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CAMPBELL'S CONJECTURE ON A MAJORIZATION-SUBORDINATION RESULT FOR CONVEX FUNCTIONS

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Let S denote the set of all normalized analytic univalent functions f, $f(z) = z + \cdots$, in the open unit disc U. Let f, F, and w be analytic in $|z| < r$. We say that f is majorized by F, $f \ll F$, in $|z| < r$, if $|f(z)| \leqq |F(z)|$ in $|z| < r$. We say that f is subordinate to F, $f \prec F$, in $|z| < r$ if $f(z) = F(w(z))$ where $|w(z)| \leqq |z|$ in $|z| < r$.

Majorization-subordination theory begins with Biernacki who showed in 1936 that if $f'(0) \geqq 0$ and $f \prec F (F \in S)$, in U, then $f \ll F$ in $|z| < 1/4$. In the succeeding years Goluzin, Tao Shah, Lewandowski and MacGregor examined various related problems (for greater detail see [1]).

In 1951 Goluzin showed that if $f'(0) \geqq 0$ and $f \prec F\,(F \in S)$ then $f' \ll F'$ in $|z| < 0.12$. He conjectured that majorization would always occur for $|z| < 3 - \sqrt{8}$ and this was proved by Tao Shah in 1958.

In a series of papers [1, 2, 3], D. Campbell extended a number of the results to the class $\mathscr{U}_\alpha$ of all normalized locally univalent ($f'(z) \neq 0$) analytic functions in U with order $\leqq \alpha$ where $\mathscr{U}_1 = K$ is the class of convex functions in S. In particular in [3] he showed that if $f'(0) \geqq 0$ and $f \prec F (F \in \mathscr{U}_\alpha)$ then $f' \ll F'$ in $|z| < \alpha + 1 - (\alpha^2 + 2\alpha)^{1/2}$ for $1.65 \leqq \alpha < \infty$ where $\alpha = 2$ yields $3 - \sqrt{8}$. Note that $\alpha = 1$ yields $2 - \sqrt{3}$, the radius of convexity for S. Campbell's proof breaks down for $1 \leqq \alpha < 1.65$ because of two different bounds being used for the Schwarz function with different ranges of α. Nevertheless, he conjectured that the result is true for all $\alpha \geqq 1$.

In this paper we combine a subordination result of Ruscheweyh's, some variational techniques and some tedious computations to verify the conjecture for $\alpha = 1$, i.e., we show that if $f'(0) \geqq 0$ and $f \prec F (F \in K)$ in U then $f' \ll F'$ for $|z| \leqq 2 - \sqrt{3}$. We note that our method of proof relies on the convexity of F in a number of places so that it is unlikely that it would extend to larger α's.

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THEOREM. *Let $f \prec F$ with $f'(0) \geqq 0$. Then $f' \ll F'$ in $|z| \leqq 2-\sqrt{3}$ for all F in K and the result is sharp.*

PROOF. Sharpness follows by considering $F(z) = z/(1 - z)$ and $f(z) = z^2/(1 - z^2)$. A Schwarz function is a function w analytic on U with $|w(z)| \leqq |z|$. Let $|z| \leqq 2 - \sqrt{3} = r_0$ and w a fixed but arbitrary Schwarz function with $w'(0) \geqq 0$. We must show that

$$\max_{|z|\leqq r_0} \operatorname{mas}_{F\in K} \left|\frac{F'(w(z))\cdot w'(z)}{F'(z)}\right| \leqq 1.$$

Ruscheweyh has proved in [5, p. 277] that if g is in S^*, the normalized starlike functions on U, then $tg(sz)/sg(tz) \prec (1 - tz/1 - sz)^2$ for all $|s| \leqq 1$, $|t| \leqq 1$. Letting $t = 1$ it follows that

$$\max_{|z|\leqq r} \max_{g\in S^*} \left|\frac{g(sz)}{g(z)}\right| \leqq \max_{|z|\leqq r} \frac{\left|\dfrac{sz}{(1 - sz)^2}\right|}{\left|\dfrac{z}{(1 - z)^2}\right|}$$

for $|s| \leqq 1$. Since F is convex, $zF'(z)$ is starlike. So, it follows that

$$\max_{|z|\leqq r_0} \max_{F\in K} \left|\frac{F'(w(z))w'(z)}{F'(z)}\right| = \max_{|z|\leqq r_0} \max_{F\in K} \left|\frac{\dfrac{zw'(z)}{w(z)}\cdot wF'(w)}{zF'(z)}\right|$$
$$\leqq \max_{|z|\leqq r_0} \left|\frac{\dfrac{zw'(z)}{w(z)}\cdot\dfrac{w(z)}{(1-w(z))^2}}{\dfrac{z}{(1 - z)^2}}\right|.$$

Therefore the theorem will be proved if for all $|z| \leqq r_0$ and all Schwarz functions w, $w'(0) \geqq 0$, we have

$$\left|\frac{w'(z)\,(1 - z)^2}{(1 - w(z))^2}\right| \leqq 1. \tag{1}$$

This follows from Lemma 1 and concludes the proof of the theorem.

