigma'})$ is the set of all pseudo-upper semicontinuous multifunctions from D into Σ'. If $\mathscr{E}$ is the family of all Borel subsets of D of additive class α, with $0 \leqq \alpha < \omega_1$, and if Σ' is separable, then $\mathscr{F}_{\mathscr{E}}(D, 2^{\Sigma'})$ (resp., $\mathscr{F}_{\mathscr{E}}(D, \Sigma')$) is the set of all multifunctions (resp., functions) from D into Σ' of lower class α (resp., of class α). If $\mathscr{E}$ is a σ-algebra and if Σ' is separable, then $\mathscr{F}_{\mathscr{E}}(D, 2^{\Sigma'})$ (resp., $\mathscr{F}_{\mathscr{E}}(D, \Sigma')$) is the set of all $\mathscr{E}$-measurable multifunctions (resp., functions) from D into Σ'. Moreover, if D and E are two topological spaces, $F \in \mathscr{F}(D, \mathscr{P}(E))$ is said to be an upper semicontinuous generalized multifunction if the set $F^-(T)$ is closed for every closed set $T \subseteqq E$.

Given a set $D' \neq \emptyset$ and a vector space Σ_1, we will regard $\mathscr{F}(D', \Sigma_1)$ as a vector space in the usual way. Moreover, if Σ_1 is a normed space, we will consider Σ_1 with the metric induced by the norm and we put

$$\mathscr{F}_{\ell}(D', \Sigma_1) = \{f \in \mathscr{F}(D', \Sigma_1): \sup_{t \in D'} \|f(t)\| < +\infty\}$$

and $\|f\|_{\ell} = \sup_{t \in D'} \|f(t)\|$ for every $f \in \mathscr{F}_{\ell}(D', \Sigma_1)$.

We now give three definitions which are explained in §3. In these definitions it is always understood that A, B are two non-empty sets and that (Σ, d) is a generalized metric space.

DEFINITION 1.1 Let $\mathscr{A} \in 2^{\mathscr{F}(A, \Sigma)}$ and $F \in \mathscr{F}(A, 2^\Sigma)$. We say that the multifunction F is $\mathscr{A}$-stable if the following two conditions are satisfied:

(i) $\mathscr{S}(F) \cap \mathscr{A} \neq \emptyset$;

(ii) for every ε, $r \in \mathbf{R}^+$ and every $f \in \mathscr{A}$ such that $F(a) \cap K_d(f(a), r) \neq \emptyset$ for all $a \in A$, there exists $g \in \mathscr{A}$ such that $g(a) \in F(a) \cap K_d(f(a), r + \varepsilon)$ for all $a \in A$.

DEFINITION 1.2. Let $\mathscr{A} \in 2^{\mathscr{F}(A, \Sigma)}$, $\mathscr{E} \in 2^{\mathscr{P}(B)}$ and $\{F_a\}_{a \in A}$ be a family of multifunctions from B into Σ. We say that the multifunctions of the family $\{F_a\}_{a \in A}$ are equi-regular with respect to the pair $(\mathscr{A}, \mathscr{E})$ if for every $f \in \mathscr{A}$ and every $r \in \mathbf{R}^+$ the set

$$\bigcup_{\varepsilon \in]0, r[} \bigcap_{a \in A} F_a^-(K_d(f(a), r - \varepsilon))$$

belongs to $\mathscr{E}$.

DEFINITION 1.3. Let

$$\mathscr{B} \in 2^{\mathscr{F}(B, 2^{\mathscr{F}(A, \Sigma)})},$$

$\mathscr{E} \in 2^{\mathscr{P}(B)}$, $\mathscr{D}$, $\mathscr{G} \in \mathscr{P}(\mathscr{P}(B))$. We say that the quadruple $(\mathscr{B}, \mathscr{E}, \mathscr{D}, \mathscr{G})$ satisfies property (P) if for every $H \in \mathscr{B} \cap \mathscr{F}_{\mathscr{E}}(B, 2^{\mathscr{F}(A, \Sigma)})$, every $D \in \mathscr{D}$ and every $\bar{h} \in \mathscr{S}(H|_D) \cap \mathscr{F}_{\mathscr{I}_D}(D, \mathscr{F}(A, \Sigma))$, where $\mathscr{I}_D = \{E \in \mathscr{P}(D)$: $\exists E^* \in \mathscr{E}$: $E = E^* \cap D\}$, there exists $h \in \mathscr{S}(H) \cap \mathscr{F}_{\mathscr{G}}(B, \mathscr{F}(A, \Sigma))$ such that, for every fixed $a \in A$, the function $b \to h(b)(a)$ belongs to $\mathscr{F}_{\mathscr{G}}(B, \Sigma)$ and $h|_D = \bar{h}$.

Finally, given $\mathscr{E} \in 2^{\mathscr{P}(B)}$ and a family $\{g_a\}_{a \in A}$ of functions belonging to $\mathscr{F}_{\mathscr{E}}(B, \Sigma)$, we say that these functions are equi-belonging to $\mathscr{F}_{\mathscr{E}}(B, \Sigma)$ if the function defined h by putting for every $b \in B$, $h(b) = g(\cdot, b)$, where $g(a, b) = g_a(b)$, belongs to $\mathscr{F}_{\mathscr{E}}(B, \mathscr{F}(A, \Sigma))$.

2. The main abstract result. Our main abstract result is the following theorem.

THEOREM 2.1. *Let A, B be two non-empty sets, (Σ, d) be a generalized metric space, $\mathscr{A} \in 2^{\mathscr{F}(A, \Sigma)}$, $\mathscr{B} \in 2^{\mathscr{F}(B, 2^{\mathscr{F}(A, \Sigma)})}$, $\mathscr{E} \in 2^{\mathscr{P}(B)}$, $\mathscr{D}$, $\mathscr{G} \in \mathscr{P}(\mathscr{P}(B))$, $G \in \mathscr{F}(A \times B, 2^{\Sigma})$. Suppose that:*

(α) for every $b \in B$, the multifunction $G(\cdot, b)$ is $\mathscr{A}$-stable;

(β) the multifunction H defined by putting $H(b) = \mathscr{S}(G(\cdot, b)) \cap \mathscr{A}$ for every $b \in B$, belongs to $\mathscr{B}$;

(γ) the multifunctions of the family $\{G(a, \cdot)\}_{a \in A}$ are equi-regular with respect to the pair $(\mathscr{A}, \mathscr{E})$;

(δ) the quadruple $(\mathscr{B}, \mathscr{E}, \mathscr{D}, \mathscr{G})$ satisfies property (P).

