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A CHARACTERIZATION OF RATIONAL AND ELLIPTIC REAL ALGEBRAIC CURVES IN TERMS OF THEIR SPACE OF ORDERINGS*

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1. Introduction. D.O.P. fields were introduced by Dubois and Recio in [3] as a class of fields whose orderings are all "almost" isomorphic (see 2.1 for a precise definition). There they prove that finitely generated pure transcendental extensions of real closed fields are D.O.P.

In a more general setting, we study here under what conditions the field of rational functions of an algebraic variety V of $\mathbf{R}^n$ is D.O.P. We obtain a necessary condition for arbitrary dimension (prop. 2.4) and a necessary and sufficient condition for curves (theorem 3.1).

The D.O.P. property turns out to be closely related to the automorphism group of the field of rational functions of the variety and consequently to Klein's surfaces and N.E.C. groups (see [2] and [5] for the basic definitions).

It will be seen in the proof of (3.2.1) that given two points of a real elliptic curve C there exists a birational morphism of C sending one of them to the other. This is a well known result if the ground field is algebraically closed (see [4]), but it is an open problem (as far as I know) working over real closed ground fields (different from $\mathbf{R}$).

2. Preliminaries and necessary condition. First we introduce some notation. Let K be a formally real field. We let Ω be the order space of K endowed with Harrison's topology. Let G stand for the automorphism group of K, and $x_1, \ldots, x_n$ for indeterminates.

DEFINITION 2.1. A formally real field K is D.O.P. (or has the dense orbits property) if for each ordering $\alpha \in \Omega$ and each open $H \subset \Omega$ there exists an automorphism $\sigma \in G$ such that $\sigma(\alpha) \in H$.

From [3] we need the following result.

PROPOSITION 2.2. *If R is a real closed field, then $R(x_1, \ldots, x_n)$ is* D.O.P.

On the other hand, in dealing with algebraic varieties we shall consider

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either bounded or unbounded birational models. The next lemma gives the convenient method.

LEMMA 2.3. *Let K be a real function field over* **R**. *Then* (a) *There exists a compact algebraic variety V over* **R** *whose field of rational functions is K.*

(b) *There exists a non-compact algebraic variety V over* **R** *whose field of rational functions is K.*

PROOF. (a) Take an algebraic variety $W \subsetneqq \mathbf{R}^n$ whose field of rational functions $\mathbf{R}(W)$ is K. Let ϕ be the inversion in $\mathbf{R}^n$ with respect to a sphere of center $a \in \mathbf{R}^n \backslash W$ and radius r small enough so that $W \cap \{x \in \mathbf{R}^n: \|x - x\| \leqq r\} = \varnothing$. Then $V = \phi(W)$ is the solution.

(b) Take an algebraic variety $W \subsetneqq \mathbf{R}^n$ with $\mathbf{R}(W) = K$. Consider the projective variety $W^* \subset \mathbf{P}_n(\mathbf{R})$ corresponding to W and a regular point $a \in W^*$. Then it is enough to take as V the affine algebraic variety of $\mathbf{R}^n$ obtained from W^* by choosing as infinite hyperplane the tangent to W^* at a. Moreover, the set of central points V_c of V is non-compact.

Now we obtain the result.

PROPOSITION 2.4. *Let V be a* D.O.P. *algebraic variety over* **R**. *Then the automorphism group $G =$* Aut $\mathbf{R}(V)$ *is infinite.*

PROOF. By (2.3.b) we may assume $V_c \subsetneqq \mathbf{R}^n$ unbounded. Let us suppose G finite, $G = \{\sigma_1, \ldots, \sigma_s\}$ and consider the corresponding continuous functions: σ_j: $V^* \to \mathbf{R}^n$ where V^* is an open dense subset of V_c. We choose $h \in \mathbf{R}[V]$ such that $W = \{a \in V: h(a) \geqq 0\}$ is compact, contained in V^* and $W \cap V_c \neq \varnothing$ (for instance consider the closure of a small open ball around a point $a_0 \in V^* \cap V_c$). Clearly the clopen $H = H(h) \subset \Omega$ is non-empty. Since V is D.O.P. we have $\Omega = \sigma_1(H) \cup \cdots \cup \sigma_s(H)$, and from this we deduce (see [3]) $V_c \subset \sigma_1(W) \cup \cdots \cup \sigma_s(W)$. As W is compact, so is V_c and we obtain an absurdity. This ends the proof.

