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INDUCING LATTICE MAPS BY SEMILINEAR ISOMORPHISMS

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In this paper all modules are left modules and all module homomorphisms act on the right. Ring homomorphisms are written on the left.

If M is a module, let $\underline{\underline{L}}(M)$ denote the lattice of submodules of M. The Fundamental Theorem of Projective Geometry asserts that if D and K are two division rings and $\lambda: \underline{\underline{L}}(D^{(3)}) \to \underline{\underline{L}}(K^{(3)})$ is a lattice isomorphism between two three-dimensional free modules, then λ is induced by a semilinear isomorphism. This means that there is an additive isomorphism $L: D^{(3)} \to K^{(3)}$ and a ring isomorphism $\sigma: D \to K$ such that $(X)L = \lambda(X)$ for each $X \in \underline{\underline{L}}(D^{(3)})$ and $(dV)L = \sigma(d)(V)L$ for all $V \in D^{(3)}$ and $d \in D$. For convenience the phrase "lattice isomorphism $\lambda: A \to B$" will be used to mean $\lambda: \underline{\underline{L}}(A) \to \underline{\underline{L}}(B)$ is a lattice isomorphism.

There has been some interest in generalizing this theorem to larger classes of rings. We prove here:

COROLLARY 6. *Let $n \geqq 3$. Let R be any one of the following:*

1) *A serial ring (i.e., a finite product of rings, each of which has linearly ordered lattice of left ideals);*
2) *A semihereditary ring; or*
3) *An integral domain (not assumed to be commutative). Let $\lambda: R^{(n)} = \sum_{i=1}^{n} \oplus Ri_k \to S^{(n)}$ be a lattice isomorphism where the $\{i_k\}$ form a basis for $R^{(n)}$ and $Sa \approx S$, with $\lambda(Ri_k) = Sa$, for some k. Then λ is induced by a semilinear isomorphism.*

We also show that if R is an artinian ring of composition length N and if $n \geqq N + 2$, then any lattice isomorphism $\lambda: R^{(n)} \to S^{(n)}$ which preserves cyclic submodules must be induced by a semilinear isomorphism. We actually need only that λ preserves a small subset of the set of cyclic submodules of $R^{(n)}$. Note, since division rings have composition length 1, this generalizes the Fundamental Theorem. We also observe in the remarks before Lemma 1 that modulo lattice maps induced by certain kinds of projective modules, all such lattice maps preserve enough cyclic modules.

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In fact, many attempts to generalize this theorem have been made. The results of this work seem to be of some interest, but they also seem to be not generally well known. For example, Stephenson [11] pointed out that von Neumann essentially proved in his Continuous Geometry [12] that for rings R and S and an integer $n \geqq 3$, $\underline{\underline{L}}(S^{(n)}) \approx \underline{\underline{L}}(R^{(n)})$ if and only if $R_n \approx S_n$, where R_n is the ring of $n \times n$ matrices over R. Baer [2] extended the Fundamental Theorem of Projective Geometry to abelian groups and obtained results in which the groups need merely contain a free group (of rank 2, in fact) or an appropriate torsion group. Stephenson, in [11] and in his thesis [10], proved a very general theorem in which he showed that if $M = \sum_{i=1}^{n} \oplus M_i$ is a module with $n \geqq 3$ and each M_i contains a copy of a module P_R, and if there is a lattice isomorphism λ: $M_R \to N_S$ then, letting $Q = \lambda(P)$ (this can be shown to be independent of the copy of P chosen), $T_1 = \mathrm{End}(P) \approx \mathrm{End}(Q) = T_2$. Let σ be the isomorphism between $\mathrm{End}(P)$ and $\mathrm{End}(Q)$. Then there is a semilinear isomorphism (L, σ): $(\mathrm{Hom}_R(P, M_R), \mathrm{End}_R(P)) \approx (\mathrm{Hom}_S(Q, N_S), \mathrm{End}_S(Q))$. If we let M be a free module and let $M_i \approx P \approx R$, then the left side of the above is just $R^{(n)}$, and we may ask if the semilinear map described actually induces λ. It is clear that if λ is induced by a semilinear map, then λ must take cyclic submodules of $R^{(n)}$ to cyclic submodules of $S^{(n)}$.

A result on when a lattice map is induced by a semilinear map was obtained by Stephenson-Skornyakov [11].

The following definition and Theorem may be found in Stephenson's thesis [10] and the result is a generalization of work in [9].

DEFINITION. (Stephenson-Skornyakov). S_1: For any $x, y, z \in M$ with $Rx \cap Ry = 0$ there is a free element w in M such that $(Rx + Ry) \cap Rw = (Ry + Rz) \cap Rw = (Rx + Rz) \cap Rw = 0$.

S_2: If $t \in M$ and u, x, y are free elements of M with $(Ru + Rt) \cap Rx = (Ru + Rt) \cap Ry = 0$ and $Rx \cap Ry \neq 0$, $Ru \cap Rt \neq 0$, then there is a free element $w \in M$ such that $Ru \cap Rw = Rt \cap Rw = Rx \cap Rw = Ry \cap Rw = 0$.

