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SPACES FORMED BY SPECIAL ATOMS, I

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"Dedicated to Ismenia Sales de Souza, my wife"

1. Introduction. C. Fefferman and E. M. Stein [3] and R. R. Coifman [1] observed that a real-valued function f in $L_1(T)$, (where T is the perimeter of the unit disk in the complex plane) is the real part of a boundary function $F \in H^1(\mathbf{D})$ ($F \in H^1(\mathbf{D})$ if and only if $\|F\|_{H^1} = \text{Sup}_{r<1} \int_T |F(re^{i\theta})|\, d\theta < \infty$, where $\mathbf{D} = \{z \in C;\ |z| < 1\}$) if and only if there is a sequence (a_n), of atoms and a sequence (c_n), of numbers, such that $\sum_{n=1}^{\infty} |c_n| < \infty$ and $f(t) = \sum_{n=1}^{\infty} c_n a_n(t)$. (A real valued function defined on T is called an *atom* whenever a is supported on an interval $I \subset T$, $|a(t)| \leqq |I|^{-1}$ and $\int_I a(t)dt = 0$.) Moreover, letting $\lambda(f)$ equal the infimum of $\sum_{n=1}^{\infty} |c_n|$ over all such representations of f, there exist absolute constants M and N such that $M\|F\|_{H^1} \leqq \lambda(f) \leqq N\|F\|_{H^1}$; we shall denote the set of all such f as ReH^1 and $\|f\|_{ReH^1} = \lambda(f)$.

ReH^1 is well known as the atomic decomposition of $H^1(\mathbf{D})$. C. Fefferman and E. M. Stein in their famous paper [3], proved that the dual of ReH^1 is the space $BMO = \{f \in L_1(T);\ \|f\|_{BMO} = \text{Sup}_{I \subset T}(1/|I|) \int_I |f(t) - f_I| dt < \infty\}$ where $f_I = (1/|I|) \int_I f(t)dt$, originally introduced by F. John and L. Nirenberg [4]. *BMO* stands for bounded mean oscillation and I is an interval.

In this paper we introduce a new function space B defined by $B = \{f\colon T \to \mathbf{R}, f(t) = \sum_{n=1}^{\infty} c_n b_n(t);\ \sum_{n=1}^{\infty} |c_n| < \infty\}$. Each b_n is a special atom, that is, a real-valued function b, defined on T, which is either $b(t) \equiv 1/2\pi$ or $b(t) = -(1/|I|)\chi_R(t) + (1/|I|)\chi_L(t)$, where I is an interval on T, L is the left half of I and R is the right half. $|I|$ denotes the length of I and χ_E the characteristic function of E. B is endowed with the norm $\|f\|_B = \text{Inf} \sum_{n=1}^{\infty} |c_n|$, where the infimum is taken over all representations of f, which becomes a Banach space. At this point, a natural question is: Is B topologically equivalent to ReH^1? In other words, do there exist positive constants C and D such that $C\|f\|_B \leqq \lambda(f) \leqq D\|f\|_B$? Regarding

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this matter one could ask another question, whose answer might help solve the first one: Is it true that $L_2(T)$ is continuously contained in B?

In trying to answer this question, we were led to the computation of the dual space of B, which is the key result of this paper. In fact, right after the duality theorem, we have applications which provide the answer to the above questions. In general in this paper we describe some interesting properties of the space B, as a Banach space.

To make the presentation reasonably self-contained, we shall include a resumé of pertinent results and definitions.

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2. Preliminaries.

DEFINITON 2.1. A function f belongs to $L_p(T)$ for $1 \leqq p < \infty$ if and only if $\|f\|_p = (\int_T |f(t)|^p \, dt)^{1/p} < \infty$. For $p = \infty$ we take $\|f\|_\infty =$ Ess $\text{Sup}_{t \in T}|f(t)|$.