3. The main theorem. Now, we can state the result.

THEOREM 3.1. *Let C be an algebraic curve over* **R**. *Then V is* D.O.P. *if and only if V is either rational or elliptic.*

PROOF. First of all we shall prove the result.

(3.2) If C has genus greater than or equal to 2, then C is not D.O.P.

Indeed, by (2.4) it suffices to prove that $G =$ Aut $\mathbf{R}(C)$ is finite. Let $\mathscr{X}$ be the Klein surface corresponding to C (see [2]. Since $\mathbf{R}(C)$ is formally real, $\mathscr{X}$ has non-empty boundary. Then, by a result of Preston [7], there exists an N.E.C. group Γ such that $\mathscr{X} = \mathrm{D}/\Gamma$ where D is the hyperbolic plane, and consequently G is a subgroup of the automorphism group G'

of D/Γ. Let Δ be the group of isometries of D. We know from **[6]** that $G \approx N/\Gamma$, where N is the normalizer of Γ in Δ. Since N is N.E.C. and the algebraic genus of D/Γ is greater than or equal to 2, we conclude by **[5]** that G is finite. This proves (3.1).

On the other hand, if V is rational, then $\mathbf{R}(C) = \mathbf{R}(x_1)$ and C is D.O.P. by (2.2). Finally it just remains to show

(3.3) If C has genus 1, then C is D.O.P.

By (2.3.a) we may assume that C is compact. Now let α be an ordering in $\mathbf{R}(C)$. Since C is compact, there exists $a \in C$ such that every $f \in R[C]$ with $f(a) > 0$ is positive in α. Let H be a non-empty open set in the order space Ω. We may assume $H = H(f_1, \ldots, f_r)$ for some $f_1, \ldots, f_r \in R[C]$, and then

$$S = \{x \in C : f_1(x) > 0, \ \ldots, f_r(x) > 0\} \neq \varnothing.$$

We shall later prove the following result.

(3.3.1) There exists a birational morphism $\sigma : C \to C$ such that $\sigma(a) \in S$.

Then let $\sigma: \mathbf{R}(C) \to \mathbf{R}(C)$ be the automorphism of $\mathbf{R}(C)$ corresponding to σ. We claim $\sigma^{-1}(\alpha) \in H$.

For, if $\sigma^{-1}(\alpha) \notin H$, some f_i, say f_1, is negative in $\sigma^{-1}(\alpha)$ and consequently $-\sigma(f_1) \in \alpha$. This implies $\sigma(f_1)(a) \leqq 0$ or, equivalently, $f_1(\sigma(a)) \leqq 0$. As $\sigma(a) \in S$, this is impossible.

Finally, let us prove (3.3.1).

Consider the Klein surface $\mathscr{X}$ associated to C. Since the genus of C is 1, there exists a lattice $L_\tau = \mathbf{Z} \oplus \tau Z$ of $\mathbf{C}$ such that $\mathscr{X} = \mathbf{C}/L_\tau$.

Let $\pi: \mathbf{C} \to \mathscr{X}$ be the canonical morphism and let G_τ be the group of automorphisms σ of $\mathbf{C}$ compatible with π (see **[1]**).

Then for each $\sigma \in G_\tau$ there exists a unique automorphism σ^* of $\mathscr{X}$ such that

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{\ \sigma\ } & \mathbf{C} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathscr{X} & \xrightarrow{\ \sigma^*\ } & \mathscr{X} \end{array}$$

commutes.

From 13.15 in **[1]** we know that

$$\pi_* \text{ (Trans } \mathbf{R}) \subset \operatorname{Aut}^+_\xi(\mathscr{X}') \approx \operatorname{Aut} \mathscr{X}$$

where $\pi_*: G \to \operatorname{Aut}(\mathscr{X}): \sigma \mapsto \sigma^*$, Trans $\mathbf{R}$ is the set of automorphisms $f_b: \mathbf{C} \to \mathbf{C}: z \mapsto z + b$, ($b$ a real number), $(\mathscr{X}', \xi)$ is the double covering of $\mathscr{X}$ and $\operatorname{Aut}^+_\xi(\mathscr{X}')$ is the set of automorphisms of $\mathscr{X}'$ preserving orientation and such that $f \circ \xi = \xi \circ f$.

Thus, given two points a, b in C and identifying C with the boundary $\partial\mathscr{X}$ of $\mathscr{X}$, the image by π_* of the translation associated to $b - a$ is a birational morphism sending a to b. In particular, taking b any point in S we prove (3.3.1).

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