THEOREM (Stephenson). *If M is a module satisfying S_1 and S_2 and if λ: ${}_RM \to {}_SN$ is a lattice isomorphism such that $\lambda(Ru) = Su'$ for two free elements u and u', then λ is induced by a semilinear map.*

Now, the main thrust of the above work is that by suitably generalizing the Fundamental Theorem we can associate to every lattice map a semilinear map which we denote by (L, σ). It is not in any way obvious however that (L, σ) induces λ. We need here some special properties of (L, σ). Technically, it would be possible to refer the reader to Stephenson [11], but his construction is very general and would make understanding here difficult. Therefore, we have made the first part of this paper expository, and written it, we hope, for maximum accessibility. Our ideas were first derived from [5] and were apparent in [11], but it is clear that the underlying idea is the same as the usual proof of the Fundamental Theorem, see [7].

Now, projective modules induce lattice maps, and the theme here is that in going from fields to rings these are the only nonsemilinear maps one needs to be concerned with. Specifically, let ${}_RP$ be a finitely generated projective module, and let T be its endomorphism ring.

The functor Hom(P,___) takes left R-modules to left T-modules. It is well known that this functor preserves lattices. In particular, if ${}_RR^{(n)} \approx P^{(m)}$, Hom($P$,___) induces a lattice isomorphism between ${}_RR^{(n)}$ and $\mathrm{Hom}(P, P^{(m)}) \approx {}_TT^{(m)}$. One should not in general expect this isomorphism to be semilinear. It is reasonable, however, to expect a lattice isomorphism to factor into a composition of a semilinear isomorphism and one induced by a projective module (actually the inverse of the above). Now, let $R^{(n)}$ and $S^{(m)}$ be free modules over rings R and S and let λ be a lattice isomorphism. Then, by simple arguments below, $S^{(m)} = [\lambda(R)]^{(n)}$. So $\lambda(R)$ is projective, say isomorphic to ${}_SP$. If $T = \mathrm{End}\ ({}_SP)$ then the functor Hom(${}_RP$,___) gives a lattice isomorphism $S^{(m)})$ to $T^{(n)}$, and the composition of the two takes $R^{(n)}$ to $T^{(n)}$ and takes the n-th coordinate of $R^{(n)}$ to the n-th coordinate of $T^{(n)}$ when the isomorphisms are constructed in the usual way (as below). It turns out that this composition preserves enough cyclic submodules to prove that it is semilinear in a large number of cases.

In what follows, those not interested in this level of generality may assume U is a free module so that $A = R$. It cannot be assumed that $B \approx S$.

LEMMA 1. *Let $X = A \oplus B$. Then $A \approx B$ if and only if there is a $D \subset X$ such that $X = A \oplus D = B \oplus D$.*

PROOF. If $f\colon A \to B$ is an isomorphism, take $D = A(1 - f)$. To prove sufficiency, let f be the projection onto B along D, restricted to A.

PROPOSITION 2. *Let R and S be rings. Let ${}_RU = \sum_{k=1}^{n} \oplus A_k$, $n \geqq 3$, where $A_k \approx A$ for some fixed A. Let ${}_SV$ be a left S module and suppose there is a lattice isomorphism $\lambda\colon {}_RU \to {}_SV$. Then V decomposes into a sum $V = \sum_{i=1}^{n} \oplus B_k$ with $\lambda(A_k) = B_k$, and all the B_k isomorphic to a fixed left S-module B. Further, for any set of isomorphisms $\{i_k\}$, $i_k\colon A \to A_k$, there is a set of isomorphisms $\{\varepsilon_k\colon B \to B_k\}$ and a ring isomorphism $\sigma\colon$ End $A \to$ End B such that whenever $X = A(f_1 i_1 + \cdots + f_n i_i)$ with some $f_k = 1$,*

$$\lambda(X) = B(\sigma(f_1)\, \varepsilon_1 + \cdots + \sigma(f_n)\, \varepsilon_n).$$

In particular, $\mathrm{End}({}_RU) \approx \mathrm{End}({}_SV)$.

PROOF. We divide the proof into several steps. In what follows, $F =$

End(A), G = End(B). We note at the outset that for the $\{i_k\}$ and $\{\varepsilon_k\}$, Hom(A, U) $= \sum \oplus Fi_k$ and Hom(B, V) $= \sum \oplus G\varepsilon_k$. In what follows, $f_k \in F$ and $g_k \in G$ always.

1) $B_k \approx B_\ell \approx B$. By one implication of Lemma 1, $A_k \oplus A_\ell = D \oplus A_k = D \oplus A_\ell$ so that, applying λ, $B_k \oplus B_\ell = \lambda(D) \oplus B_k = \lambda(D) \oplus B_\ell$ and by the other implication of the lemma, $B_k \approx B_\ell$.

Let $\{\varepsilon_k\}$ be any set of isomorphisms from B to B_k.

2) $X = A(\sum f_k i_k)$ with $f_\ell = 1$ if and only if X is a complement for $\sum_{k \neq \ell} \oplus A_k = Y$.

