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ON UNIFORM ERGODIC THEOREMS FOR MARKOV OPERATORS ON C(X)

ROBERT E. ATALLA

ABSTRACT. H. P. Lotz recently proved a uniform ergodic theorem from which he derives, by rather elaborate and indirect arguments, the following Theorem. Let T be a Markov operator on $C(X)$ where X is σ-stonian. If every invariant probability for T is sequentially order continuous, then T is uniformly ergodic. Using the theory of P-sets we prove a variant of Lotz's uniform ergodic theorem, from which the theorem in question can be derived in a more direct fashion.

1. Introduction. Throughout, X will be a compact space, $C(X)$ the space of complex valued continuous functions on X, and T a Markov operator on $C(X)$. This means $T \geqq 0$ and $T1 = 1$. Let $T_n = (1/n)(I + \cdots + T^{n-1})$. T is called strongly ergodic if T_n converges to some projection P in the strong operator topology, and uniformly ergodic if the convergence is in the uniform operator topology. A regular Borel measure m is sequentially order continuous if whenever we have f_n in $C(X), f_1 \geqq f_2 \geqq \cdots \geqq 0$, and $\wedge f_n = 0$, then $|m|(f_n) \to 0$. A Markov operator is sequentially order continuous if for each sequence as in the last sentence we have $\wedge Tf_n = 0$.

In 1969 T. Ando proved the following theorem.

THEOREM A. [**1**, Theorem 2]. *If X is σ-stonian and T a sequentially order continuous Markov operator on $C(X)$, the following are equivalent:*

(a) *Every invariant Borel measure for T is sequentially order continuous,*

(b) *T is strongly ergodic, and* $\dim F(T) < \infty$, *where* $F(T) = \{f$ *in* $C(X)$: $Tf = f\}$.

(For generalizations, see [**3**] and [**9**].) In his thesis [**4**, Theorem 4.9], D. Axmann showed that in Ando's theorem strong ergodicity can be replaced by uniform ergodicity. His proof is based on spectral theory. (Actually, his results are more general, since they are stated for general AM spaces.)

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Recently, H. P. Lotz has shown that T need not be sequentially order continuous.

THEOREM B. [**8**, Corollary 8]. *If T is any Markov operator on $C(X)$, where X is σ-stonian, and every positive T-invariant measure is sequentially order continuous, then T is uniformly ergodic.*

Lotz's proof is a rather long and involved derivation from the following result (whose proof is quite interesting).

THEOREM C. [**8**, Theorem 1]. *Let T be a Markov operator on $C(X)$ which satisfies*

i) *T' is strongly ergodic,*

ii) *The fixed space $F(T')$ of T' has a weak order unit,*

iii) *Every T-invariant probability has non-meager support. Then T is quasi-compact and hence uniformly ergodic, and* $\dim F(T) = \dim F(T') < \infty$.

In this paper we use the theory of P-sets to prove a variation of Theorem C (Theorem 2.3 below) from which Theorem B may be derived by more elementary methods than those of [**8**].

1.1. Some terminology. A compact space X is σ-stonian if either of the equivalent conditions hold: (1) if A is an open F_σ set, then closure(A) is clopen (= open and closed), (2) $C(X)$ is a conditionally σ-complete vector lattice. X is stonian under the equivalent conditions: (1) if A is open then closure(A) is clopen, (2) $C(X)$ is conditionally complete as a vector lattice.

A positive Borel measure m is order-continuous under the equivalent conditions: (1) if $\{f_a\}$ is a directed family in $C(X)$ and $f = \sup f_a$, then $\int f d|m| = \lim \int f_a d|m|$, (2) if F is a closed nowhere dense set, then $m(F) = 0$. Let Σ be the Banach space of such measures on X.

A compact space X is hyperstonian if any of the equivalent conditions hold: (1) $C(X)$ is the Banach space dual of Σ, (2) X is stonian and Σ is weak-* dense in $C(X)^*$, (3) X is stonian and the union of the supports of measures in Σ is dense in X [10, p. 121–122].

If m is a positive measure, the closed support is denoted by sup(m).