Under such hypotheses, for every $D \in \mathscr{D}$ and every $\varphi \in \mathscr{S}(G|_{A \times D})$ such that $\varphi(\cdot, b) \in \mathscr{A}$ for all $b \in D$ and that the functions of the family $\{\varphi(a, \cdot)\}_{a \in A}$ are equi-belonging to $\mathscr{F}_{\mathscr{I}_D}(D, \Sigma)$ there exists $g \in \mathscr{S}(G)$ such that:

(α') for every $b \in B$, the function $g(\cdot, b)$ belongs to $\mathscr{A}$;

(β') *the functions of the family* $\{g(a, \cdot)\}_{a\in A}$ *are equi-belonging to* $\mathscr{F}_{\mathscr{G}}(B, \Sigma)$;

(γ') $g|_{A\times D} = \varphi$.

PROOF. Let us prove that the multifunction H, defined in (β), belongs to $\mathscr{F}_{\mathscr{E}}(B, 2^{\mathscr{A}})$. To this end, let $\psi \in H(B)$ and $r \in \mathbf{R}^+$. It is easily seen that

$$H^-(K_{\rho_d}(\psi, r)) = \{b \in B: \exists \psi_1 \in \mathscr{A}, \varepsilon \in]0, r[: \psi_1(a) \in G(a, b) \cap K_d(\psi(a), r - \varepsilon)\ \forall a \in A\}. \tag{1}$$

Fix $\varepsilon^* \in]0, r[$ and $b^* \in \bigcap_{a\in A} G_a^-(K_d(\psi(a), r - \varepsilon^*))$. Since $\psi \in \mathscr{A}$, by (α), it follows that there exists $\psi_1 \in \mathscr{A}$ such that $\psi_1(a) \in G(a, b^*) \cap K_d(\psi(a), r - \varepsilon^*/2)$ for every $a \in A$. From this fact and from (1) it follows that

$$H^-(K_{\rho_d}(\psi, r)) = \bigcup_{\varepsilon\in]0,r[} \bigcap_{a\in A} G_a^-(K_d(\psi(a), r - \varepsilon)). \tag{2}$$

Now, our claim follows from the fact that the set on the right-hand side of (2), by (γ), belongs to $\mathscr{E}$.

At this point, let $D \in \mathscr{D}$ and let $\bar{h} \in \mathscr{F}_{\mathscr{G}_D}(D, \mathscr{F}(A, \Sigma))$ be such that $\varphi(a, b) = \bar{h}(b)(a)$ for every $a \in A$, $b \in D$. Since $\varphi \in \mathscr{S}(G|_{A\times D})$ and $\varphi(\cdot, b) \in \mathscr{A}$ for every $b \in D$, we have that $\bar{h} \in \mathscr{S}(H|_D)$. But then, as the quadruple ($\mathscr{B}$, $\mathscr{E}$, $\mathscr{D}$, $\mathscr{G}$) satisfies property (P), there exists $h \in \mathscr{S}(H) \cap \mathscr{F}_{\mathscr{G}}(B, \mathscr{A})$ such that, for every fixed $a \in A$, the function $b \to h(b)(a)$ belongs to $\mathscr{F}_{\mathscr{G}}(B, \Sigma)$ and $h|_D = \bar{h}$. If we put $g(a, b) = h(b)(a)$ for every $(a, b) \in A \times B$, the function g has the desired properties.

3. Explanations of Definitions 1.1, 1.2, 1.3. In this section we establish some results which explain the definitions put in §1. We shall use the same notations. With regard to Definition 1.1 the following proposition is useful.

PROPOSITION 3.1. *Let* $\mathscr{R} \in 2^{\mathscr{P}(A)}$ *and* $F, G \in \mathscr{F}_{\mathscr{R}}(A, 2^{\Sigma})$. *Let one of the following two conditions be satisfied*:

(α) *A is a topological space and $\mathscr{R}$ is the family of all open subsets of A.*

(β) *$F(A)$ and $G(A)$ are separable and $\mathscr{R}$ is closed under finite intersection and countable union.*

Then, if $r \in \mathbf{R}^+$ *is such that* $I_r(a) = F(a) \cap K_d(G(a), r) \neq \emptyset$ *for every* $a \in A$, *we have* $I_r \in \mathscr{F}_{\mathscr{R}}(A, 2^{\Sigma})$.

PROOF. The proof is exactly that of Proposition 2.1 of [1]. We include it here for the sake of completeness. Put $P(a) = F(a) \times G(a)$ for every $a \in A$. Then, for every $\Omega', \Omega'' \in 2^{\Sigma}$, we have

$$P^-(\Omega' \times \Omega'') = F^-(\Omega') \cap G^-(\Omega''). \tag{3}$$

Moreover, for every $\Omega \in 2^{\Sigma}$, we have

(4) $\quad I_r^-(\Omega) = P^-(\{(x, y) \in F(A) \times G(A): d(x, y) < r\} \cap (\Omega \times \Sigma)).$

The assertion follows from (3) and (4), since any open set in $F(A) \times G(A)$ is the union of a family (countable if (β) holds) of sets of type $\Omega' \times \Omega''$, with Ω', Ω'' open sets, respectively, in $F(A)$ and $G(A)$.

We now establish the following result.

PROPOSITION 3.2. *Let $\mathscr{R} \in 2^{\mathscr{P}(A)}$ and $F \in \mathscr{F}_{\mathscr{R}}(A, 2^{\Sigma})$. Let one of the following two sets of conditions be satisfied:*

(α) *A is a paracompact topological space; $\mathscr{R}$ is the family of all open subsets of A; Z is a subset of A, with $\dim_A(Z) \leqq 0$; Σ is a Banach space, if $A \neq Z$; the set $F(a)$ is closed (resp., complete, if $A = Z$) for every $a \in A$ and convex for every $a \in A \backslash Z$.*

(β) *$F(A)$ is separable; there exists $\mathscr{X} \in 2^{\mathscr{P}(A)}$, closed under finite union and difference, such that, $\mathscr{R} = \{D \in \mathscr{P}(A): \exists\{D_n\}, D_n \in \mathscr{X}\ \forall\, n \in \mathbf{N},: D = \bigcup_{n=1}^{\infty} D_n\}$; the set $F(a)$ is complete for every $a \in A$.*

Then, if we put

$$\mathscr{A} = \begin{cases} \mathscr{F}_{\mathscr{R}}(A, \Sigma) & \text{if } (\alpha) \text{ holds,} \\ \{f \in \mathscr{F}_{\mathscr{R}}(A, \Sigma): f(A) \quad \text{is separable}\} & \text{if } (\beta) \text{ holds,} \end{cases}$$

the multifunction F is $\mathscr{A}$-stable.