DEFINITION 2.2. The Zygmund space Λ_* is defined by $\Lambda_* = \{g: T \to \mathbf{R},$ continuous, $g(x + h) + g(x - h) - 2g(x) = O(h)\}$. The Λ_* norm is given by

$$\|g\|_{\Lambda_*} = \sup_{\substack{h>0 \\ x}} \left| \frac{g(x + h) + g(x - h) - 2g(x)}{2h} \right|.$$

For more details about this space the reader may refer to A. Zygmund **[12]**.

Observe that if we consider the space $B_0 = \{f\colon T \to \mathbf{R};\ f(t) = \sum_{finite} c_n b_n(t)\}$ where the b_n's are special atoms, and $\|f\|_{B_0} = \text{Inf} \sum_{finite} |c_n|$, where the infimum is taken over all possible representations of f, then we regard the space B as the completion of B_0 under the norm $\| \ \|_{B_0}$.

3. Some Properties of B. In this section we state and prove some properties of the space B.

LEMMA 3.1. *B is an embedding in ReH^1, that is, the inclusion mapping is a bounded linear operator.*

PROOF. It is obvious from the definition on ReH^1 and B that $\|f\|_{ReH^1} \leqq \|f\|_B$.

LEMMA 3.2. *B is an embedding in $L_1(T)$.*

PROOF. From the definition of the L_1-norm it is clear that $\|f\|_1 \leqq \|f\|_B$.

We will prove later on that the inclusion in Lemma 3.1 is proper, that is, there is an f in ReH^1 such that f does not belong to B.

LEMMA 3.3. *Let (f_n) be a sequence in B, and f such that f_n converges to f in B-norm. Then for any $g \in L_\infty(T)$ we have*

$$\lim_{n\to\infty} \int_T f_n(t)g(t)dt = \int_T f(t)g(t)dt.$$

PROOF. It follows easily from $|\int_T (f_n(t)-f(t))g(t)dt| \leqq \|f_n - f\|_1 \cdot \|g\|_\infty$ and Lemma 3.2.

LEMMA 3.4. *If $f \in B$ and $g \in \Lambda_*$ then $\lim_{r\to 1} \int_T f(t)g_r'(t)dt$ exists, where $g_r = P_r * g$, P_r is the Poisson kernel and the dash means derivative.*

PROOF. This lemma follows easily from the fact that $g_r \to g$ uniformly as $r \to 1$ and $\|g_r\|_{\Lambda_*} \leqq \|g\|_{\Lambda_*}$.

THEOREM 3.5. (HÖLDER'S TYPE INEQUALITY). *If $f \in B$ and $g \in \Lambda_*$ then $|\lim_{r\to 1} \int_T f(t)g_r'(t)dt| \leqq \|f\|_B \cdot \|g\|_{\Lambda_*}$ where g_r is as in the previous lemma.*

PROOF. Observe that we just need to prove this theorem for a non-constant special atom, say for

$$f(t) = -\frac{1}{2h}\chi_{[\alpha-h,\alpha)}(t) + \frac{1}{2h}\chi_{[\alpha,\alpha+h]}(t).$$

In fact, we have $\lim_{r\to 1} \int_T f(t)g_r'(t)dt = (g(\alpha+h) + g(\alpha-h) - 2g(\alpha))/2h$ and thus by definition of Λ_*-norm we get $|\lim_{r\to 1} \int_T f(t)g_r'(t)dt| \leqq \|g\|_{\Lambda_*}$, and since $\|f\|_B = 1$, we have $|\lim_{r\to 1} \int_T f(t)g_r'(t)dt| \leqq \|f\|_B\|g\|_{\Lambda_*}$, the proof for the constant special atom is trivial and also for a finite linear combination of special atoms; consequently, the extension to any f in B follows easily from the definition of B and Lemma 3.3.

The next result gives us a different way to define a norm in the Zygmund space Λ_*.

COROLLARY 3.6. *If $f \in B$ and $g \in \Lambda_*$, then*

$$\|g\|_{\Lambda_*} = \sup_{\|f\|_B \leqq 1} \left| \lim_{r\to 1} \int_T f(t)g_r'(t)dt \right|$$

where g_r is as before.