The condition is obviously sufficient. On the other hand, let f be the projection from U onto Y along X. Then, $X = U(1-f) = (A_\ell \oplus Y)(1-f)$ so $X = A_\ell(1-f)$, since $Y(1-f) = 0$. But $A_\ell = Ai_\ell$ so $X = A(i_\ell - i_\ell f)$, where $i_\ell f \in$ Hom(A, Y). Thus $i_\ell f$ is a linear combination of the i_k, $i \neq \ell$, so 2) is established. Notice also that 2) holds for B with the f_k replaced by g_k and the i_k replaced by ε_k.

3) Let $X = A(\sum f_k i_k)$, with some $f_\ell = 1$, then $\lambda(X) = B(\sum g_k \varepsilon_k)$, with $g_\ell = 1$.

This follows because X is a complement for $\sum_{k \neq \ell} A_k$ if and only if $\lambda(X)$ is a complement for $\sum_{k \neq \ell} B_k$. Note we have not proved that for an arbitrary $X = A(\sum f_k i_k)$, $\lambda(X)$ has the form $B(\sum g_k \varepsilon_k)$.

4) The ε_k may be modified so that $\lambda A(i_1 + i_k) = B(\varepsilon_1 + \varepsilon_k)$. To see this, note $\lambda A(i_1 + i_k)$ is a complement for $B\varepsilon_1$ and $B\varepsilon_k$ in $B\varepsilon_1 + B\varepsilon_k$ by 2) so that

$$\lambda A(i_1 + i_k) = B(\varepsilon_1 + g_k\varepsilon_k) = B(g_1\varepsilon_1 + \varepsilon_k).$$

We claim $g_1 g_k = g_k g_1 = 1$. To see this, let $b \in B$, then there is a $b' \in B$ such that $b(\varepsilon_1 + g_k\varepsilon_k) = b'(g_1\varepsilon_1 + \varepsilon_k)$, so $b\varepsilon_1 = b'g_1\varepsilon_1; bg_k\varepsilon_k = b'\varepsilon_k$. Cancelling the ε_1 and ε_k gives $b = b'g_1$ and $b' = bg_k$. So $b = bg_k g_1$ and $b' = b'g_1 g_k$. Since we also can find b, given b', this proves the claim.

We now change the ε_k to $g_k\varepsilon_k$ so that 4) is established.

5) For each $k > 1$ there is a bijection σ_k: $F \to G$ such that $\lambda A(i_1 + fi_k) = B(\varepsilon_1 + \sigma_k(f)\varepsilon_k)$.

For any f, $\lambda A(i_1 + fi_k) = B(\varepsilon_1 + g\varepsilon_k)$ by 2). It is easy to see that $B(\varepsilon_1 + g_1\varepsilon_k) = B(\varepsilon_1 + g_2\varepsilon_k)$ if and only if $g_1 = g_2$, so that the map $\sigma_k(f) = g$ is well defined. Since the modules in question are exactly the complements of Ai_k in $Ai_1 + Ai_k$ (respectively $B\varepsilon_k$ in $B\varepsilon_1 + B\varepsilon_k$), and λ is a lattice isomorphism with $\lambda Ai_k = B\varepsilon_k$, σ must also be one-to-one and onto.

6) $\lambda A(i_1 + f_2 i_2 + \cdots + f_n i_n) = B(\varepsilon_1 + \sigma_2(f_2)\varepsilon_2 + \cdots + \sigma_n(f_n)\varepsilon_n)$.

By 3), the right hand side of the above is of the form $B(\varepsilon_1 + g_2\varepsilon_2 + \cdots + g_n\varepsilon_n)$. But

$$A(i_1 + f_2 i_2 + \cdots + f_n i_n) \oplus A(\sum_{\substack{t \neq 1 \\ t \neq k}} f_t i_t)$$

$$= A(i_1 + f_k i_k) \oplus A(\sum_{\substack{t\neq 1\\ t\neq k}} f_t i_t).$$

So, applying λ,

$$B(\varepsilon_1 + g_2\varepsilon_2 + \cdots + g_n\varepsilon_n) \oplus \lambda(A \sum_{\substack{t\neq 1\\ t\neq k}} f_t\varepsilon_t)$$
$$= B(\varepsilon_1 + \sigma_k(f_k)\varepsilon_k) \oplus \lambda(A \sum_{\substack{t\neq 1\\ t\neq k}} f_t i_t).$$

Therefore, if $b \in B$, $b(\varepsilon_1 + g_2\varepsilon_2 + \cdots + g_k\varepsilon_k + \cdots + g_n\varepsilon_n) = b_1(\varepsilon_1 + \sigma_k(f_k)\,\varepsilon_k) + b_2$ where $b_2 \in \lambda(A \sum_{t\neq 1,k} f_t i_t) \subseteq \sum_{t\neq 1,k} B_t$. So, $b = b_1$ and $bg_k = b_1\sigma_k(f_k) = b\sigma_k(f_k)$, so $g_k = \sigma_k(f_k)$.

7) $\lambda A(i_k + i_\ell) = B(\varepsilon_\ell + \varepsilon_k)$.

