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EXISTENCE AND UNIQUENESS OF SOLUTIONS OF RIGHT FOCAL POINT BOUNDARY VALUE PROBLEMS FOR THIRD AND FOURTH ORDER EQUATIONS

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1. Introduction. In this paper we consider the existence and uniqueness of solutions of right focal point boundary value problems for third and fourth order equations. To relate this to earlier results concerning right focal point boundary value problems, we shall first formulate such problems for equations of arbitrary order; thus, we shall be concerned with solutions of the equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \tag{1}$$

satisfying boundary conditions of the form $y^{(i-1)}(x_i) = y_i$, $1 \leqq i \leqq n$, where $a < x_1 \leqq x_2 \leqq \cdots \leqq x_n < b$. Such a problem is called a right focal point boundary value problem for (1) on (a, b). To be more precise, we give the definition, as it appears in [6, 7], of a right $(m_1, \ldots, m_r)$-focal point boundary value problem for (1) on (a, b).

DEFINITION. Let $2 \leqq r \leqq n$ and let m_i, $1 \leqq i \leqq r$, be positive integers such that $\sum_{i=1}^r m_i = n$. Let $s_0 = 0$ and for $1 \leqq k \leqq r$, let $s_k = \sum_{i=1}^k m_i$. A boundary value problem for (1) with boundary conditions

$$y^{(i)}(x_k) = y_{ik}, \ s_{k-1} \leqq i \leqq s_k - 1, \ 1 \leqq k \leqq r,$$

where $a < x_1 < \cdots < x_r < b$, is called a right $(m_1, \ldots, m_r)$-focal point boundary value problem for (1) on (a, b).

In addition to [6, 7], for results related to this type of boundary value problem, see for example Muldowney [13, 14], Nehari [15], Elias [2, 3], Jackson [10], and Peterson [16. 17]. Commas appear in the notation $(m_1, \ldots, m_r)$, instead of semicolons, to distinguish this concept from a similar but different concept used by Peterson [18].

Now if (1) is linear, the uniqueness of solutions of a particular right $(m_1, \ldots, m_r)$-focal point boundary value problem implies the existence of solutions of the same type problem for any assignment of y_{ik}. In [7], a type of "uniqueness implies existence" result for right focal point bound-

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ary value problems is established for nonlinear equations. More specifically, in [7], we prove the following theorem.

THEOREM 1.1. *Assume that with respect to* (1) *the following conditions are satisfied*:
(A) $f(x, y_1, y_2, \ldots, y_n)$ *is continuous on* $(a, b) \times \mathbf{R}^n$,
(B) *Solutions of initial value problems for* (1) *are unique*,
(C) *Solutions of initial value problems for* (1) *extend to* (a, b),
(D_1) *Each right* $(1, 1, \ldots, 1)$-*focal point boundary value problem for* (1) *on* (a, b) *has at most one solution, and*
(E) *If* $\{y_k(x)\}$ *is a sequence of solutions of* (1) *and* $[c, d]$ *is a compact subinterval of* (a, b) *such that* $\{y_k(x)\}$ *is uniformly bounded on* $[c, d]$, *then there exists a subsequence* $\{y_{k_j}(x)\}$ *such that* $\{y_{k_j}^{(i)}(x)\}$ *converges uniformly on* $[c, d]$, *for each* $0 \leqq i \leqq n - 1$.
Then all right $(m_1, \ldots, m_r)$-*focal point boundary value problems*, $2 \leqq r \leqq n$, *for* (1) *on* (a, b) *have unique solutions.*

Theorem 1.1 is analogous to a uniqueness implies existence result for k-point boundary value problems due to Hartman [**4**, **5**] and Klaasen [**12**]. Moreover, for k-point boundary value problems, consideration has been given to converse questions. In particular, Jackson [**9**] proved that for the nonlinear third order equation

$$y''' = f(x, y, y', y''), \tag{2}$$

if conditions (A) and (C) are satisfied and if all 2-point boundary value problems for (2) on (a, b) have at most one solution, then all 3-point and all 2-point boundary value problems for (2) on (a, b) have solutions which are unique.

Then in [**8**], Henderson and Jackson proved a theorem which gives sufficient conditions under which the existence of solutions of all 2-point boundary value problems for (1) on (a, b) implies the existence of solutions of all k-point boundary value problems for (1) on (a, b), for each $2 \leqq k \leqq n$. In particular, the following theorm was proven.