REMARK. $\dim_A(Z) \leqq 0$ means that $\dim(T) \leqq 0$ for every $T \in \mathscr{P}(Z)$ which is closed in A, where $\dim(T)$ denotes the covering dimension of T.

PROOF. First, we notice that for any $G \in \mathscr{F}(A, 2^{\Sigma})$, if we put $G^*(a) = \overline{G(a)}$ for every $a \in A$, we have $G^-(\Omega) = G^{*-}(\Omega)$ for every open set $\Omega \subseteqq \Sigma$. Now, let $r \in \mathbf{R}^+$ and $f \in \mathscr{A}$ be such that $I_r(a) = F(a) \cap \bar{K}_d(f(a), r) \neq \emptyset$ for every $a \in A$. By Proposition 3.1, in any case, the multifunction I_r, and so also I_r^*, belongs to $\mathscr{F}_{\mathscr{R}}(A, 2^{\Sigma})$. Clearly, $I_r^*(a) \subseteqq \overline{F(a)} \cap K_d(f(a), r)$. Now, our assertion follows from Theorem 1.1 of **[4]**, if (α) holds, or from Theorem 3.2 of **[5]**, if (β) holds. In particular, if (α) holds and if $A = Z$, take into account footnote 2 of **[4]** and observe that Theorem 2 of **[6]** remains valid for any lower semicontinuous multifunction, with complete values, from a space as our A into a (generalized) metric space.

Another result about Definition 1.1 is the following proposition.

PROPOSITION 3.3. *Let A be an extremally disconnected Hausdorff topological space, $\mathscr{R}$ the family of all open subsets of A and F an upper semicontinuous multifunction from A into Σ, with compact values.*

Then, the multifunction F is $\mathscr{F}_{\mathscr{R}}(A, \Sigma)$-stable.

PROOF. Given $r \in \mathbf{R}^+$ and $f \in \mathscr{F}_{\mathscr{R}}(A, \Sigma)$, put $G_r(a) = \bar{K}_d(f(a), r)$ for every $a \in A$. We have $G_r^-(\Omega) = f^{-1}(\bar{K}_d(\Omega, r))$ for every $\Omega \in 2^\Sigma$. Thus, the multifunction G_r is upper semicontinuous. If $I_r'(a) = F(a) \cap G_r(a) \neq \emptyset$ for every $a \in A$, by Theorem 1 on p. 180 of [7], the multifunction I_r' is upper semicontinuous. Now, our assertion follows from Theorem 1.1 of [**8**].

Consider now Definition 1.2. With regard to this definition, a first result is the following proposition.

PROPOSITION 3.4. *Let B be a topological space, $\mathscr{E}$ the family of all open subsets of B, $\mathscr{A} \in 2^{\mathscr{F}(A, \Sigma)}$, $\{F_a\}_{a \in A}$ a family of multifunctions from B into Σ such that, for every $b \in B$, the multifunction $a \to F_a(b)$ is $\mathscr{A}$-stable. Suppose, furthermore, that for every $b \in B$ and every selection φ of the multifunction $a \to F_a(b)$ belonging to $\mathscr{A}$, one has*

$$\lim_{c \to b} \sup_{a \in A} d(\varphi(a), F_a(c)) = 0. \tag{5}$$

Under such hypotheses, the multifunctions of the family $\{F_a\}_{a \in A}$ are equi-regular with respect to the pair $(\mathscr{A}, \mathscr{E})$.

PROOF. Given $f \in \mathscr{A}$ and $r \in \mathbf{R}^+$, fix $b \in \bigcup_{\varepsilon \in]0, r[} \bigcap_{a \in A} F_a^-(K_d(f(a), r - \varepsilon))$. Therefore, there exists $\varepsilon^* \in]0, r[$ such that $F_a(b) \cap K_d(f(a), r - \varepsilon^*) \neq \emptyset$ for every $a \in A$. Since the multifunction $a \to F_a(b)$ is $\mathscr{A}$-stable, there exists $\varphi \in \mathscr{A}$ such that $\varphi(a) \in F_a(b) \cap K_d(f(a), r - \varepsilon')$ for every $a \in A$, where $\varepsilon' \in]0, \varepsilon^*[$. By (5), there exists a neighbourhood V of b such that $d(\varphi(a), F_a(c)) < \varepsilon'/2$ for every $c \in V$, $a \in A$. Fix $\bar{c} \in V$, $\bar{a} \in A$. Let $y \in F_{\bar{a}}(\bar{c})$ be such that $d(\varphi(\bar{a}), y) < \varepsilon'/2$. Since $d(\varphi(\bar{a}), f(\bar{a})) < r - \varepsilon'$, we have $d(y, f(\bar{a})) < r - \varepsilon'/2$ and so $y \in F_{\bar{a}}(\bar{c}) \cap K_d(f(\bar{a}), r - \varepsilon'/2)$. Thus, we have proved that the neighbourhood V of b is contained in the set $\bigcup_{\varepsilon \in]0, r[} \bigcap_{a \in A} F_a^-(K_d(f(a), r - \varepsilon))$ which, therefore, is open.

It is easy to check that condition (5) is certainly satisfied whenever A is finite and each F_a is lower semicontinuous.

Still with regard to Definition 1.2 we have the following result.