PROOF. By Theorem 3.5 we get

$$\sup_{\|f\|_B \leqq 1} \left| \lim_{r\to 1} \int_T f(t)g_r'(t)dt \right| \leqq \|g\|_{\Lambda_*}.$$

On the other hand, if

$$f(t) = -\frac{1}{2h}\chi_{[\alpha-h,\alpha)}(t) + \frac{1}{2h}\chi_{[\alpha,\alpha+h]}(t)$$

then

$$\sup_{\|f\|_B \leq 1} \left| \lim_{r\to 1} \int_T f(t) g_r'(t) dt \right| \geqq \left| \frac{g(\alpha + h) + g(\alpha - h) - 2g(\alpha)}{2h} \right|$$

so that

$$\sup_{\|f\|_B \leq 1} \left| \lim_{r\to 1} \int_T f(t) g_r'(t) dt \right| \geqq \|g\|_{\Lambda_*}$$

and so combining these two inequalities involving Λ_*-norm we get the desired result.

4. Duality. Consider the mapping ψ_g: $B \to \mathbf{R}$ defined by $\psi_g(f) = \lim_{r\to 1} \int_T f(t) g_r'(t) dt$, where g is a fixed function in Λ_* and g_r as before. One can easily see that ψ_g is a linear functional on B. Moreover, theorem 3.5 (Hölder's Type Inequality) tells us that $|\psi_g(f)| \leqq \|g\|_{\Lambda_*} \cdot \|f\|_B$, and therefore ψ_g is also bounded. Consequently, we see that for each $g \in \Lambda_*$, ψ_g is a bounded linear functional on B. At this point, a natural question is: Are these all the linear functionals on B? We anticipate that the answer is yes; in order to formulate the theorem which leads to this answer, we need the next two results.

THEOREM 4.1. *If we define* $\Lambda_*' = \{g' ; g \in \Lambda_*\}$ *and put the norm on* Λ_*' *by setting* $\|g'\|_{\Lambda_*'} = \|g\|_{\Lambda_*}$, *then* Λ_*' *endowed with* $\| \ \|_{\Lambda_*'}$ *is a Banach space. The dash means the derivative.*

Before proving this theorem we would like to point out that the concept of derivative that is being used is the general notion given to us by the theory of distribution, that is, we say $g' = h$ if $\int_T g(t)\psi'(t)dt = -\int_T h(t)\psi(t)dt$ for all ψ infinitely differentiable functions ψ on T. Integration by parts shows us that this is indeed the relation that we would expect if g had continuous derivative, and $g' = h$ would have the usual meaning.

PROOF OF THEOREM 4.1. Follows easily from the fact that $(\Lambda_*, \| \ \|_{\Lambda_*})$ is a Banach space.

THEOREM 4.2. If χ_I *is the characteristic function of an interval and* $I \subset [0, 2\pi]$, *then* $\chi_I \in B$, *moreover* $\|\chi_I\|_B \leqq C\,|I| \log(2\pi/|I|)$ *where* C *is an absolute constant.*

PROOF. One can easily observe that it suffices to prove this theorem for $I = [0, 2\pi/2^N]$ where N is a fixed non-negative integer. The idea is to expand χ_I in Haar-Fourier series on $[0, 2\pi]$. In fact, we recall that the Haar system on $[0, 2\pi]$ is defined by

$$\psi_{nk}(t) = \begin{cases} \left(\frac{2^n}{2\pi}\right)^{1/2} \text{ on } \left[\frac{k-1}{2^n} 2\pi, \frac{k-1/2}{2^n} 2\pi\right), \\ -\left(\frac{2^n}{2\pi}\right)^{1/2} \text{ on } \left[\frac{k-1/2}{2^n}, \frac{k}{2^n} 2\pi\right], \\ 0 \text{ elsewhere.} \end{cases}$$