THEOREM 1.2. *If* (1) *satisfies* (A), (B), (C) *and* (E), *if all 2-point boundary value problems for* (1) *on* (a, b) *have solutions, and if all* $(n - 1)$-*point boundary value problems for* (1) *on* (a, b) *have at most one solution, then all k-point boundary value problems*, $2 \leqq k \leqq n$, *for* (1) *on* (a, b) *have solutions which are unique.*

In §2 of this paper, we prove the corresponding theorem of Jackson [**9**] for right focal point boundary value problems for (2). Then in §3, we prove the analogue of Theorem 1.2 for right focal point boundary value problems for the fourth order equation

(3) $$y^{(4)} = f(x, y, y', y'', y''').$$

Although, the complete analogue of Theorem 1.2 for equations of arbitrary order is not obtained here, some generalizations of the result established for (3) are given in §4.

Moreover, sufficient conditions are given under which the compactness condition (E) is valid for the respective equations (2) and (3).

2. The third order equation. Using techniques similar to those employed by Jackson [9], we obtain a converse to Theorem 1.1 for the third order equation

(2) $$y''' = f(x, y, y', y'').$$

We will assume at time at times all or some of the conditions, (A), (B) and (C) are satisfied with respect to (2). In addition, we will have need of the condition:

(D_2) Each right (m_1, m_2)-focal point boundary value problem for (2) on (a, b) has at most one solution.

THEOREM 2.1. *Assume that equation* (2) *satisfies conditions* (A), (B), (C), *and* (D_2). *Then* (2) *satisfies condition* (D_1); *that is, each right* (1, 1, 1)-*focal point boundary value problem for* (2) *on* (a, b) *has at most one solution.*

PROOF. Assume the hypotheses of the Theorem are satisfied, but assume that the conclusion is false. Then there are points $a < x_1 < x_2 < x_3 < b$ and distinct solutions $y(x)$ and $z(x)$ of (2) satisfying $y^{(i-1)}(x_i) = z^{(i-1)}(x_i)$, $i = 1, 2, 3$. As a consequence of (D_2), we may assume that $y'(x_1) > z'(x_1)$. Further, we may assume that $y^{(i)}(x) - z^{(i)}(x) \neq 0$ in (x_i, x_{i+1}), for $i = 1, 2$. From this and (D_2), it follows that $y'(x) > z'(x)$ on $[x_1, x_2)$ and $y''(x) < z''(x)$ on $[x_2, x_3)$. Consequently $y'(x) < z'(x)$ on $(x_2, x_3]$.

Now let $y_\varepsilon(x)$, for $\varepsilon > 0$, denote the solution of (2) which satisfies $y_\varepsilon^{(i)}(x_1) = y^{(i)}(x_1)$, $i = 0, 1$, and $y_\varepsilon''(x_1) = y''(x_1) + \varepsilon$. One can then establish from condition (D_2) and continuous dependence of solutions on initial conditions that, for each $\varepsilon > 0$, there exists an interval $[x_2(\varepsilon), x_3(\varepsilon)] \subset (x_2, x_3)$ such that $y_\varepsilon'(x) > z'(x)$ on $[x_1, x_2(\varepsilon))$, $y'(x_2(\varepsilon)) = z'(x_2(\varepsilon))$, $y'(x) < y_\varepsilon'(x) < z'(x)$ on $(x_2(\varepsilon), x_3(\varepsilon)]$, $y_\varepsilon''(x_3(\varepsilon)) = z''(x_3(\varepsilon))$, and $y''(x) < y_\varepsilon''(x) < z'(x)$ on $[x_2(\varepsilon), x_3(\varepsilon))$. Furthermore, the intervals are nested in that $[x_2(\varepsilon_2), x_3(\varepsilon_2)] \subset (x_2(\varepsilon_1), x_3(\varepsilon_1))$, whenever $0 < \varepsilon_1 < \varepsilon_2$. Such intervals are obtained by simply choosing $x_2(\varepsilon)$ to be the first zero of $y_\varepsilon'(x) - z'(x)$ in (x_1, x_3) and then choosing $x_3(\varepsilon)$ to be the first zero of $y_\varepsilon''(x) - z''(x)$ in $(x_2(\varepsilon), x_3)$.