PROPOSITION 3.5. *Let $\mathscr{E}$, $\mathscr{G}^* \in 2^{\mathscr{P}(B)}$, with $\mathscr{G}^* \subseteq \mathscr{E}$ and $\mathscr{E}$ closed under countable union, and let $\{F_a\}_{a \in A}$ be a family of multifunctions from B into Σ, with $F_a \in \mathscr{F}_{\mathscr{G}^*}(B, 2^\Sigma) \cup \bar{\mathscr{F}}_{\mathscr{G}^*}(B, 2^\Sigma)$ for every $a \in A$. Moreover, let one of the following two sets of conditions be satisfied*:

(α) *For every family $\{E_i\}_{i \in I}$ in $\mathscr{G}^*$, one has $\bigcap_{i \in I} E_i \in \mathscr{E}$;*

(β) *For every sequence $\{E_n\}$ in $\mathscr{G}^*$, one has $\bigcap_{n=1}^\infty E_n \in \mathscr{E}$; A is a separable topological space; there exists a topology τ' on Σ such that*: (i) *Σ is τ'-normal*; (ii) *every set $F_a(b)$ and every closed ball of Σ are τ'-closed*; (iii) *for every $b \in B$, the multifunction $a \to F_a(b)$ is τ'-upper semicontinuous.*

Under such hypotheses, if we put

$$\mathscr{A} = \begin{cases} \mathscr{F}(A, \Sigma) & \textit{if } (\alpha) \textit{ holds,} \\ \{f \in \mathscr{F}(A, \Sigma) : f \textit{ is d-continuous}\} & \textit{if } (\beta) \textit{ holds,} \end{cases}$$

the multifunctions of the family $\{F_a\}_{a \in A}$ *are equi-regular with respect to the pair* $(\mathscr{A}, \mathscr{E})$.

PROOF. Given $A' \in 2^A$, put $A'_1 = \{a \in A' : F_a \in \mathscr{F}_{\mathscr{G}^*}(B, 2^\Sigma)\}$. Given $f \in \mathscr{F}(A, \Sigma)$ and $r \in \mathbf{R}^+$, for every $n \in \mathbf{N}$ such that $n \geqq [1/r] + 1$, put

$$G_n(a) = \begin{cases} K_d(f(a), r - 1/n) & \text{if } a \in A'_1 \\ \bar{K}_d(f(a), r - 1/n) & \text{if } a \in A' \backslash A'_1. \end{cases}$$

We have

$$\bigcup_{\varepsilon \in]0, r[} \bigcap_{a \in A'} F_a^-(K_d(f(a), r - \varepsilon)) = \bigcup_{n = [\frac{1}{r}]+1} \bigcap_{a \in A'} F_a^-(G_n(a)). \tag{6}$$

Now, if (α) holds, our claim follows immediately from (6), by taking $A' = A$. Suppose that (β) holds. Consider a countable dense set $D \subseteqq A$. Given $f \in \mathscr{A}$ and $r \in \mathbf{R}^+$, we claim that

$$\bigcup_{\varepsilon \in]0, r[} \bigcap_{a \in A} F_a^-(K_d(f(a), r - \varepsilon)) = \bigcup_{\varepsilon \in]0, r[} \bigcap_{a \in D} F_a^-(K_d(f(a), r - \varepsilon)). \tag{7}$$

Obviously, the set on the left-hand side of (7) is contained in that on the right-hand side. We show the opposite inclusion. Therefore, let $b_0 \in B$ and $\varepsilon^* \in]0, r[$ such that $F_a(b_0) \cap K_d(f(a), r - \varepsilon^*) \neq \emptyset$ for every $a \in D$. From our hypotheses and from Theorem 1 on p. 180 of [7], it follows that the generalized multifunction $a \to F_a(b_0) \cap \bar{K}_d(f(a), r - \varepsilon^*)$ is τ'-upper semicontinuous in A. By this fact and since D is dense in A, it follows that $F_a(b_0) \cap \bar{K}_d(f(a), r - \varepsilon^*) \neq \emptyset$ for every $a \in A$, and so b_0 belongs to the set on the left-hand side of (7). Finally, our thesis follows from (6) and (7), by taking $A' = D$.

Consider, finally, Definition 1.3. We have the following result.

PROPOSITION 3.6. *Let one of the following three sets of conditions be satisfied:*

(α) *B, $\mathscr{E}$, Z, Σ are, respectively, as A, $\mathscr{R}$, Z, Σ in*(α) *of Proposition* 3.2; $\mathscr{B} = \{H \in \mathscr{F}(B, 2^{\mathscr{F}(A, \Sigma)}) : H(B) \subseteqq \mathscr{F}_{\prime}(A, \Sigma)$ *if* $B \neq Z$ *and* $H(b)$ *is closed (resp., complete, if* $B = Z$*) for every* $b \in B$ *and convex for every* $b \in B \backslash Z\}$; $\mathscr{D} = \{D \in 2^B : D$ *is closed*$\}$; $\mathscr{G} = \mathscr{E}$.

(β) *B is a paracompact topological space which, moreover, is either separable or σ-compact;* $\mathscr{B} = \{H \in \mathscr{F}(B, 2^{\mathscr{F}(A, \Sigma)}) : H(b)$ *is complete for every* $b \in B\}$; $\mathscr{E}$, $\mathscr{D}$ *are as in* (α); $\mathscr{G}$ *is the additive class* 1 *of Borel subsets of* B.

(γ) $\mathscr{B} = \{H \in \mathscr{F}(B, 2^{\mathscr{F}(A, \Sigma)}) : H(B)$ *is separable and* $H(b)$ *is complete* *for every* $b \in B\}$; $\mathscr{E}$ *is as* $\mathscr{R}$ *in* (β) *of Proposition* 3.2; $\mathscr{D} = \{D \in \mathscr{E}\colon B\backslash D \in \mathscr{E}\}$; $\mathscr{G} = \mathscr{E}$.