Start with $A(i_1 + i_\ell + i_k) \oplus A(i_\ell + i_k) = Ai_1 \oplus A(i_\ell + i_k)$. So, applying 6) and using the fact that $\sigma_t(1) = 1$, we have $B(\varepsilon_1 + \varepsilon_\ell + \varepsilon_k) \oplus \lambda A(i_\ell + i_k) = B\varepsilon_1 \oplus \lambda A(i_\ell + i_k)$. But also by 3) (for $U = A\, i_\ell \oplus Ai_k$), $\lambda A(i_\ell + i_k) = B(\varepsilon_\ell + g_k\varepsilon_k) = B(g_\ell\varepsilon_\ell + \varepsilon_k)$ (as in the proof of 4), $g_1 g_k = g_k g_\ell = 1$). So $B(\varepsilon_1 + \varepsilon_\ell + \varepsilon_k) \oplus B(\varepsilon_\ell + g_k\varepsilon_k) = B\varepsilon_1 + B(\varepsilon_\ell + g_k\varepsilon_k)$. Let $b \in B$; then $b(\varepsilon_1 + \varepsilon_\ell + \varepsilon_k) = b_1\varepsilon_1 + b_2(\varepsilon_\ell + g_k\varepsilon_k)$. Then $b = b_1$, $b = b_2$ and $b = b_2 g_k$. So, in particular, $b = bg_k$. Therefore $g_k = 1$.

8) $\sigma_\ell = \sigma_k$.

We have $A(i_1 + fi_\ell + fi_k) \oplus A(i_\ell + i_k) = Ai_1 \oplus A(i_\ell + i_k)$. So using 6) and applying λ, we have $B(\varepsilon_1 + \sigma_\ell(f)\varepsilon_\ell + \sigma_k(f)\varepsilon_k) \oplus B(\varepsilon_k + \varepsilon_\ell) = B\varepsilon_1 \oplus B(\varepsilon_k + \varepsilon_\ell)$. Write $b(\varepsilon_1 + \sigma_\ell(f)\varepsilon_\ell + \sigma_k(f)\varepsilon_k) = b_1\varepsilon_1 + b_2(\varepsilon_k + \varepsilon_\ell)$. Cancel ε_ℓ and ε_k as before to get $b\sigma_\ell(f) = b_2 = b\sigma_k(f)$.

We denote the common value of the σ_k by σ. Fix some t.

9) Let $\tau_k(f)$ be defined by $\lambda A(i_t + fi_k) = b(\varepsilon_t + \tau_k(f)\varepsilon_k)$. Then $\tau_k(f) = \sigma_k(f)$.

First by 7), $\tau_k(1) = 1$. Then since the τ_k are defined in a manner analogous to the σ_k, 6) allows us to conclude that

$$\lambda A(f_1 i_1 + \cdots + i_t \cdots + f_n i_n)$$
$$= B(\tau_1(f_1)\,\varepsilon_1 + \cdots + \varepsilon_t + \cdots + \tau_n(f_n)\varepsilon_n).$$

Therefore, for any k,

$$\lambda A({}_1 i + i_1 + fi_k) = B(\tau_1(1)\varepsilon_1 + \varepsilon_t + \tau_k(f)\varepsilon_k)$$
$$= B(\varepsilon_1 + \sigma_t(f)\,\varepsilon_t + \sigma_k(f)\varepsilon_k).$$

So $B(\varepsilon_1 + \varepsilon_t + \tau_k(f)\varepsilon_k) = B(\varepsilon_1 + \varepsilon_t + \sigma_k(f)\varepsilon_k)$ and $\tau_k(f) = \sigma_k(f)$. We therefore have,

10) Given any $\{i_k\}$ as in the proposition, there is a bijection $\sigma\colon F \to G$, such that $\sigma(1) = 1$, and a set of isomorphisms $\{\varepsilon_k\}$ as in the proposition such that

$$\lambda A(f_1 i_1 + \cdots + f_n i_n) = B(\sigma(f_1)\varepsilon_1 + \cdots + \sigma(f_n)\varepsilon_n),$$

whenever some $f_k = 1$.

11) We now claim that σ as defined in 10) is a ring isomorphism. Note that

$$A(i_1 + (f_a + f_b)\, i_2 + i_3) \subset A(i_1 + f_a i_2) + A(f_b i_2 + i_3).$$

So, applying λ, we have

$$B(\varepsilon_1 + \sigma(f_a + f_b)\varepsilon_2 + \varepsilon_3) \subset B(\varepsilon_1 + \sigma(f_a)\varepsilon_2) + B(\sigma(f_b)\varepsilon_2 + \varepsilon_3).$$

So, if $b_1 \in B$,

$$b_1(\varepsilon_1 + \sigma(f_a + f_b)\varepsilon_2 + \varepsilon_3) = b_2(\varepsilon_1 + \sigma(f_a)\varepsilon_2) + b_3(\sigma(f_b)\varepsilon_2 + \varepsilon_3).$$

So $b_1 = b_2 = b_3$ and $b_1\sigma(f_a + f_b) = b_1(\sigma(f_a) + \sigma(f_b))$, so σ is additive. Also,

$$A(i_1 + f_a f_b i_2 + f_a i_3) \subset A(i_1) + A(f_b i_2 + i_3),$$

so we have

$$B(\varepsilon_1 + \sigma(f_a f_b)\, \varepsilon_2 + \sigma(f_a)\varepsilon_3) \subset B\varepsilon_1 + B(f_b\, \varepsilon_2 + \varepsilon_3).$$

Thus if $b_1 \in B$,

$$b_1(\varepsilon_1 + \sigma(f_a f_b)\, \varepsilon_2 + \sigma(f_a)\, \varepsilon_3) = b_2\varepsilon_2 + b_3(\sigma(f_b)\, \varepsilon_2 + \varepsilon_3).$$

Then $b_1\sigma(f_a f_b) = b_3\sigma(f_b)$ and $b_3 = b_1\sigma(f_a)$, so that $b_1\sigma(f_a f_b) = b_1\sigma(f_a)\sigma(f_b)$. Thus σ is multiplicative and is a ring isomorphism. Also, $\mathrm{End}(U) \approx F_n$; $\mathrm{End}(V) \approx G_n$, so $\mathrm{End}(U) \approx \mathrm{End}(V)$. This establishes the proposition.