Then there is a point x_0 in $\bigcap_{k=1}^{\infty} [x_2(k), x_3(k)]$, and the sequences $\{y_k'(x_0)\}_{k=1}^{\infty}$ and $\{y_k''(x_0)\}_{k=1}^{\infty}$ are bounded since $y'(x_0) < y_k'(x_0) < z'(x_0)$ and $y''(x_0) < y_k''(x_0) < z''(x_0)$, for all $k \geqq 1$. Moreover, from (D_2), for each $k \geqq 1$, $y''(x) < y_k''(x)$ on $[x_1, b)$ and thus, $y^{(i)}(x) < y_k^{(i)}(x)$ on

(x_1, b), $i = 0, 1$, and both $y_k'(x) - y'(x)$ and $y_k(x) - y(x)$ are increasing on $[x_1, b)$. Since $\{y_k'(x_0)\}$ is a bounded sequence, it follows that there is an $M > 0$ such that $|y_k'(x)| \leqq M$, for all $x \in [x_1, x_0]$ and all $k \geqq 1$. Furthermore, $y_k(x_1) = y(x_1)$, for all $k \geqq 1$; thus

$$\begin{aligned} |y_k(x_0)| &\leqq \int_{x_1}^{x_0} |y_k'(s)|ds + |y(x_1)| \\ &\leqq M(b - a) + |y(x_1)|, \end{aligned}$$

for all $k \geqq 1$. In particular, we now have that the sequences $\{y_k^{(i)}(x_0)\}_{k=1}^{\infty}$, $i = 0, 1, 2$, are bounded sequences. It follows that there is a subsequence $\{y_{k_j}(x)\}$ such that $\{y_{k_j}^{(i)}(x)\}$ converges uniformly on each compact subinterval of (a, b). for each $i = 0, 1, 2$. This is contradictory to the fact that $y_{k_j}''(x_1) = y''(x_1) + k_j \to +\infty$. Hence our assumption concerning the existence of distinct solutions $z(x)$ and $y(x)$ is false, and the proof is complete.

Although we did not have to make use of the compactness condition (E) in the proof of Theorem 2.1, it nevertheless is satisfied under the conditions of the Theorem. The proof proceeds much like the one given in [**11**]. We state then without proof the following theorem.

THEOREM 2.2. *Assume that with respect to* (2), *conditions* (A), (C), *and* (D_2) *are satisfied. If* $[c, d]$ *is a compact subinterval of* (a, b) *and if* $\{y_k(x)\}$ *is a sequence of solutions of* (2) *such that* $|y_k(x)| \leqq M$ *on* $[c, d]$, *for some* $M > 0$ *and all* $k \geqq 1$, *then there is a subsequence* $\{y_{k_j}(x)\}$ *such that* $\{y_{k_j}^{(i)}(x)\}$ *converges uniformly on* $[c, d]$, *for each* $i = 0, 1, 2$.

In view of Theorem 1.1, and as a consequence of Theorem 2.1, our analogue of Jackson's [**9**] converse theorem for right focal point boundary value problems for (2) is immediate.

THEOREM 2.3. *Assume that* (2) *satisfies conditions* (A), (B), (C), *and* (D_2). *Then all right* (m_1, m_2)*-and all right* $(1, 1, 1)$*-focal point boundary value problems for* (2) *on* (a, b) *have solutions which are unique.*

3. The fourth order equation. In this section we prove the analogue of Theorem 1.2 for right focal point boundary value problems for fourth order equations. Hence, we will now be concerned with solutions of

$$y^{(4)} = f(x, y, y', y'', y'''). \tag{3}$$

With respect to (3), we will have need of hypotheses (A), (B), and (C). Moreover, the following condition will be required at times.

(D_3) Each right (m_1, m_2, m_3)-focal point boundary value problem for (3) on (a, b) has at most one solution.

In some of the proofs in this section reference is made to the continuous dependence of solutions of (3) on boundary conditions. Verification of this dependence relies on (B), uniqueness of solutions of a particular type of boundary value problem, and the Brouwer Invariance of Domain Theorem [**1**, p. 156]. Typical arguments verifying this dependence can be found in for example, [**6**] or [**8**].