Under such hypotheses, the quadruple ($\mathscr{B}$, $\mathscr{E}$, $\mathscr{D}$, $\mathscr{G}$) *satisfies property* (P)

PROOF. Given $h \in \mathscr{F}(B, \mathscr{F}(A, \Sigma))$ and $\bar{a} \in A$, put $h_{\bar{a}}(b) = h(b)(\bar{a})$ for every $b \in B$. Moreover, given $z_0 \in \Sigma$ and $r \in \mathbf{R}^+$, for every $b \in B$, put

$$f_b(a) = \begin{cases} z_0 & \text{if } a = \bar{a}, \\ h(b)(a) & \text{if } a \neq \bar{a},\ a \in A. \end{cases}$$

We have

$$h_{\bar{a}}^{-1}(K_d(z_0, r)) = \bigcup_{b \in B} h^{-1}(K_{\rho_d}(f_b, r)). \tag{8}$$

Suppose now that $h(B)$ is separable. Denote by $\{\varphi_n\}$ a sequence dense in $h(B)$ and put

$$f_n(a) = \begin{cases} z_0 & \text{if } a = \bar{a} \\ \varphi_n(a) & \text{if } a \neq \bar{a},\ a \in A. \end{cases}$$

In this case, we have

$$h_{\bar{a}}^{-1}(K_d(z_0, r)) = \bigcup_{n=1}^{\infty} h^{-1}(K_{\rho_d}(f_n, r)). \tag{9}$$

Now, let $H \in \mathscr{B} \cap \mathscr{F}_{\mathscr{E}}(B, 2^{\mathscr{F}(A, \Sigma)})$, $D \in \mathscr{D}$ and $\bar{h} \in \mathscr{S}(H|_D) \cap \mathscr{F}_{\mathscr{G}_D}(D, \mathscr{F}(A, \Sigma))$. Put

$$H_1(b) = \begin{cases} H(b) & \text{if } b \in B\backslash D \\ \{\bar{h}(b)\} & \text{if } b \in D. \end{cases}$$

Plainly, for every $\Omega \in \mathscr{P}(\mathscr{F}(A, \Sigma))$, we have

$$H_1^-(\Omega) = [H^-(\Omega) \cap (B\backslash D)] \cup \bar{h}^{-1}(\Omega). \tag{10}$$

In any case, $H_1 \in \mathscr{B} \cap \mathscr{F}_{\mathscr{E}}(B, 2^{\mathscr{F}(A, \Sigma)})$. Precisely, this follows from Example 1.3* of **[9]**, if (α) or (β) holds, or from (10), if (γ) holds. Now, suppose that (α) or (γ) holds. Then, respectively by Theorem 1.1 of **[4]** of by Theorem 3.2 of **[5]**, we have $\mathscr{S}(H_1) \cap \mathscr{F}_{\mathscr{E}}(B, \mathscr{F}(A, \Sigma)) \neq \emptyset$. Therefore, in these two cases, to finish the proof it suffices to take into account (8) or (9), respectively. Suppose now that (β) holds. By Theorem 1.1 of **[10]**, there exist $H_2, H_3 \in \mathscr{F}(B, 2^{\mathscr{F}(A, \Sigma)})$, with $H_2 \in \mathscr{F}_{\mathscr{E}}(B, 2^{\mathscr{F}(A, \Sigma)})$ and H_3 upper semicontinuous, such that, for every $b \in B$, the sets $H_2(b)$, $H_3(b)$ are compact and $H_2(b) \subseteq H_3(b) \subseteq H_1(b)$. The set $H_2(B)$ is, therefore, separable. Precisely, this follows from Proposition 2.2 of **[1]**, if B is separable, or from Theorem 3 on p. 116 of **[11]** if B is σ-compact. By Theorem 2 of **[12]**, there exists $h \in \mathscr{S}(H_2) \cap \mathscr{F}_{\mathscr{G}}(B, \mathscr{F}(A, \Sigma))$. In particular, $h \in \mathscr{S}(H_1)$ and $h(B)$ is separable. Now, the thesis follows from (9).

4. Consequences of Theorem 2.1 and remarks. In the present section, by using the preceding propositions, we shall derive several consequences of Theorem 2.1. We give here their statements one after the other, while various remarks on them will be made at the end. In the sequel, the meaning of T, X, Y, F is that of the Introduction.

In the next five theorems X is a paracompact topological space, Z is a subset of X, with $\dim_X(Z) \leqq 0$, and Y is a Banach space or simply a metric one provided that $X = Z$ and, in Theorem 4.4, $T = S$.

THEOREM 4.1. *Let $(T, \mathscr{T})$ be a measurable space. Suppose that:*

(1) *the set $F(t, x)$ is closed (resp. complete. if $X = Z$) for every $(t, x) \in T \times X$ and convex for every $(t, x) \in T \times (X \backslash Z)$;*

(2) *for every $x \in X$, the set $F(T, x)$ is separable and, if $X \neq Z$, also bounded, the multifunction $F(\cdot, x)$ is $\mathscr{T}$-measurable and for each of its $\mathscr{T}$-measurable selections ψ, one has*

$$\lim_{\xi \to x} \sup_{t \in T} d(\psi(t), F(t, \xi)) = 0 \tag{11}$$

Under such hypotheses, for every closed set $D \subseteqq X$ and every $\varphi \in \mathscr{S}(F|_{T \times D})$ such that, for all $x \in D$, the function $\varphi(\cdot, x)$ is $\mathscr{T}$-measurable and that the functions of the family $\{\varphi(t, \cdot)\}_{t \in T}$ are equi-continuous, there exists $f \in \mathscr{S}(F)$ such that:

(1′) *for every $x \in X$, the function $f(\cdot, x)$ is $\mathscr{T}$-measurable;*

(2′) *the functions of the family $\{f(t, \cdot)\}_{t \in T}$ are equi-continuous;*

(3′) *$f|_{T \times D} = \varphi$.*

PROOF. Apply Theorem 2.1 by taking: $A = T$; $B = X$; $\Sigma = Y$; $\mathscr{A} = \{\psi \in \mathscr{F}_{\mathscr{T}}(T, Y)$: $\psi(T)$ is separable and, if $X \neq Z$, also bounded$\}$; $\mathscr{B} = \{H \in \mathscr{F}(X, 2^{\mathscr{F}(T, Y)})$: $H(X) \subseteqq \mathscr{A}$ and $H(x)$ is closed (resp., complete, if $X = Z$) for every $x \in X$ and convex for every $x \in X \backslash Z\}$; $\mathscr{E} = \{E \in \mathscr{P}(X)$: E is open$\}$; $\mathscr{D} = \{D \in \mathscr{P}(X)$: D is closed$\}$; $\mathscr{G} = \mathscr{E}$; $G = F$.

With these choices, it is possible to check that the hypotheses of Theorem 2.1 are satisfied. Precisely, conditions (α), (γ), (δ) follow, respectively, from Proposition 3.2 (part β), from Proposition 3.4 and from Proposition 3.6 (part α). Condition (β) is obvious. Our result follows then from that of Theorem 2.1.