PROPOSITION. *If R and S are rings, and ${}_RU$ and ${}_SV$ are modules with $U \approx A^{(n)}$ for some module R^A and some $n \geqq 3$ and if $\underline{\underline{L}}({}_RU) \approx \underline{\underline{L}}({}_SV)$, then* $\mathrm{End}({}_RU) \approx \mathrm{End}({}_SV)$.

The above proposition has a special case, the fact that if $R^{(n)}$ and $S^{(m)}$ are free modules with $\underline{\underline{L}}(R^{(n)}) \approx \underline{\underline{L}}(S^{(m)})$ and one of m or n greater than 3, then $R_n \approx S_m$, where R_n (resp. S_m) is the $n \times n$ (resp. $m \times m$) matrix ring over R (resp. S) It is claimed by Stephenson **[10]** that this fact is implicit in von Neumann's Continuous Geometry **[12]**.

Below is a proposition which seems to summarize the situation nicely, using a bit of folklore.

THEOREM A. *The following are equivalent, for $n \geqq 3$:*

1) $\underline{\underline{L}}(R^{(n)}) \approx \underline{\underline{L}}(S^{(m)})$;

2) $S^{(m)} = P^{(n)}$ *with* $\mathrm{End}\,(P) \approx R$.

3) $R_n \approx S_m$.

PROOF. 1) implies 2). This is a special case of Proposition 2. Let i_k be a free basis for $R^{(n)}$, take $A_k = Ri_k$, $P = B$ and we have the isomorphism given by σ.

2) implies 3). Take the endomorphism ring of both sides of 2. From the left side it is S_m. From the right side it is $(\mathrm{End}\,(P))_n \approx R_n$, so $S_m \approx R_n$.

3) implies 1). This proof must "factor through" 2). Let e_{ij} be a set of matrix units for R_n, that is, $e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$ and $\sum e_{ii} = 1$. Let θ be the isomorphism. Think of the rings as operators on $R^{(n)}$ and $S^{(m)}$, respectively. Then let $f_{ij} = \theta(e_{ij})$. The f_{ij} are matrix units in the above sense. This means $V = S^{(m)} = \sum \oplus Vf_{ii}$. Claim, $Vf_{ii} \approx Vf_{jj}$. First, $Vf_{ii} \supset Vf_{ji}f_{ii} = Vf_{ji} \supset Vf_{ij}f_{ji} = Vf_{ii}$, so $Vf_{ii} = Vf_{ji}$. Second, the map $Vf_{ji} \to Vf_{ji}f_{ij} = Vf_{jj}$ has inverse $Vf_{jj} \to Vf_{jj}f_{ji} = Vf_{ii}$ so that $Vf_{ii} \approx Vf_{jj}$.

Take $P = Vf_{ii}$; then $\mathrm{End}\,(P) = f_{ii}(\mathrm{End}\,(V))f_{ii} = f_{ii}(S_m)f_{ii} = \theta(R_{ii}R_n e_{ii}) \approx R$, which shows 3) implies 2). Now, apply the functor $\mathrm{Hom}(P, \_\_)$, we have $\underline{\underline{L}}(S^{(m)} \approx \underline{\underline{L}}\,\mathrm{Hom}(P, S^{(m)}) = \underline{\underline{L}}(\mathrm{Hom}(P, P^{(n)}) \approx \underline{\underline{L}}(\mathrm{End}\,(P^{(n)})) = \underline{\underline{L}}(R^{(n)})$. The fact that $\mathrm{Hom}(P, \_\_)$ preserves lattices is well known [1].

It is natural to ask, if $\underline{\underline{L}}(R^{(n)}) \approx \underline{\underline{L}}(S^{(m)})$ must R and S be isomorphic? We do have from folklore the following proposition.

PROPOSITION. *Let R and S be commutative semi-local rings or let one of them be semilocal* (i.e., *artinian modulo its radical*). *Then, if there are free modules ${}_RU$ and ${}_SV$ of the same rank $\geqq 3$ with $\underline{\underline{L}}(U) \approx \underline{\underline{L}}(V)$, $R \approx S$.*

PROOF. If R and S are commutative then by Theorem A, $R_n \approx S_n$. In particular, R and S are Morita Equivalent and, as is well known, if they are both commutative, $R \approx S$.

If one of them is semilocal, we may count the simples in the top (P/PJ) of P in 2) to conclude $P \approx S$ (or perhaps R) so that $R \approx \mathrm{End}(P) \approx S$.