THEOREM 3.1. *Assume that equation* (3) *satisfies* (A), (B), *and* (D_3). *Then all right* (1, 3)- *and all right* (2, 2)-*focal point boundary value problems for* (3) *on* (a, b) *have at most one solution.*

PROOF. We deal first with uniqueness of solutions of right (1, 3)-focal point problems for (3). Assume in this case that the conclusion is false. Then there exist distinct solutions $y(x)$ and $z(x)$ of (3) and points $a < x_1 < x_2 < b$ such that $z(x_1) = y(x_1)$ and $z^{(i)}(x_2) = y^{(i)}(x_2)$, $i = 1, 2, 3$. Let $(\omega^-, \omega^+) \subseteq (a, b)$ be the common interval of existence of $y(x)$ and $z(x)$. Let x_0 and x_3 be arbitrary but fixed points with $\omega^- < x_0 < x_1 < x_2 < x_3 < x < \omega^+$. As a consequence of ($D_3$) and Rolle's Theorem $z(x) - y(x)$ has a simple zero at $x = x_1$, and moreover, we may assume that $z'''(x) > y'''(x)$ on (x_2, ω^+). Given $\delta > 0$, let $u_\delta(x)$ be the solution of the initial value problem for (3) satisfying $u_\delta^{(i)}(x_2) = z^{(i)}(x_2)$, $i = 0, 1, 2$, and $u_\delta'''(x_2) = z'''(x_2) - \delta$. Solutions of (3) depend continuously upon initial conditions, and so for δ sufficiently small there exist $t_1 \in (x_0, x_2)$ and $t_2 \in (x_2, x_3)$ such that $u_\delta(t_1) = y(t_1)$ and $u_\delta'''(t_2) = y'''(t_2)$. However, we also have $u_\delta^{(i)}(x_2) = y^{(i)}(x_2)$, $i = 1, 2$, which is contradictory to (D_3). This disposes of the case concerning the uniqueness of solutions of each right (1, 3)-focal point problem.

We now deal with the right (2, 2)-focal point problem for (3). Assume again that the conclusion of the Theorem is false. Then there are distinct solutions $y(x)$ and $z(x)$ of (3) and points $a < x_1 < x_2 < b$ such that $z^{(i)}(x_1) = y^{(i)}(x_1)$, $i = 0, 1$, $z^{(i)}(x_2) = y^{(i)}(x_2)$, $i = 2, 3$, Let (ω^-, ω^+) and x_0, x_3 be as above. There are two cases to consider.

Case 1. $y(x) - z(x)$ has a zero of order 2 at $x = x_1$. From this and (D_3), we may assume that $y''(x) > z''(x)$ on $[x_1, x_2)$. Consequently $y^{(i)}(x) > z^{(i)}(x)$ on $(x_1, x_2]$, for $i = 0, 1$. Now given $\delta > 0$, let $u_\delta(x)$ be the solution of the initial value problem for (3) satisfying $u_\delta(x_2) = y(x_2) - \delta$, and $u_\delta^{(i)}(x_2) = y^{(i)}(x_2)$, $i = 1, 2, 3$. Uniqueness of solutions of right (1, 3)-focal point problems for (3) implies that $u_\delta(x) < y(x)$ on any common interval of existence to the left of x_2. Again solutions of (3) depend continuously upon initial conditions, and so it follows that, for δ sufficiently small, there exist $x_0 < t_1 < x_1 < t_2 < x_2$ such that $u_\delta(t_i) = z(t_i)$, $i = 1, 2$. Since $u_\delta^{(i)}(x_2) = z^{(i)}(x_2)$, $i = 2, 3$, it follows from Rolle's Theorem that (D_3) is contradicted. This disposes of Case 1.

Case 2. $y(x) - z(x)$ has a zero of order 3 at $x = x_1$, (thus, it is also true that $y''(x_1) = z''(x_1)$). Assume $y'''(x_1) > z'''(x_1)$. Again as a consequence of

(D_3), we may assume that $y''(x) > z''(x)$ on (x_1, x_2). There are two further subcases.

Subcase 2a. Assume $y'''(x) < z'''(x)$ on (x_2, ω^+). Then $y''(x) < z''(x)$ on (x_2, ω^+) and consequently, $y''(x) - z''(x)$ changes sign at $x = x_2$. In this case, given $\delta > 0$, let $u_\delta(x)$ be the solution of the initial value problem for (3) with initial conditions $u_\delta^{(i)}(x_1) = y^{(i)}(x_1)$, $i = 0, 1, 3$, and $u_\delta''(x_1) = y''(x_1) - \delta$. Again by (B), for δ sufficiently small, there exist $x_1 < t_1 < t_2 < x_3$ such that $u_\delta(t_1) = z(t_1)$ and $u_\delta''(t_2) = z''(t_2)$. However, in this consideration $u_\delta^{(i)}(x_1) = z^{(i)}(x_1)$, $i = 0, 1$, znd by repeated applications of Rolle's Theorem, we obtain a contradiction to the uniqueness of solutions of right (2, 1, 1)-focal point boundary value problems for (3).