THEOREM 4.2. *Let $(T, \mathscr{T})$ be a measurable space, T being finite. Suppose that:*

(1) *the set $F(t, x)$ is closed (resp., complete, if $X = Z$) and separable for every $(t, x) \in T \times X$ and convex for every $(t, x) \in T \times (X \backslash Z)$;*

(2) *for every $x \in X$, the multifunction $F(\cdot, x)$ is $\mathscr{T}$-measurable;*

(3) *for every $t \in T$, the multifunction $F(t, \cdot)$ is lower semicontinuous.*

Under such hypotheses, the conclusion of Theorem 4.1 *holds.*

PROOF. The proof follows the same pattern of that of Theorem 4.1. Take also into account the remark after Proposition 3.4.

THEOREM 4.3. *Let T be a perfectly normal topological space. Suppose that the hypotheses of Theorem* 4.1 *are satisfied, with "of lower class α" and "of class α"* ($\alpha > 0$), *respectively referred to $F(\cdot, x)$ and ψ, instead of "$\mathscr{T}$-measurable".*

Then, the conclusion of Theorem 4.1 *holds, with "of class α" instead of "$\mathscr{T}$-measurable". Theorefore, if T and X are metrizable, the function f is of class $\alpha + 1$ on $T \times X$.*

PROOF. The proof follows the same pattern of that of Theorem 4.1. In particular, in applying Proposition 3.2 (part (β)), we must assume as $\mathscr{X}$ the family of all Borel subsets of T which are ambiguous of class α (see [7], p. 347). Thus, since T is perfectly normal, $\mathscr{R}$ is the additive class α (see also [**13**], p. 401). The last assertion of our theorem follows from Theorem 2 on p. 378 of [7].

THEOREM 4.4. *Let T be a paracompact topological space and S be a subset of T, with* $\dim_T(S) \leqq 0$. *Suppose that*:

(1) *the set $F(t, x)$ is closed (resp., complete, if $T = S$ and $X = Z$) and convex for every $(t, x) \in [(T \times (X\backslash Z)) \cup ((T\backslash S) \times Z)]$*;

(2) *for every $x \in X$, the set $F(T, x)$ is bounded provided that $X \neq Z$, the multifunction $F(\cdot, x)$ is lower semicontinuous and for each of its continuous selections ψ, one has* (11).

Under such hypotheses, the conclusion of Theorem 4.1 *holds, with "continuous" instead of "$\mathscr{T}$-measurable". Therefore, the function f is continuous on $T \times X$.*

PROOF. The proof is analogous to that of Theorem 4.1. In particular, we must use part (α) of Proposition 3.2. The last part of the conclusion is immediate.

THEOREM 4.5. *Let T be an extremally disconnected Hausdorff topological space. Suppose that*:

(1) *that set $F(t, x)$ is compact for every $(t, x) \in T \times X$ and convex for every $(t, x) \in T \times (X\backslash Z)$*;

(2) *for every $x \in X$, the set $F(T, x)$ is bounded provided that $X \neq Z$, the multifunction $F(\cdot, x)$ is upper semicontinuous and for each of its continuous selections ψ, one has* (11). *Under such hypotheses, the conclusion of Theorem* 4.4 *holds.*

PROOF. The proof is exactly that of Theorem 4.4, except that we must use Proposition 3.3 instead of Proposition 3.2.

In the next four theorems, X is a paracompact and perfectly normal topological space which is, furthermore, either separable or σ-compact, and Y is a metric space, except that, in Theorem 4.8, it is assumed to be a Banach space, if $T \neq S$.

THEOREM 4.6. *Let $(T, \mathscr{T})$ be a measurable space. Suppose that:*

(1) *for every $(t, x) \in T \times X$, the set $F(t, x)$ is complete;*

(2) *for every $x \in X$, the set $F(T, x)$ is separable, the multifunction $F(\cdot, x)$ is $\mathscr{T}$-measurable and for each of its $\mathscr{T}$-measurable selections ϕ, one has* (11).

Under such hypotheses, the conclusion of Theorem 4.1 *holds, with "the functions of the family $\{f(t, \cdot)\}_{t\in T}$ are equi-belonging to the class* 1" *instead of* (2′) *of that theorem.*

Moreover, the set $\bigcup_{x\in X} \{f(\cdot, x)\}$ is separable in $\mathscr{F}(T, Y)$.

PROOF. The proof is analogous to that of Theorem 4.1. In particular, after choices by now obvious in order to apply Theorem 2.1, we must use Proposition 3.6 (part (β)), by taking into account also its proof to get the last part of our result.

THEOREM 4.7. *Let T be a pefectly normal topological space. Suppose that the hypotheses of Theorem* 4.6 *are satisfied, with "of lower class α" and "of class α"* ($\alpha > 0$), *respectively referred to $F(\cdot, x)$ and ϕ, instead of "$\mathscr{T}$-measurable".*

Then, the conclusion of Theorem 4.6 *holds, with "of class α" instead of "$\mathscr{T}$-measurable".*

Moreover, the function f is of class α on $T \times X$, provided that T, X are metrizable.

PROOF. For the first part of the conclusion see the proofs of Theorems 4.3 and 4.6. The second one follows from a more general result recently proved by Z. Grande in **[17]**.

THEOREM 4.8. *Let T, S be as in Theorem* 4.4. *Suppose that:*

(1) *the set $F(t, x)$ is closed (resp. complete, if $T = S$) for every $(t, x) \in T \times X$ and convex for every $(t, x) \in (T\backslash S) \times X$;*

(2) *for every $x \in X$, the multifunction $F(\cdot, x)$ is lower semicontinuous and for each of its continuous selections ϕ, one has* (11).

Under such hypotheses, the conclusion of Theorem 4.6 *holds, with "continuous" instead of "$\mathscr{T}$-measurable".*

Moreover, the function f is of class 1 *on $T \times X$, provided that T, X are metrizable.*

PROOF. See the proofs of Theorems 4.4 and 4.7.

THEOREM 4.9. *Let T be as in Theorem* 4.5. *Suppose that:*

(1) *for every $(t, x) \in T \times X$, the set $F(t, x)$ is compact;*

(2) *for every $x \in X$, the multifunction $F(\cdot, x)$ is upper semicontinuous and for each of its continuous selection ϕ, one has* (11).