2. A variation of Theorem C.

DEFINITION 2.1. A closed set F is a P-set if it is interior to any G_δ set which contains it. Equivalently, if A is an F_σ set disjoint from F, then closure(A) is disjoint from F.

The following properties are known, but for completeness we give proofs.

PROPOSITION 2.2. (a) *If F is a P-set and S the support of a positive regular*

Borel measure, then $F \cap S$ is relatively clopen in S. (b) *If m is a positive regular Borel measure which is sequentially order continuous, and the space X is σ-stonian, then $F = sup(m)$ is a P-set* [7, Lemma 1].

PROOF. (a) By regularity, there exist open $A_n \supset F$ with $m(A_n) < m(F) + n^{-1}$. Since F is a P-set there exists open A with $F \subset A \subset A_n$ for all n, whence $m(F) = m(A)$. Since $A \backslash F$ is an open m-null set, we get $A \cap S = F \cap S$.

(b) Let B be an F_σ set with $B \cap F = \emptyset$. Then there exists an open F_σ set A with $B \subset A$ and $A \cap F = \emptyset$. Since X is compact and totally disconnected, $A = \bigcup A_n$, where each A_n is clopen. If f_n is the characteristic function of $A_1 \cup \cdots \cup A_n$, then $0 = \int f_n \, dm$ for all n. If f is the characteristic function of the clopen set closure(A), then $f = \bigvee f_n$, so $\int f \, dm = 0$, and hence closure(A) $\cap F = \emptyset$.

THEOREM 2.3. *Let T be a Markov operator on $C(X)$ such that:*

i) *T' is strongly ergodic,*

ii) *$F(T')$ has a weak order unit,*

iii) *The support of a T-invariant probability measure is a P-set.*

Then T is uniformly ergodic and $\dim F(T) = \dim F(T') < \infty$.

PROOF. Let m be the weak order unit of $F(T')$ and let $M = \sup(m)$, a P-set. Since M is T-invariant, i.e., $f|M = 0$ implies $Tf|M = 0$, T induces in a natural way an operator on $C(M)$. (Namely if g is in $C(M)$, let $\bar{g}$ be its extension to an element of $C(X)$, and let $T_0 g = T\bar{g}|M$.) Clearly, T_0 satisfies the conditions (i) and (ii) of Lotz's Theorem C. For (iii), since the support of each invariant probability is a P-set, 2.2(a) implies that it is open in M. By Theorem C, T_0 is quasi-compact, hence uniformly ergodic.

We now use the fact that M is a P-set to show that $\{T_n\}$ is Cauchy sequence, so T is uniformly ergodic. Since

$$M = \text{closure} \bigcup \{\sup(r): r \text{ in } F(T')\},$$

[2, Lemma 3.3] implies $(T^n)'\delta_x(X\backslash M) \to 0$ uniformly in x. Let $\varepsilon > 0$, and choose k so that $(T^k)'\delta_x(X\backslash M) < \varepsilon/8$ for all x. Since $\|T\| = 1$, $\lim(n \to \infty)$ $\|T_n(T^k - I)\| = 0$, so there exists N such that if $m, n \geqq N$,

$$\|T_n T^k - T_n\| < \varepsilon/4 \tag{1}$$

$$\|(T_0)_n - (T_0)_m\| < \varepsilon/4. \tag{2}$$

Then, $m, n \geqq N$ implies

$$\begin{aligned}\|T_n - T_m\| &\leqq \|T_n - T_n T^k\| + \|T_n T^k - T_m T^k\| + \|T_m T^k - T_m\| \\ &< \varepsilon/2 + \|T^k(T_n - T_m)\|.\end{aligned}$$

We now show that if $\|f\| \leqq 1$ and $m, n \geqq N$, then $\|T^k(T_n f - T_m f)\| \leqq \varepsilon/2$. But (writing $(T^k)' \, \delta_x = t_x^k$),

$$\begin{aligned}|T^k(T_nf - T_mf)(x)| &\leqq \int_M |T_nf - T_mf|\, dt_x^k \\ &\quad + \int_{X\setminus M} |T_nf - T_mf|\, dt_x^k \\ &\leqq \varepsilon/4t_x^k(M) + 2t_x^k(X\setminus M) \\ &\leqq \varepsilon/4 + 2\,\varepsilon/8 = \varepsilon/2.\end{aligned}$$