Subcase 2b. Assume $y'''(x) > z'''(x)$ on (x_2, ω^+). Then $y''(x) > z''(x)$ on (x_2, ω^+), and hence $y''(x) - z''(x)$ does not change sign at $x = x_2$. It follows that $y''(x) - z''(x)$ attains a positive maximum at some point $c_0 \in (x_1, x_2)$, and thus, there are points $c_0 < t_1 < x_2 < t_2 < x_3$ such that $y'''(t_1) < z'''(t_1)$ and $y'''(t_2) > z'''(t_2)$, (recall $y'''(x) > z'''(x)$ on (x_2, ω^+)). Now for each $\delta > 0$, we let $u_\delta(x)$ be the solution of the initial value problem for (3) such that $u_\delta^{(i)}(x_1) = y^{(i)}(x_1)$, $i = 0, 1, 3$, and $u_\delta''(x_1) = y''(x_1) - \delta$. Again using continuous dependence of solutions of (3) upon initial conditions, we conclude that there exist $x_1 < t_3 < t_1 < t_4 < t_2$ such that $u_\delta(t_3) = z(t_3)$ and $u_\delta'''(t_4) = z'''(t_4)$. Since $u_\delta^{(i)}(x_1) = z^{(i)}(x_1)$, $i = 0, 1$, Rolle's Theorem leads to a contradiction of (D_3).

This shows that case 2 is also impossible and thus the conclusion concerning uniqueness of solutions of right (2, 2)-focal point boundary value problems for (3) is valid.

We now give conditions sufficient for the validity of hypothesis (D_1) with respect to equation (3).

THEOREM 3.2. *If equation* (3) *satisfies conditions* (A), (B), (C), *and* (D_3), *and if all right* (m_1, m_2)*-focal point boundary value problems for* (3) *on* (a, b) *have solutions, then all right* (1, 1, 1, 1)*-focal point boundary value problems for* (3) *on* (a, b) *have at most one solution.*

PROOF. As a consequence of the hypotheses and Theorem 3.1, all right (1, 3)- and all right (2, 2)-focal point problems for (3) have unique solutions on (a, b).

Now assume that the conclusion of the Theorem is false. Let $y(x)$ and $z(x)$ be distinct solutions of (3) such that, for some $a < x_1 < x_2 < x_3 < x_4 < b$, $y^{(i-1)}(x_i) = z^{(i-1)}(x_i)$, $i = 1, 2, 3, 4$. Because of (D_3), x_i is a simple zero of $y^{(i-1)}(x) - z^{(i-1)}(x)$, for $i = 1, 2, 3$. We may assume without loss of generality that $y^{(i)}(x) - z^{(i)}(x) \neq 0$ in (x_i, x_{i+1}), for $i = 1, 2, 3$. Moreover, let's assume the case where $y'(x_1) < z'(x_1)$. It follows that $y''(x_2) > z''(x_2)$, $y'''(x_3) < z'''(x_3)$, that $y'(x) < z'(x)$ on $[x_1, x_2)$, $y''(x) >$

$z''(x)$ on $[x_2, x_3)$, and that $y'''(x) < z'''(x)$ on $[x_3, x_4)$. Notice here that this last inequality implies $y''(x) < z''(x)$ on $(x_3, x_4]$.

The argument now proceeds much like the one presented in Theorem 2.1. Let $y_\varepsilon(x)$, for $\varepsilon > 0$, denote the solution of (3) which satisfies the right (1, 3)-focal point boundary conditions $y_\varepsilon(x_1) = y(x_1)$, $y_\varepsilon^{(i)}(x_2) = y^{(i)}(x_2)$, $i = 1, 2$, and $y_\delta'''(x_2) = y'''(x_2) + \varepsilon$. From (B), uniqueness of solutions of right (1, 3)-focal point problems, and an application of the Brouwer Invariance of Domain Theorem, one can conclude that solutions of right (1, 3)-focal point problems depend continuously upon this type of boundary condition. From this and the fact that right (1, 1, 2)-focal point boundary value problems have at most one solution, one can establish that, for each $\varepsilon > 0$, there is an interval $[x_3(\varepsilon), x_4(\varepsilon)] \subset (x_3, x_4)$ such that $y_\varepsilon''(x) > z''(x)$ on $[x_2, x_3(\varepsilon))$, $y''(x_3(\varepsilon)) = z''(x_3(\varepsilon))$, $y''(x) < y_\varepsilon''(x) < z''(x)$ on $(x_3(\varepsilon), x_4(\varepsilon)]$, $y_\varepsilon'''(x_4(\varepsilon)) = z'''(x_4(\varepsilon))$, and $y'''(x) < y_\varepsilon'''(x) < z'''(x)$ on $[x_3(\varepsilon), x_4(\varepsilon))$. Again, the intervals are nested in that $[x_3(\varepsilon_2), x_4(\varepsilon_2)] \subset (x_3(\varepsilon_1), x_4(\varepsilon_1))$, whenever, $0 < \varepsilon_1 < \varepsilon_2$.