Remarkably, M. Isaacs (personal-communication) has proved that if $R_n \approx S_n$, and only one of the rings is assumed to be commutative, then $R \approx S$. In fact, if R is commutative he has shown the conclusion follows if $R_n \approx S_m$ and $n \leqq m$.

An example of two rings R and S for which $R_n \approx S_n$ but R and S not isomorphic is given by Plastiras in [8]. This example has a certain naturalness about it, but verifying its correctness here would take us too far afield. A sketch of this example is the following.

Let K be a field and V an infinite dimensional vector space over K. In $V \oplus V$ look at the ring generated by the linear transformations of the form $T \oplus T$ with $T \in \mathrm{End}\,{}_KV$ together with the transformations of finite rank. Call this ring R.

In $K \oplus V \oplus V$ look at the linear transformations of the form $0 \oplus T \oplus T$ together with the transformations of finite rank. Call this ring S. Then, $R_2 \approx S_2$ but R and S are not isomorphic. It is also asserted that this example may be made to work for R_n and S_n where n is any even integer.

In general, as the previous theorem shows, $R_n \approx S_n$ if and only if $R^{(n)} = P^{(n)}$ with $S = \text{End}\,(P_R)$. The existence or nonexistence of such projective modules P can be a deep and difficult matter.

Semilinearity. Here we address the question, if λ: $R^{(n)} \to S^{(n)}$ is a lattice isomorphism, is λ induced by a semilinear isomorphism? We show here that for large classes of rings such maps are either semilinear or are determined by a semilinear map and a projective module. In what follows the reader who is interested only in rings may let $A = R$ and assume $\{i_k\}$ is a basis for $R^{(n)}$.

We begin with a discussion which justifies the above paragraph.

Let λ: $R^{(n)} \to S^{(m)}$. Then let us write $R^{(n)} = \sum_{k=1}^{n} \oplus Ri_k$, and let $P_k = \lambda(Ri_k)$. Then, as we have shown previously, $S^{(n)} = \sum_{k=1}^{n} \oplus P_k$ and all the P_k are isomorphic to a projective S-module P. Let $T = \text{End}\,({}_SP)$. Then, the function Hom(P,___) induces a lattice isomorphism from $S^{(n)}$ to Hom(P, $S^{(n)}$) = Hom(P, $\sum \oplus P_k$) $= \sum_{k=1}^{n} \oplus$ Hom(P, P_k) but Hom(P, P_k) $= T$ and, choosing a basis ε_k of isomorphisms ε_k: $P \to P_k$, Hom(P, $\sum \oplus P_k$) $= \sum_{k=1}^{n} \oplus T\varepsilon_k$, and it is clear that the composition of the two lattice maps Hom(P,___) o λ takes Ri_k to $T\varepsilon_k$. Below we will be concerned with lattice maps which preserve enough cyclic modules. The composition above will always satisfy our hypotheses, and will turn out in these cases to be induced by a semilinear map. Let us isolate this as a proposition.

Proposition. *Write* $S^{(n)} = \sum_{k=1}^{n} \oplus Si_k$. *Let* λ: $R^{(n)} \to S^{(m)}$ *be a lattice isomorphism. Then there is a projective module* ${}_SP$ *and a lattice isomorphism* Hom(P,___): $S^{(m)} \to \sum_{k=1}^{n} \oplus T\varepsilon_k$, *where* $T = \text{End}(P)$, *such that* (Hom(P,___) 0 $\lambda(Ri_k) = T\varepsilon_k$.

Proposition 3. *The equation* $\lambda(X) = B(\sigma(f_1)\varepsilon_1 + \cdots + \sigma(f_n)\varepsilon_n)$ *of Proposition* 2 *holds if some* $f_i = 0$.

Proof. Without loss of generality, $f_1 = 0$. Start with $Ai_1 + A(i_1 + f_2i_2 + \cdots + f_ni_n) = Ai_1 + A(f_2i_2 + \cdots + f_ni_n)$. Applying λ, we obtain the following:

1) $B\varepsilon_1 + B(\varepsilon_1 + \sigma(f_2)\,\varepsilon_2 + \cdots + \sigma(f_n)\varepsilon_n) = B\varepsilon_1 + \lambda A(f_2i_2 + \cdots + f_ni_n)$. Now, given $b \in B$ there is a $b_1 \in B$ and $x \in \lambda A(f_2i_2 + \cdots + f_ni_n)$ with

2) $b(\varepsilon_1 + \sigma(f_2)\,\varepsilon_2 + \cdots + \sigma(f_n)\varepsilon_n) = b_1\varepsilon_1 + x$, and conversely, given any such x, a b_1 and b can be found. But then, because $A(f_2i_2 + \cdots + f_ni_n) \subset \sum_{k=2}^{n} \oplus Ai_n$, $A(f_2i_2 + \cdots + f_ni_n) \subset \sum_{k=2}^{n} \oplus B\varepsilon_k$. Thus, in equation 2, we have always $b = b_1$ and $x = b(\sigma(f_2)\,\varepsilon_2 + \cdots + \sigma(f_n)\varepsilon_n)$. Since x and b can each be found from the other,

$$\lambda A(f_2i_2 + \cdots + f_ni_n) = B(\sigma(f_2)\,\varepsilon_2 + \cdots + \sigma(f_n)\varepsilon_n).$$

PROPOSITION 4. *Let* $X = A(\sum_{k=1}^{n} f_k i_k)$. *Suppose there are distinct indices* k, ℓ *such that* $X \cap (Ai_k + Ai_\ell) = 0$. *Then* $\lambda(X) = A(\sum \sigma(f_k)\varepsilon_k)$.