Under such hypotheses, the conclusion of Theorem 4.8 *holds.*

PROOF. See the proofs of Theorems 4.5, 4.6 and 4.7.

In the next three theorems, X is a compact metric space, with metric d'.

THEOREM 4.10. *Let* $(T, \mathscr{T})$, Z, Y *be as in Theorem* 4.1, Y *being also separable. Suppose that*:

(1) *the set* $F(t, x)$ *is closed (resp., complete, if* $X = Z$*) for every* $(t, x) \in T \times X$ *and convex for every* $T \times (X \backslash Z)$;

(2) *for every* $x \in X$, *the multifunction* $F(\cdot, x)$ *is* $\mathscr{T}$*-measurable*;

(3) *for every* $t \in T$, *the multifunction* $F(t, \cdot)$ *is lower semicontinuous*;

(4) *there exists a topology* τ *on* X, *stronger than the one induced by* d', *and a topology* τ' *on* $F(T \times X)$ *such that*:

(i_1) X *is* τ*-separable*;

(i_2) $F(T \times X)$ *is* τ'*-normal*;

(i_3) *every closed ball in* $F(T \times X)$ *and every set* $F(t, x)$ *are* τ'*-closed*;

(i_4) *for every* $t \in T$, *the multifunction* $F(t, \cdot)$ *is* (τ, τ')*-upper semicontinuous.*

Under such hypotheses, for every $E \in \mathscr{T}$ *and every* $\varphi \in \mathscr{S}(F|_{E\times X})$, *such that, for all* $t \in E$, *the function* $\varphi(t, \cdot)$ is $(d', \|\cdot\|)$*-continuous and that the functions of the family* $\{\varphi(\cdot, x)\}_{x\in X}$ *are equi-*$\mathscr{T}$*-measurable, there exists* $f \in \mathscr{S}(F)$ *such that*:

(1') *for every* $t \in T$, *the function* $f(t, \cdot)$ *is* $(d', \|\cdot\|)$*-continuous*;

(2') *the functions of the family* $\{f(\cdot, x)\}_{x\in X}$ *are equi-*$\mathscr{T}$*-measurable*;

(3') $f|_{E\times X} = \varphi$.

PROOF. Apply Theorem 2.1 by taking: $A = X$; $B = T$; $\Sigma = Y$; $\mathscr{A} = \{\psi \in \mathscr{F}(X, F(T \times X))$: ψ is $(d', \|\cdot\|)$-continuous ; $\mathscr{B} = \{H \in \mathscr{F}(T, 2^{\mathscr{F}(X,Y)})$: $H(T)$ is separable and $H(t)$ is complete for every $t \in T\}$; $\mathscr{E} = \mathscr{D} = \mathscr{G} = \mathscr{T}$; $G(x, t) = F(t, x)$ for every $(x, t) \in X \times T$.

Thus, conditions (α) and (δ) of Theorem 2.1 follow, respectively, from Proposition 3.2 (part (α)) and from Proposition 3.6 (part (γ)). Condition (β) follows from the Theorem on p. 244 of [7]. Finally, from Proposition 3.5 (part (β)), it follows that the multifunctions of the family $\{F(\cdot, x)\}_{x\in X}$ are equi-regular with respect to the pair $(\mathscr{A}', \mathscr{T})$, where $\mathscr{A}' = \{\psi \in \mathscr{F}(X, F(T \times X))$: ψ is $(\tau, \|\cdot\|)$-continuous$\}$, and so, a fortiori, with respect to the pair $(\mathscr{A}, \mathscr{T})$, being $\mathscr{A} \subseteqq \mathscr{A}'$. Our result follows then from Theorem 2.1.

THEOREM 4.11. *Let* T, Z, Y *be as in Theorem* 4.3, Y *being also separable. Suppose that* F *satisfies conditions* (1), (3), (4) *of Theorem* 4.10 *and the following one*:

(2) *for every* $x \in X$ *and every closed ball* Ω *of* Y *the set* $\{t \in T$: $F(t, x) \cap \Omega \neq \emptyset\}$ *is of multiplicative class* (α-1 $\alpha > 1$).

Then, the conclusion of Theorem 4.10 *holds, with E ambiguous of class α and with "equi-belonging to the class α" instead of "equi-$\mathcal{T}$-measurable". Therefore, if T is metrizable, the function f is of class α on $T \times X$.*

PROOF. The proof of the first part of the conclusion is exactly that of Theorem 4.10, except that, in applying Proposition 3.5 (part (β)), we must assume, relatively to T, as $\mathcal{E}$ the additive class α and as $\mathcal{G}^*$ the multiplicative class $\alpha - 1$; moreover, in applying Proposition 3.6 (part (γ)), $\mathcal{E}$ is always the additive class α, $\mathcal{G} = \mathcal{E}$, and $\mathcal{D}$ is the family of all ambiguous sets of class α. Let us prove now the second part of the conclusion. Therefore, let T be metrizable. For every $t \in T$, put $g(t) = f(t, \cdot)$. Denote by $C(X, Y)$ the set of all $(d', \|\cdot\|)$-continuous functions from X into Y. Moreover, put $\psi(t, x) = (g(t), x)$ for every $(t, x) \in T \times X$, and $\omega(h, x) = h(x)$ for every $(h, x) \in C(X, Y) \times X$. Thus, we have $f(t, x) = \omega(\psi(t, x))$. It is clear that the function ω is continuous. Moreover, by Theorem V.1 on p. 377 and Theorem 1 on p. 382 of [7], the function ψ is of class α. Hence, our result follows from Theorem 2 on p. 376 of [7].

THEOREM 4.12. *Let T, Z, Y be as in Theorem* 4.11. *Suppose that F satisfies conditions* (1), (3) *of Theorem* 4.10 *and the following one*:

(2) *for every $x \in X$, the multifunction $F(\cdot, x)$ is pseudo-upper semicontinuous.*

Under such hypotheses, the conclusion of Theorem 4.11 *holds with $\alpha = 1$.*

PROOF. The proof is exactly that of Theorem 4.11, except that we must use part (α) of Proposition 3.5, by taking as $\mathcal{G}^*$ the family of all closed of T.

Now, we make various remarks on the preceding theorems.