Then there is a point x_0 in $\bigcap_{k=1}^{\infty} [x_3(k), x_4(k)]$, and the sequences $\{y_k''(x_0)\}_{k=1}^{\infty}$ and $\{y_k'''(x_0)\}_{k=1}^{\infty}$ are bounded. Arguing as in Theorem 2.1, the sequence $\{y_k'(x_0)\}_{k=1}^{\infty}$ is also bounded. Then there exists a subsequence of positive integers $\{k_j\}_{j=1}^{\infty} \subset \{k\}_{k=1}^{\infty}$, and there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$ such that, $y_{k_j}^{(i)}(x_0) \to \alpha_i$, $i = 1, 2, 3$.

Now let $u(x)$ denote the solution of the right (1, 3)-focal point boundary value problem for (3) such that $u(x_1) = y(x_1)$, and $u^{(i)}(x_0) = \alpha_i$, $i = 1, 2, 3$. It follows from the continuous dependence on boundary conditions of solutions of right (1, 3)-focal point problems that $\{y_{k_j}^{(i)}(x)\}$ converges uniformly to $u^{(i)}(x)$ on each compact subinterval of (a, b), for $i = 0, 1, 2, 3$. This contradicts the fact that $y_{k_j}'''(x_2) \to +\infty$, and hence our initial assumption concerning the existence of distinct solutions $y(x)$ and $z(x)$ is false. The conclusion of the Theorem follows.

In [**8**], it is mentioned that the compactness condition (E) holds under the other hypotheses of Theorem 1.2 for fourth order equations. The corresponding statement is true here for right focal point boundary value problems for (3) as the next two results show.

LEMMA 3.3. *Assume the hypotheses of Theorem* 3.2. *Then all right* (2, 1, 1)*-focal point boundary value problems for* (3) *on* (a, b) *have unique solutions.*

PROOF. Uniqueness of any such solution is from (D_3). Let $a < x_1 < x_2 < x_3 < b$ and $y_i \in \mathbf{R}$, $i = 1, 2, 3, 4$, be given. As a consequence of the hypotheses and Theorem 3.1, there exists a unique solution, $z(x)$ of the right (2, 2)-focal point boundary value problem for (3) such that

$$z(x_1) = y_1,$$
$$z'(x_1) = y_2,$$
$$z''(x_3) = 0,$$
and
$$z'''(x_3) = y_4.$$

Define $S(z) \equiv \{y(x) \mid y(x)$ is a solution of (3) with $y(x_1) = z(x_1)$, $y'(x_1) = z'(x_1)$, and $y'''(x_3) = z'''(x_3)\}$. Then let $S = \{y''(x_2) \mid y(x) \in S(z)\}$. S is a nonempty set since $z''(x_2) \in S$.

S is also an open set. To see this, let $s \in S$ and let $y_s(x) \in S(z)$ where $y_s''(x_2) = s$. It follows from (B), (D_3) and the Brouwer Invariance of Domain Theorem that there is a $\delta_0 > 0$ such that $|x_j - t_j| < \delta_0$ for $j = 1, 2, 3$, $|y_s^{(i-1)}(x_1) - c_i| < \delta_0$, for $i = 1, 2$, $|y_s''(x_2) - c_3| < \delta_0$ and $|y_s'''(x_3) - c_4| < \delta_0$ imply that there is a solution $u(x)$ of (1) satisfying

$$u^{(i-1)}(t_1) = c_i,\ i = 1, 2,$$
$$u''(t_2) = c_3,$$
and
$$u'''(t_3) = c_4.$$

In particular, $(s - \delta_0, s + \delta_0) \subseteq S$ and hence S is an open set.