3. Proof of Theorem B. Most of the proof consists of lemmas which show that under the hypotheses of Theorem B, conditions i), ii), and iii) of Theorem 2.3 are fulfilled. Our proofs are direct and elementary, in contrast to [**8**], which relies heavily on Banach space theory. (See, e.g., Lemmas 1 and 3 and Corollaries 7 and 8 of [**8**].) Some of the methods here are slight modifications of [**3**].

LEMMA 3.1. *If X is σ-stonian, m a sequentially order continuous regular Borel probability measure, and S =* sup(m), *then the relative topology on S is hyperstonian.*

PROOF. By [**11**, Theorem 2.2], the relative topology is stonian, and it sufficies to prove that the restriction of m to S is order continuous. Let F be a nowhere dense closed subset of S. We must prove $m(F) = 0$ [**10**, page 149, exercise 24]. First we find a nowhere dense closed G_δ set W in X such that $F \subset W$. By regularity of m there exist clopen V_n with $F \subset Z = \bigcap V_n$ and $m(Z\setminus F) = 0$. Now $Z \cap S$ is nowhere dense in the relative topology of S. (Otherwise there exists clopen B in X with $\emptyset \neq B \cap S \subset Z \cap S$. Since F is closed and nowhere dense in S, $B \cap S \not\subset F$, so $(B\setminus F) \cap S \neq \emptyset$, whence $m(B\setminus F) > 0$. But $(B\setminus F) \cap S \subset (Z\setminus F) \cap S$, so $m(Z\setminus F) > 0$, a contradiction.) Since X is σ-stonian, A = interior(Z) is clopen. Since $Z \cap S$ is nowhere dense in S, $A \cap S = \emptyset$, so $W = Z\setminus A$ is a closed G_δ set nowhere dense in X with $W \cap S = Z \cap S \supset F$. Since m (as a measure on X) is sequentially order continuous, $m(W) = 0$, so $m(F) = 0$.

LEMMA 3.2. *Let T be a Markov operator on $C(X)$, where X is σ-stonian. If m_1 and m_2 are two distinct extreme invariant probabilities for T which are both sequentially order continuous, then* $\sup(m_1) \cap \sup(m_2) = \emptyset$.

PROOF. Let $S = \sup(m_1) \cup \sup(m_2) = \sup(m_1 + m_2)$. By Lemma 3.1, S is hyperstonian and m_1, m_2 are order continuous on S. (Note that by Proposition 2.2, $\sup(m_1)$ and $\sup(m_2)$ are clopen in S.) Since $m_1 \neq m_2$ there exists f in $C(S)$ with $m_1(f) \neq m_2(f)$. Since $\{T_nf\colon n \geqq 1\}$ is a norm bounded sequence in $C(S)$ and $C(S)$ is the dual of the order continuous measures on S, the sequence is pre-compact. Let g in $C(S)$ be a weak-* cluster point and $\{T_{n(a)}f\colon a \text{ in } A\}$ a subnet with $T_{n(a)}f \to g$ weak-*. Then for $i = 1, 2$ we have $m_i(f) = m_i(T_{n(a)}f) \to m_i(g)$, so $m_1(g) \neq m_2(g)$. But since $Tg = g$ (Proof:

$$\|T_{n(a)}Tg - T_{n(a)}g\| \leqq n(a)^{-1} \|T^{n(a)}g - g\| \\ \leqq 2n(a)^{-1} \|g\| \to 0,$$

whence $p(Tg) = p(g)$ for each order continuous measure p. Since these are weak-* dense in $C(S)'$, we have $Tg = g$), and m_1 and m_2 are extreme invariant probabilities on S, [**12**, Theorem 1.11] implies g is constant on $\sup(m_1)$ and on $\sup(m_2)$. Hence the sets are disjoint.