Consequently, the expansion of χ_I is

$$\chi_I(t) = \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} a_{nk}\psi_{nk}(t),$$

where

$$a_{nk} = \int_{I_{nk}} \chi_I(t)\psi_{nk}(t)dt, \; I_{nk} = \left[\frac{k-1}{2^n} 2\pi, \frac{k}{2^n} 2\pi\right].$$

If we split I_{nk} as in the definition of (ψ_{nk}) then the geometry of I_{nk} and $[0, 2\pi/2^N]$ shows that $a_{nl} \neq 0$ for $0 \leqq n < N$ and $a_{nk} = 0$ otherwise, Thus the expansion of χ_I in Haar-Fourier series becomes

$$\chi_I(t) = \sum_{n=0}^{N-1} a_{nl}\psi_{nl}(t), \tag{4.3}$$

so by computing the coefficients a_{nl} we get $a_{nl} = (2^n/2\pi)^{1/2} \cdot s$ where $s = 2\pi/2^N$. Substituting these values into (4.3), we have

$$\chi_I(t) = \sum_{n=0}^{N-1} s\left(\frac{2^n}{2\pi}\right)^{1/2} \psi_{nl}(t).$$

Now observe that $b_n(t) = (2^n/2\pi)^{1/2} \psi_{nl}(t)$ are special atoms for $n = 0, 1, 2, \ldots, N-1$, therefore $\chi_I \in B$; moreover by definition of B-norm we have $\|\chi_I\|_B \leqq \sum_{n=0}^{N-1} s$, that is $\|\chi_I\|_B \leqq Ns$, since $s = 2\pi/2^N$, then $N = \log_2(2\pi/s)$ and thus $\|\chi_I\|_B \leqq s \log_2(2\pi/s)$. Consequently $\|\chi_I\|_B \leqq Cs \log(2\pi/s)$ where $C = 1/\log 2$ and therefore we have $\|\chi_I\|_B \leqq C\,|I|\,\log(2\pi/|I|)$. Now if $I = [0, s]$ where s is an arbitrary number in $(0, 2\pi]$, we can write the dyadic expansion of s and apply the above argument. Finally if I is any interval, say $I = (\alpha, \beta]$, $0 < \alpha < \beta \leqq 2\pi$, then $\chi_I = \chi_{[0,\beta]} - \chi_{[0,\alpha]}$, and so $\chi_I \in B$. On the other hand, observe that the operator $T_a f = f^a$ where $f^a(x) = f(x-a)$, maps B boundedly into B, in fact $\|T_a f\|_B \leqq \|f\|_B$, so if we take $f(t) = \chi_{(0,\beta-\alpha]}(t)$ then $f^\alpha(t) = \chi_{(\alpha,\beta]}(t)$ and therefore

$$\|\chi_{(\alpha,\beta]}\|_B \leqq \|\chi_{(0,\beta-\alpha]}\|_B \leqq C(\beta-\alpha)\log\frac{2\pi}{\beta-\alpha}\,;$$

therefore the theorem is proved.

The next result answers the question formulated right before the two previous theorems determining the linear functionals on B. It can be regarded as one of the most relevant results of this paper.

Throughout this paper X^* will denote the dual space of X, that is, the space of bounded linear functionals ψ on X with the norm

$$\|\psi\| = \sup_{\|f\|_X \leq 1} |\psi(f)|.$$

THEOREM 4.4. (DUALITY THEOREM). *If* $\psi \in B^*$ *then there is a unique*

$g \in \Lambda_*$ such that $\phi = \phi_g$, that is, $\phi(f) = \lim_{r\to 1} \int_T f(t)g_r'(t)dt$ for all $f \in B$, where g_r is as before; moreover, $\|\phi\| = \|g\|_{\Lambda_*}$. Conversely if $\phi(f) = \lim_{r\to 1} \int_T f(t)g_r'(t)dt$ then $\phi \in B^*$. Furtheremore the mapping φ: $\Lambda_*' \to B^*$ defined by $\varphi(g') = \phi_g$ is an isometric isomorphism.