We claim moreover that S is also a closed subset of the reals. Assume that this is not the case. Then S has a limit point $r_0 \notin S$, and there is a strictly monotone sequence $\{r_n\} \subset S$ which converges to r_0. We will deal with the case where $r_n \uparrow r_0$ since the argument for the other case is similar. From the manner in which S is defined, it follows that there is a sequence of solutions $\{y_n(x)\} \subset S(z)$ such that $y_n''(x_2) = r_n$, for $n \geqq 1$. (D_3) and the uniqueness results of Theorem 3.1 concerning right (2, 2)-focal point problems imply that $y_n''(x) < y_{n+1}''(x)$ on $(x_1, x_3]$.

We claim now that $\{y_n''(x)\}$ is not bounded above on any compact subinterval of $(x_1, x_3]$. Assume that this claim is also false. Then there exists $[c, d] \subset (x_1, x_3]$ and $M > 0$ such that $|y_n''(x)| \leqq M$, for all $x \in [c, d]$ and all $n \geqq 1$. Consequently, for each $n \geqq 1$, there exists $x_n \in (c, d)$ such that $|y_n'''(x_n)| \leqq 2M/(d - c)$. Then there is a subsequence $\{n_k\}_{k=1}^{\infty} \subseteq \{n\}_{n=1}^{\infty}$, $x_0 \in [c, d]$, and $\alpha_1, \alpha_2 \in \mathbf{R}$ such that $x_{n_k} \to x_0$, $y_{n_k}''(x_{n_k}) \to \alpha_1$, and $y_{n_k}'''(x_{n_k}) \to \alpha_2$.

Now let $v(x)$ be the solution of the right (2, 2)-focal point boundary value problem for (3) with boundary conditions $v^{(i)}(x_1) = z^{(i)}(x_1)$, $i = 0, 1$, $v''(x_0) = \alpha_1$, and $v'''(x_0) = \alpha_2$. From the hypotheses, uniqueness of solutions of right (2, 2)-focal point problems, and the Brouwer Invariance of Domain Theorem, we conclude that $\{y_{n_k}^{(i)}(x)\}$ converges uniformly to $v^{(i)}(x)$ on each compact subinterval of (a, b), for $i = 0, 1, 2, 3$. This implies however that $v''(x_2) = r_0$ and $v'''(x_3) = z'''(x_3)$, which in turn yields the contradiction, $r_0 \in S$. Thus the claim $\{y_n''(x)\}$ is not bounded above on any compact subinterval of $(x_1, x_3]$ is valid.

Returning now to our argument that S is also closed, let $u(x)$ be the solution of the right (2, 2)-focal point problem for (3) given by $u^{(i)}(x_1) = z^{(i)}(x_1)$, $i = 0, 1$, $u''(x_2) = r_0$, and $u'''(x_2) = 0$. Due to the monotoneity and unboundedness conditions of $\{y_n''(x)\}$ on $(x_1, x_3]$ and due to the fact that $u''(x_2) = r_0 > y_n''(x_2)$ for $n \geqq 1$, it follows that there exist points $x_1 < t_1 < x_2 < t_2 < x_3$ and $n_0 \geqq 1$ such that $y_{n_0}''(t_i) = u''(t_i)$, $i = 1, 2$. Hence there is a third point $t_1 < t_3 < t_2$ such that $y_{n_0}'''(t_3) = u'''(t_3)$. Furthermore, it is also the case here that $y_{n_0}^{(i)}(x_1) = y^{(i)}(x_1)$, $i = 0, 1$, but this is a contradiction to (D_3).

Consequently, it must be the case that S is also closed and due to the connectedness of the real line, it follows that $S \equiv \mathbf{R}$. Thus, by choosing $r = y_3 \in S$, the Lemma is proven.

THEOREM 3.4. *If we assume the hypotheses of Theorem* 3.2, *then condition* (E) *with respect to* (3) *is satisfied.*