LEMMA 3.3. *Under the hypotheses of Theorem B, there exist at most a finite number of distinct extreme invariant probabilities for T.*

PROOF. Suppose $\{m_k\}$ is an infinite sequence of distinct extreme probabilities, and let $S_k = \sup(m_k)$. Then $\{S_k\}$ is a sequence of pairwise disjoint P-sets, and a routine induction gives a sequence of pairwise disjoint clopen sets $\{A_n\}$ with $S_k \subset A_k$. If f_n is the characteristic function of $B_n = \text{closure} \bigcup \{A_k: k \geqq n\}$, then $\bigwedge f_n = 0$, while if m is a weak-* cluster point of $\{m_k\}$, then $m(f_n) = 1$ for all n. Thus m is an invariant probability which is not sequentially order continuous, contrary to hypothesis.

3.4. Proof of Theorem B. By 2.2 (b), condition iii) of 2.3 holds. If $m_1, \ldots, m_R$ are the extreme invariant probabilities for T, then $m = R^{-1}(m_1 + \cdots + m_R)$ is a weak order unit for $F(T')$, so ii) holds. For i), we first show that T is strongly ergodic. Now for each i, $S_i = \sup(m_i)$ is a T-invariant set, and the operator induced by T on $C(S_i)$ has m_i as its unique invariant probability (by lemma 3.2). Hence by [**12**, Theorem 2.7], this operator is strongly ergodic. It follows that the operator induced by T on the P-set $S = \bigcup S_i$ is strongly ergodic. Finally if f is in $C(X)$ we must prove $\{T_n f\}$ is uniformly Cauchy. Now

$$\|T_n f - T_m f\| \leqq \|T_n f - T^k T_n f\| + \|T^k T_n f - T^k T_m f\| \\ + \|T^k T_m f - T_m f\|.$$

As in the proof of 2.3, for fixed k the first and last terms can be made uniformly small, while the middle term is handled as in the end of that proof, again using [**2**, Lemma 3.3]. (Cf. [**2** lemma 3.4].) Since T is strongly ergodic and X is σ-stonian, T' is strongly ergodic [**1**, lemma 2].

4. Final remark. W. Bartoszek has shown in [**5**] that condition (ii) on weak order unit is not needed in Lotz' uniform ergodic theorem, and in [**6**] that it is not needed in Theorem 2.3 above.

REFERENCES

1. T. Ando, *Positive Kontraktionen in C(X)*, Studia Math. **31** (1968), 173–187.

2. R. Atalla, *Generalized almost convergence vs. matrix summability*, Colloq. Math. **47** (1982), 103–111.

3. ———, *Markov operators and quasi-stonian spaces*, Proc. Amer. Math. Soc. **82** (1981), 613–618.

4. D. Axmann, *Struktur-und Ergodentheorie irreduzibler Operatoren auf Banachverbänden*, dissertation, Tübingen, 1980.

5. W. Bartoszek, *On a theorem of H. P. Lotz on quasi-compactness of Markov operators*, Colloq. Math., to appear.

6. ———, *On quasi-compactness and invariant measures of Markov operators on C(X)*, Bull. de l'acad. Polonaise des sci., to appear.

7. S. Lloyd, *Two lifting theorems*, Proc. Amer. Math. Soc. **42** (1974), 128–134.

8. H. Lotz, *Uniform ergodic theorems for Markov operators in C(X)*, Math. Z. **178** (1981), 145–156.

9. R. Sato, *Invariant measures and ergodic theorems for positive operators*, Math. J. Okayama Univ. **22** (1979), 77–90.

10. H. Schaefer, *Banach Lattices and Positive Operators*, Springer, 1974.

11. G. Seever, *Measures on F-spaces*, Trans. Amer. Math. Soc. **133** (1968), 267–280.

12. R. Sine, *Geometric theory of a single Markov operator*, Pac. J. Math. **27** (1968), 155–166.

DEPARTMENT OF MATHEMATICS, OHIO UNIVERSITY, ATHENS, OH 45701