PROOF. If $\phi(f) = \lim_{r\to 1} \int_T f(t)g_r'(t)dt$, then we already have seen that the theorem 3.5 implies that ϕ is a bounded linear functional, that is, $\phi \in B^*$, so it remains to prove the first part. In fact, let $\phi \in B^*$ and define $g(s) = \phi(\chi_{[0,s]})$ for $s \in [0, 2\pi]$. Observe that $g(s+h) - g(s) = \phi(\chi_{(s,s+h]})$ and thus Theorem 4.2 and the boundedness of ϕ tell us that g is continuous. On the other hand, using the definition of g we get

$$\frac{g(s+h) + g(s-h) - 2g(s)}{2h} = \phi\left(\frac{1}{2h}\chi_{(s,s+h]} - \frac{1}{2h}\chi_{(s-h,s]}\right). \tag{4.5}$$

Consequently, since $b(t) = (1/2h)\,\chi_{(s,s+h]}(t) - (1/2h)\,\chi_{(s-h,s]}(t)$ is a special atom, we have that $\|b\|_B = 1$. Therefore using the boundedness of ϕ in (4.5) we get $|g(s+h) + g(s-h) - 2g(s)|/|2h| \leqq \|\phi\| < \infty$, so that $\|g\|_{\Lambda_*} < \infty$ and therefore $g \in \Lambda_*$.

Let $b(t) = (1/2h)\,\chi_{[\alpha-h,\alpha)}(t) - (1/2h)\,\chi_{[\alpha,\alpha+h]}(t)$ and $P = \{0 = t_0 < t_1 < \cdots < t_n = 2\pi\}$ be a partition of $T = [0, 2\pi]$, we may assume that $\alpha - h$, α, $\alpha + h$ belong to P, otherwise we consider a new partition P' having them inserted.

Observe now that we may write

$$b(t) = \sum_{i=1}^{n} b(t_{i-1})[\chi_{[0,t_i]}(t) - \chi_{[0,t_{i-1}]}(t)]$$

for $t \in [0, 2\pi]$, therefore since ϕ is linear, we get that $\phi(b) = \sum_{i=1}^{n} b(t_{i-1})[g(t_i) - g(t_{i-1})]$. Thus we have that $\lim_{n\to\infty} \sum_{i=1}^{n} b(t_{i-1})[g(t_i) - g(t_{i-1})]$ exists, which we denote by

$$\lim_{n\to\infty} \sum_{i=1}^{n} b(t_{i-1})[g(t_i) - g(t_{i-1})] = \int_T b(t)dg(t)$$

therefore

$$\phi(b) = \int_T b(t)dg(t). \tag{4.6}$$

Since $g \in \Lambda_*$ is not necessarily a differentiable function in the usual sense, we consider a smooth regularization of g, namely $g_r = P_r * g$ where P_r is the Poisson kernel, Then $dg(t) = g_r'(t)dt$ and (4.6) becomes

$$\phi(b) = \lim_{r\to 1} \int_T b(t)g_r'(t)dt. \tag{4.7}$$

Observe that Lemma 3.4 tells us that this limit indeed exists, so that the functional representation for B is proved for special atoms and so for a finite linear combination of them. Therefore the extension for any $f \in B$ follows from the definition of B and lemma 3.3. Furthermore from corollary 3.6 it follows that $\|\varphi\| = \|g\|_{\Lambda_*}$ and thus by the definition of Λ'_*-norm we get φ as an isometry, so that the theorem is proved.

5. Applications. As applications of the duality theorem we shall use it to answer the questions posed in the introduction.

APPLICATION 1. *ReH^1 is not topologically equivanlet to B.*

PROOF. If ReH^1 is continuously contained in B, then B^* is continuously contained in $(ReH^1)^*$ so by the C. Eefferman and E. M. Stein duality theorem in [3] and theorem 4.4 we get Λ'_* is continuously contained in BMO. So that for every $g \in \Lambda_*$, it follows that $g' \in BMO$. But for $g(x) = \sum_{n=1}^{\infty} b^{-n} \cos b^n x$, $b > 1$, the Weierstrass function (which belongs to Λ_*, see [12]), would imply that $g' \in BMO$ which is absurd, since g is wellknown to be nowhere differentiable, indeed g' is a distribution, but BMO is a space of functions.