PROOF. Assume the hypotheses of the Theorem, and suppose there is a sequence of solutions $\{y_k(x)\}$ of (3), a compact subinterval $[c, d] \subset (a, b)$, and a constant $M > 0$ such that $|y_k(x)| \leqq M$, for all $x \in [c, d]$ and all $k \geqq 1$. By repeated applications of the Mean Value Theorem, there are disjoint subintervals $[t_1, t_2]$, $[t_3, t_4]$, and $[t_5, t_6]$ of $[c, d]$, where $c = t_1 < t_2 < t_3 < t_4 < t_5 < t_6 = d$, sequences $\{w_k\} \subset (t_1, t_2)$, $\{x_k\} \subset (t_3, t_4)$, and $\{z_k\} \subset (t_5, t_6)$, and constants $M_1, M_2, M_3 > 0$ such that, $|y_k(w_k)| \leqq M$, $|y_k'(w_k)| \leqq M_1$, $|y_k''(x_k)| \leqq M_2$, and $|y_k'''(z_k)| < M_3$. Now there is a subsequence of positive integers $\{k_j\}_{j=1}^{\infty} \subset \{k\}_{k=1}^{\infty}$ and there are points $w_0 \in [t_1, t_2]$, $x_0 \in [t_3, t_4]$ and $z_0 \in [t_5, t_6]$, and $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in \mathbf{R}$ such that, $w_{k_j} \to w_0$, $x_{k_j} \to x_0$, $z_{k_j} \to z_0$, $y_{k_j}(w_{k_j}) \to \alpha_0$, $y_{k_j}'(w_{k_j}) \to \alpha_1$, $y_{k_j}''(x_{k_j}) \to \alpha_2$, and $y_{k_j}'''(z_{k_j}) \to \alpha_3$.

As a consequence of Lemma 3.3, there is a unique solution, $u(x)$, of (3) satisfying $u^{(i)}(w_0) = \alpha_i$, $i = 0, 1$, $u''(x_0) = \alpha_2$, and $u'''(z_0) = \alpha_3$. Again the hypotheses, the uniqueness of solutions of right (2, 1, 1)-focal point problems for (3), and the Brouwer Invariance of Domain Theorem imply that $\{y_{k_j}^{(i)}(x)\}$ converges uniformly to $u^{(i)}(x)$ on each compact subinterval of (a, b), and in particular on $[c, d]$, for each $i = 0, 1, 2, 3$. This completes the proof.

The anlogue of Theorem 1.2 for right focal point problems for (3) is now easily established.

THEOREM 3.5. *If equation* (3) *satisfies conditions* (A), (B), (C), *and* (D_3), *and if all right* (m_1, m_2)*-focal point boundary value problems for* (3) *on* (a, b) *have solutions, then all right* $(m_1, \ldots, m_r)$*-focal point boundary value problems,* $r = 2, 3, 4$, *for* (3) *on* (a, b) *have solutions which are unique.*

PROOF. The hypotheses of the Theorem are the same as those of Theorem 3.2. As a consequence of Theorems 3.2 and 3.4, conditions (D_1) and (E) of Theorem 1.1 are satisfied with respect to (3). The conclusion of the Theorem then follows from Theorem 1.1.

4. Generalizations. The requirement of establishing uniqueness of right $(1, 1, \ldots, 1)$-focal point boundary value problems in the general case is an obstacle to the extension of results in §3 and in complete analogy to Theorem 1.2 for equations of arbitrary order. Yet, if we assume

(D_{n-1}) Each right $(m_1, m_2, \ldots, m_{n-1})$-focal point boundary value problem for (1) on (a, b) has at most one solution;

then by modifying slightly the arguments of Theorem 3.2, we have the following theorem.

THEOREM 4.1. *If equation* (1) *satisfies conditions* (A), (B), (C), *and* (D_{n-1}), *and if all right* $(m_1, m_2, \ldots, m_{n-2})$*-focal point boundary value problems for* (1) *on* (a, b) *have unique solutions, then all right* $(1, 1, \ldots, 1)$*-focal point boundary value problems for* (1) *on* (a, b) *have at most one solution.*

As in Lemma 3.3, if we assume the hypotheses of Theorem 4.1, we can establish the existence of unique solutions of all right $(2, 1, \ldots, 1)$-focal point boundary value problems for (1) on (a, b) and this in turn can be employed, as in Theorem 3.4, to show that the compactness hypothesis (E) is satisfied by (1).

In a similar manner, a generalization of Theorem 3.5 can be stated.

THEOREM 4.2. *If equation* (1) *satisfies conditions* (A), (B), (C), *and* (D_{n-1}), *and if all right* $(m_1, m_2, \ldots, m_{n-2})$*-focal point boundary value problems for* (1) *on* (a, b) *have solutions which are unique, then all right* $(m_1, \ldots, m_r)$*-focal point boundary value problems,* $2 \leqq r \leqq n$, *for* (1) *on* (a, b) *have solutions which are unique.*

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