REMARK 4.1. Theorems 4.1 and 4.2 essentially specify the analogous results of [**1**].

REMARK 4.2. We don't know, in general, whether, by the equicontinuity of the functions of the family $\{f(t, \cdot)\}_{t \in T}$, the function f in the conclusion of Theorem 4.3 is actually of class α on $T \times X$. In certain cases, this happens (see, for instance, [**14**], p. 241).

REMARK 4.3. Observe that the Michael theory on continuous selections cannot be directly applied to obtain the existence of a continuous selection of a multifunction F satisfying the hypotheses of Theorem. 4.4 or 4.5. In fact, in Theorem 4.4, condition (2) implies that F is lower semicontinuous on $T \times X$, but, as is well-known, the product $T \times X$ need not be paracompact. In Theorem 4.5, in general, F is neither lower semicontinuous nor upper semicontinuous on $T \times X$.

REMARK 4.4. It is easily seen that, in Theorems 4.1–4.5, relation (11) is necessary, that is to say, it is implied by the conclusion.

REMARK 4.5. Observe that Theorem 2 of **[2]**, when it is applied to multifunctions Γ (our F) having weakly compact values and such that $\Gamma(\Omega \times T)$ (our $F(T \times X)$) equipped with the weak topology is a normal space, becomes a particular case of Theorem 4.10. To see this, it suffices to apply Theorem II.20 and Theorem III.30 of **[15]**.

Observe, moreover, that, whenever X is countable, condition (4) of Theorem 4.10 is trivially satisfied, since as τ we can take, in this case, the discrete topology on X and as τ' the same topology induced by the metric on $F(T \times X)$.

REMARK 4.6. Theorem 4.12 improves in several directions Theorem 1 of **[3]**, whenever X (just our X) is a compact metric space. In particular, with regard to condition (2), it is clear that there are pseudo-upper semicontinuous (even single-valued) multifunctions which are not upper semicontinuous.

To conclude, as an application of Theorem 4.12, we present an existence theorem for differential inclusions in separable and reflexive Banach spaces, which, in the finite-dimensional case, reduces to an improved version of Theorem 2 of **[3]**.

Let B be a separable and reflexive Banach space, $x_0 \in B$, $t_0 \in \mathbf{R}$, $a, r \in \mathbf{R}^+$, $F \in \mathscr{F}(I \times \bar{K}_d(x_0, r), 2^B)$, where $I = [t_0, t_0 + a]$.

Denote by d' a metric inducing on $\bar{K}_d(x_0, r)$ the weak topology.

We have the following result

THEOREM 4.13. *Let* $Z \subseteqq (\bar{K}_d(x_0, r))$, *with*

$$\dim_{(\bar{K}_d(x_0, r), d')}(Z) \leqq 0.$$

Suppose that:

(1) *the set* $F(t, x)$ *is closed for every* $(t, x) \in I \times \bar{K}_d(x_0, r)$ *and convex for every* $(t, x) \in I \times (\bar{K}_d(x_0, r) \setminus Z)$;

(2) *the set* $F(I \times \bar{K}_d(x_0, r))$ *is bounded*;

(3) *for every* $x \in X$, *the multifunction* $F(\cdot, x)$ *is pseudo-upper semicontinuous*;

(4) *for every* $t \in I$, *the multifunction* $F(t, \cdot)$ *is* $(d', \|\cdot\|)$*-lower semicontinuous.*

Under such hypotheses, there exists $\delta > 0$ *and at least a strongly absolutely continuous function* φ *from* $[t_0, t_0 + \delta]$ *into* B, *such that* $\varphi'(t) \in F(t, \varphi(t))$ a.e. *in* $[t_0, t_0 + \delta]$, $\varphi(t_0) = x_0$.

PROOF. It suffices to apply the Theorem on p. 438 of **[16]** to the Carathéodory selection f of F whose existence is assured by Theorem 4.12.

REFERENCES

1. B. Ricceri, *Carathéodory's selections for multifunctions with non-separable range*, Rend. Sem. Mat. Univ. Padova, **67** (1982), 185–190.

2. C. Castaing, *Sur l'existence des sections séparément mesurables et séparément continues d'une multi-application*, Séminaire d'Analyse Convexe, Montpellier, Exposé n. **14** (1975).

3. A. Cellina, *A selection theorem*, Rend. Sem. Mat. Univ. Padova, **55** (1976), 143–149.

4. E. Michael, C. Pixley, *A unified theorem on continuous selections*, Pacific J. Math., **87** (1980), 187–188.

5. S.J. Leese, *Measurable selections and the uniformization of Souslin sets*, Amer. J. Math., **100** (1978), 19–41.

6. E. Michael, *Selected selections theorems*, Amer. Math. Monthly, **73** (1956), 233–238.

7. K. Kuratowski, *Topology*, Vol. I, Academic Press (1966).

8. M. Hasumi, *A continuous selection theorem for extremally disconnected spaces*, Math. Ann., **179** (1969), 83–89.

9. E. Michael, *Continuous selections, I*, Ann. of Math., **63** (1956), 361–382.

10. E. Michael, *A theorem on semi-continuous set-valued functions*, Duke Math. J., **26** (1959), 647–651.

11. C. Berge, *Espaces topologiques*, Dunod (1959).

12. R. Engelking, *Selectors of the first Baire class for semicontinuous set-valued functions*, Bull. Acad. Polon. Sci., Sér. sci. math. astronom. et phys., **16** (1968), 277–282.

13. K. Kuratowski, C. Ryll-Nardzewski, *A general theorem on selectors*, Bull. Acad. Polon. Sci., Sér. sci. math. astronom. et phys., **13** (1965), 397–403.

14. Z. Grande, *Quelques remarques sur les classes de Baire des functions de deux variables*, Math. Slovaca, **26** (1976), 241–246.

15. C. Castaing, M. Valadier, *Convex analysis and measurable multifunctions*, Lecture notes in Math., vol. **580**, Springer-Verlag (1977).

16. W.J. Knight, *Solutions of differential equations in B-spaces*, Duke Math. J., **41** (1974), 437–442.

17. Z. Grande, *Sur un problème de Ricceri*, Coll. Math, to appear.

SEMINARIO MATEMATICO, CITTA UNIVERSITARIA, VIALE A. DORIA 6, 95125 CATANIA, ITALY