APPLICATION 2. *$L_2(T)$ is not continuously contained in B.*

PROOF. Use the same argument as in Application 1.

APPLICATION 3. As a third application we will give a proof of a remark made by A. Zygmund and E. M. Stein [11]. That is, *if $g \in BMO$, then $G(x) = \int_0^x g(t)dt$ is in the Zygmund space Λ_*, moreover $\|G\|_{\Lambda_*} \leqq M\|g\|_{BMO}$.*
In fact, by Lemma 3.1, B is continuously contained in ReH^1 so that BMO is continuously contained in Λ'_*; then $g \in BMO$ implies $g = G'$ for some $G \in \Lambda_*$, that is, $G(x) = \int_0^x g(t)dt \in \Lambda_*$. By boundedness of the inclusion operator, we have $\|G'\|_{\Lambda'_*} \leqq M\|g\|_{BMO}$ and therefore by definition of Λ'_*-norm, we get $\|G\|_{\Lambda_*} \leqq M\|g\|_{BMO}$.

6. More Results About B. In this section we point out some results, which we believe are important for better understanding of the space B.

THEOREM 6.1. *If $g \in BMO$ and $f \in B$, then $|\int_T f(t)g(t)dt| \leqq \|f\|_B \cdot \|G\|_{\Lambda_*}$, where $G(x) = \int_0^x g(t)dt$.*

PROOF. The proof of this theorem follows the same idea as the proof of Theorem 3.5.

COROLLARY 6.2. *If $f \in B$ and $g \in BMO$ then*

$$\|G\|_{\Lambda_*} = \sup_{\|f\|_B \leq 1} \left| \int_T f(t)\, g(t)dt \right|.$$

PROOF. Use the same idea as in Corollary 3.6.

Note that after application 3, these two results are basically included in

Theorem 3.5 and Corollary 3.6; however, for use as references we prefer to state them explicitly here.

As we have seen in Lemma 3.1, $B \subset ReH^1$. Then a very natural question to ask is: "How big" is B in comparison with ReH^1? The answer to this question is contained in the following result.

THEOREM 6.3. *B is a dense subset of ReH^1, that is, given $h \in ReH^1$ and a positive ε, there exists an $f \in B$ such that $\|f - h\|_{ReH^1} \leqq \varepsilon$. That is, ReH^1 is the closure of B under the ReH^1-norm. We may write $\bar{B} = ReH^1$.*

PROOF. By Lemma 3.1, we have $B \subset ReH^1$, so that by taking the closure in ReH^1, we get $\bar{B} \subset ReH^1$.

On the other hand, if $f_0 \in ReH^1$ and $f_0 \notin \bar{B}$, as $\bar{B}$ is closed, then by a corollary of the Hahn-Banach theorem, there is a bounded linear functional ϕ on ReH^1 such that $\phi(f) = 0$ on $\bar{B}$ and $\phi(f_0) = 1$. Now the duality theorem for ReH^1 implies the existence of a unique $g \in BMO$ such that $\phi(f) = \int_T f(t)g(t)dt$, so that $\int_T f(t)g(t)dt = 0$ for all $f \in \bar{B}$, in particular for $f \in B$. By Corollary 6.2 we have that

$$\|G\|_{\Lambda_*} = \operatorname*{Sup}_{\|f\|_B \leq 1} \left| \int_T f(t)g(t)dt \right| \text{ where } G(x) = \int_0^x g(t)dt,$$

so that $\|G\|_{\Lambda_*} = 0$, which implies $G(x) = \int_0^x g(t)dt = \text{constant}$, hence $g(x) = 0$ almost everywhere. Thus $\phi \equiv 0$, which is absurd since $\phi(f_0) = 1$. Hence, we have proved $\bar{B} = ReH^1$.