PROOF. Assume $X = Ax$, $x = \sum_{k=1}^{n} f_k i_k$, and $X \cap (Ai_1 \oplus Ai_2) = 0$. Start with $Ax \oplus Ai_1 = Ai_1 \oplus A(f_2 i_2 + \cdots + f_n i_n)$. Applying λ and using Proposition 3 for the right hand side, we obtain

$$\lambda(Ax) \oplus B\varepsilon_1 = B\varepsilon_1 \oplus B(\sigma(f_2)\,\varepsilon_2 + \cdots + \sigma(f_n)\varepsilon_n).$$

We show first that there is a $g \in G$ with

$$\lambda(Ax) = B(g\varepsilon_1 + \sigma(f_2)\,\varepsilon_2 + \cdots + \sigma(f_n)\varepsilon_n).$$

Let $y \in \lambda(Ax)$. Write

$$y = b_1\varepsilon_1 + b_2(\sigma(f_2)\,\varepsilon_2 + \cdots + \sigma(f_n)\varepsilon_n).$$

On the other hand, given b_2, the left side of

$$y - b_1\varepsilon_1 = b_2(\sigma(f_2)\,\varepsilon_2 + \cdots + \sigma(f_n)\varepsilon_n)$$

is uniquely determined. This means first that the map

$$b_2 \to b_2(\sigma(f_2)\,\varepsilon_2 + \cdots + \sigma(f_n)\,\varepsilon_n) \to b_1\varepsilon_1 \to b_1$$

is a homomorphism. ($b_1\varepsilon_1 \to b_1$ is well defined because ε_1 is a monomorphism.) Call this map g. Then,

$$y = (b_2)g\varepsilon_1 + b_2(\sigma(f_2)\varepsilon_2 + \cdots + \sigma(f_n)\varepsilon_n)$$
$$y = b_2(g\varepsilon_1 + \sigma(f_2)\varepsilon_2 + \cdots + \sigma(f_n)\varepsilon_n).$$

So, we have shown every y has the form on the right hand side. On the other hand, from the definition of g the right hand side is always contained in $\lambda(Ax)$. We wish to show $g = \sigma(f_1)$. To do this, define h, analogous to g for ε_2 and obtain $y = b_2(\sigma(f_1)\varepsilon_1 + h\varepsilon_2 + \cdots + \sigma(f_n)\varepsilon_n)$. Next observe $\lambda Ax \cap (B\varepsilon_1 + B\varepsilon_2) = 0$. Let $b \in B$, let

1) $$x = b(g\varepsilon_1 + \sigma(f_2)\varepsilon_2 + \cdots + \sigma(f_n)\varepsilon_n), \text{ and}$$

find $b' \in B$ with

2) $$x = b'(\sigma(f_1)\varepsilon_1 + h\varepsilon_2 + \cdots + \sigma(f_n)\varepsilon_n).$$

Then $b\sigma(f_n) = b'\sigma(f_n)$ for $k \geqq 2$, so $(b - b')(\sigma(f_1)\varepsilon_1 + h\varepsilon_2 + \cdots + \sigma(f_n)\varepsilon_n) \in \lambda(Ax) \cap (B\varepsilon_1 + B\varepsilon_2) = 0$.

This means $(b - b')\sigma(f_1) = 0$. Also from 1) and 2), $bg = b'\,\sigma(f_1)$. So $b\sigma(f_1) = b'\sigma(f_1)$ and $bg = b'\sigma(f_1)$. So $b\sigma(f_1) = bg$ for all $b \in B$ so $g = \sigma(f_1)$.

Let us now apply this result to lattice isomorphisms between free modules, and then indicate how to prove it in more general settings. Note, if a lattice map is to be induced by a semilinear map λ, then $\lambda(Rx)$ must be cyclic. Let $\lambda\colon R^{(n)} \to S^{(m)}$ be a lattice isomorphism.

PROPOSITION 5. *Let* $R^{(n)} = \sum_{k=1}^{n} \oplus Ri_k$, *where* $\{i_k\}$ *is a basis for* $R^{(n)}$, $n \geqq 3$ *Assuine that every submodule of* $R^{(n)}$ *is a sum of modules* A, *each of which satisfies either*

1) A *has zero projection to* Ri_k *for some* k, *or*

2) *there are distinct indices* k, ℓ *such that* $A \cap (Ri_k + Ri_\ell) = 0$.