There are some recent results about B, whose proofs are not included here, but the interested reader can consult the references. One of these results is as follows:

$$\|f\|_B = \operatorname*{Sup}_{\|g\|_{\Lambda_*} \leq 1} \left| \lim_{r \to 1} \int_T f(t)g_r'(t)dt \right|,$$

where g_r is as before.

We would like to point out that the space B has been generalized by the author in two directions, namely if we define a special p-atom by setting

$$b(t) = \frac{-1}{|I|^{1/p}} \chi_R(t) + \frac{1}{|I|^{1/p}} \chi_L(t),$$

where R, L and I are as before and $1/2 < p < \infty$. Now consider the space B^p defined by $B^p = \{f: T \to \mathbf{R}, f(t) = \sum_{n=1}^\infty c_n b_n(t);\ \sum_{n=1}^\infty |c_n| < \infty\}$. We endow B^p with the norm $\|f\|_{B^p} = \operatorname{Inf} \sum_{n=1}^\infty |c_n|$ where the infimum is taken over all possible representations of f. Similarly we define the space C^p by

$$C^p = \{f: T \to \mathbf{R}, f(t) = \textstyle\sum_{n=1}^\infty c_n b_n(t); \sum_{n=1}^\infty |c_n|^p < \infty\}.$$

C^p is endowed with the "norm", $\|f\|_{C^p} = \text{Inf} \sum_{n=1}^{\infty} |c_n|^p$, where the infimum is taken over all possible representations of f.

We are discussing these spaces in our paper, "Spaces formed by special atoms, II", which will be ready soon. However, we refer the interested reader to [5] and [6].

One of the important features of the spaces B^p for $1/2 < p < 1$ is that B^p can be identified with the space of analytic functions for the disk $\mathbf{D} = \{z \in C; |z| < 1\}$ satisfying

$$\int_0^1 \int_0^{2\pi} |f(re^{i\theta})| (1 - r)^{1/p-2} \, d\theta \, dr < \infty.$$

These spaces were introduced by P. L. Duren, B. W. Romberg and A. L. Shields in [2], and for $p = 1$, B is identified with the pre-dual of the Bloch functions, namely, the space S of those analytic functions g on $\mathbf{D}$ such that $\int_0^1 \int_0^{2\pi} |g'(re^{i\theta})| \, d\theta \, dr < \infty$; for these observations the reader may refer to [6] and [10].

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References

1. R.R. Coifman, *A real variable characterization of H^p*, Studia Math., **51** (1974), 269–274.

2. P.L. Duren, B.W. Romberg and A.L. Shields, *Linear functionals on H^p with $0 < p < 1$*, J. Reine Angen Math., **238** (1969), 32–60.

3. C. Fefferman and E.M. Stein, *H^p-spaces of several variables*, Acta Math., **129** (1972), 137–193.

4. F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14** (1961), 415–426.

5. Geraldo Soares de Souza, *Spaces formed by special atoms*, Ph. D. dissertation, SUNY at Albany, 1980.

6. ——— and Richard O'Neil, *Spaces formed with special atoms*. To appear Proceedings Conference on Harmonic Analysis, 1980, Italy.

7. ———, *A class of functions in B*, preprint.

8. ———, *The dyadic special atom space*. To appear Proceedings Conference on Harmonic Analysis, Minneapolis.

9. ———, *A subspace of H^1 stable under the Hilbert transform*. Presented in the 792nd meeting of The AMS, Cincinnati, Ohio, under the title "*A relationship with L $\log^+ L$ and the Hilbert transform*".

10. ——— and Gary Sampson, *A real characterization of the pre-dual of the Bloch function*, J. London Math. Soc. (2), **27** (1983), 267–276.

11. A. Zygmund and E.M. Stein, *Boundedness of the translation operators on Hölder and L_p spaces*, Ann. of Math., **86** (1967), 337–349.

12. ———, *Trigonometric Series*, 2nd rev. ed. Vols I, II, Cambridge University Press, New York, 1959.

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