If in addition $\lambda(Ri_k) = Sa$ *where* $Sa \approx S$ *for some* k, *then* λ *is induced by a semilinear map.*

PROOF. If $\lambda(Ri_k) = Sa$, then $\lambda(Ri_\ell)$ is cyclic for all ℓ, because $Ri_k \oplus Ri_\ell = D \oplus Ri_k = D \oplus Ri_\ell$ so $\lambda(Ri_k) \approx \lambda(Ri_\ell)$. Therefore, by Propositions 2,3, and 4, there is a basis $\{\epsilon_k\}$ for $S^{(n)}$ and an isomorphism $\sigma: R \to S$ such that $\lambda R(\sum fi_k) = S(\sum \sigma(f_k)\varepsilon_k)$ whenever $(\sum f_k i_k)$ is contained in one of the above A's.

That is to say, we are given the semilinear map (L, σ); where $(\sum f_k i_k) L = \sum \sigma(f_k)\varepsilon_k$, and $\lambda(B) = (B)L$ for any B contained in any A above. However, $\omega = \{X | \lambda(X) = (X)L\}$ is a sublattice of $R^{(n)}$, and our hypothesis gives that $\omega = \underline{L}(R^{(n)})$.

COROLLARY 6. *Let* $n \geqq 3$. *Let* R *be any one of the following*:

1) *A serial ring* (i.e., *a finite product of rings, each of which has a linearly ordered lattice of left ideals*);

2) *A semihereditary ring*; *or*

3) *An integral domain* (*not assumed to be commutative*). *Let* λ: $R^{(n)} = \sum_{i=1}^{n} \oplus Ri_k \to S^{(n)}$ *be a lattice isomorphism where the* $\{i_k\}$ *form a basis for* $R^{(n)}$ *and* $Sa \approx S$, *with* $\lambda(Ri_k) = Sa$, *for some* k. *Then* λ *is induced by a semilinear isomorphism.*

PROOF. Find L, σ, ε_k as usual. We need to find submodules $A \subset R^{(n)}$ such that each A satisfies 1) or 2) of Proposition 5. In this case we use the set (or a subset of the set) of cyclic modules of $R^{(n)}$.

1) Since $R = \prod_{i=1}^{n} Re_i$ with Re serial we need only consider $A = Rx$ with Rx serial. Project to the complement of each Ri_k. The intersection of the kernels of these maps is zero. Since there are only finitely many of them, and since Rx is not zero, one of them is zero. Thus $Rx \cap \sum_{k \neq \ell} \oplus Ri_\ell = 0$ for some k. Since there are at least more than three Ri_k, condition 2) of Proposition 5 is satisfied.

2) Project Rx to $R\varepsilon_1$. This splits $Rx = T \oplus K$, where $K \subset \sum_{k=2}^{n} \oplus Ri_k$. Then K satisfies 1) of Proposition 5 while T satisfies 2).

3) Let $x = \sum f_k i_k$. If some $f_k = 0$ then Rx satisfies 1), of Proposition 5, if not, Rx satisfies 2).

To make Proposition 5 work, we need only that if $R \to \sum \oplus Ri_k$ then $\operatorname{Im} R \cap (Ri_a + Ri_b) = 0$ for some a, b. Let us say R has the bounded annihilator condition if there is an integer N such that for any finite set $X \subset R$ there are $x_1, \ldots, x_N \varepsilon X$ with $\ell(X) = \ell\{x_1, \ldots, x_N\}$.

THEOREM 7. *Let* λ: $R^{(n)} \to S^{(m)}$ *be a lattice isomorphism. If* R *satisfies the bounded annihilator condition with bound* N *then* λ *is induced by a semilinear isomorphism whenever* $n \geqq N + 2$. *In particular, if* R *is left artinian,* $\lambda = L$ *whenever* $n \geqq C({}_RR) + 2$.

We remark that there is a free module $R^{(3)}$ which satisisfies our conditions, but not Stephenson's. Let R be a local commutative ring with radical J. Assume that $J^3 = 0$ and that R has simple essential socle. We need to know that every cyclic module $R(x, y, z)$ is in the lattice generated by the modules above. If one of the coordinates is a unit, we are done by counting dimension. If not, they are all contained in J so $R(x, y, z) \approx R/I$ with $I \neq 0$. But $\mathrm{Soc}(R^{(3)})$ has Goldie dimension 3, so $R(x, y, z)$ has composition length at most 4. Note if $y = 0$, then 1) of Propositixn 5 is satisfied. Now the module $R(x, 0, 0) \oplus R(0, y, z)$ is in the set of lattice generators and by a similar argument has composition length at most 5, because the first module has length at most 2, and the second has length at most 3. Further, this module contains $R(x, y, z)$. One verifies $R(x, y, 0) \oplus R(0, 0, z) \neq R(x, 0, 0) \oplus R(0, y, z)$. We have assumed $y \neq 0$ so the right side cannot contain $R(0, 0, z)$. Therefore, the intersection of these two modules has length 4 and so is equal to $R(x, y, z)$.

This free module does not in general satisfy Stephenson's condition S_1. To see this, find two elements a and b in R with incomparable annihilators. Let $z = (0, a, b)$. Then Rz has Goldie dimension 2 and if we choose $x = (1, 0, 0)$; $y = (0, 0, 0)$ we have $Rx \oplus Rw$ is essential, so no w can exist.

All of this raises the question: When is the lattice of $R^{(3)}$ equal to the lattice generated by the modules in 1) and 2) of Proposition 5. We know of no situation where it is not, and conjecture that the two are equal except on a set of measure zero (you define the measure).

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