Before we turn to the proof of Lemma 1 we note the parallel between (1) and the ordinary Schwarz's lemma. Schwarz's lemma says that

$$\frac{|w'(z)|(1 - |z|^2)}{1 - |w(z)|^2} \leqq 1$$

throughout $|z| < 1$. It weighs information about z and $w(z)$ in a uniform manner relative to $|z| = 1$. In our case we weigh information about z and $w(z)$ relative to one point of $|z| = 1$, namely $z = 1$. In such a case we find that inequality (1) holds only for $|z| \leqq 2 - \sqrt{3}$.

LEMMA 1. *Let w, $w'(0) \geqq 0$, be a Schwarz function. Let p, with $p(z) =$*

$1 + 2az + \cdots$, $a \geqq 0$, *be a function of positive real part in U. Then for all* $|z| \leqq 2 - \sqrt{3}$,

$$(2) \qquad |(1 - z)^2 p'(z)| \leqq 2,$$

$$(3) \qquad |w'(z)(1 - z)^2/(1 - w(z))^2| \leqq 1$$

and the results are sharp.

PROOF. Let $P_1 = \{p : p(z) = 1 + 2az + \cdots, a \geqq 0, \operatorname{Re} p(z) > 0\}$. Since a Schwarz function w, $w'(0) = a \geqq 0$, is associated with the function $p(z)$ of P_1 by the relation $p(z) = (1 + w(z))/(1 - w(z))$ it is easy to check that (2) and (3) are equivalent.

We first prove that (2) holds for any p in P_1 with $p'(0) = 0$. In this case p has the form $p(z) = (1 + w)/(1 - w)$ with w a Schwarz function satisfying

$$(4) \qquad |w(z)| \leqq |z|^2, \; z \in U.$$

It follows from Goluzin's improved Schwarz's estimate given in [4, Lemma 2] with $a = 0$ that

$$(5) \qquad |w'(z)| \leqq 2r(1 - |w(z)|^2)/(1 - r^4)$$

for $|z| \leqq r$. Thus using (5) and then (4) we have

$$\begin{aligned} |p'(z)(1 - z)^2| &= |2\, w'(z)\, (1 - z)^2|/|1 - w(z)|^2 \\ &\leqq 4r(1 - |w|^2)(1 + r)^2/(1 - r^4)(1 - |w|)^2 \\ &= 4r(1 + |w|)(1 + r)/(1 - r)(1 + r^2)(1 - |w|^2) \\ &\leqq 4r/(1 - r)^2 \end{aligned}$$

which is $\leqq 2$ for $0 \leqq r \leqq 2 - \sqrt{3}$.

We now prove (2) for functions in P_1 with $p'(0) = 2a > 0$. The Pfaltzgraff-Pinchuk result [4, Thm. 7.4] guarantees that a function p_0 that maximizes for a given z in U the quantity $|(1 - z)^2 p'(z)|$ over all p in P_1 will have at most three jumps in its representing measure. We apply a variational method to show that for $|z| \leqq 2 - \sqrt{3}$ the function can have at most two jumps.

Suppose there were an $a > 0$ and a z in $|z| \leqq 2 - \sqrt{3} = r_0$ such that

$$(6) \qquad \begin{aligned} &p_0(z) = \sum_{j=1}^{3} \lambda_j \frac{1 + ze^{it_j}}{1 - ze^{it_j}}, \qquad 0 \leqq t_1 < t_2 < t_3 < 2\pi \\ &\sum_{j=1}^{3} \lambda_j = 1, \; \sum_{j=1}^{3} \lambda_j e^{it_j} = 2a > 0, \\ &0 < \lambda_1, \lambda_2, \lambda_3, < 1, \end{aligned}$$

and for all p in P_1, $|(1 - z)^2 p'(z)| \leqq |(1 - z)^2\, p_0'(z)|$.

From (6) we would have $\sum_{j=1}^{3} \lambda_j \sin t_j = 0$ and $\lambda_3 = 1 - \lambda_1 - \lambda_2$. Since $0 \leqq t_1 < t_2 < t_3 < 2\pi$, two of the t_j's say t_1 and t_3 would be such that $\sin t_1 \neq \sin t_3$. We could then solve for λ_1 as a linear function of λ_2.

$$\lambda_1 = \frac{\sin t_3}{\sin t_3 - \sin t_1} + \lambda_2\left(\frac{\sin t_2 - \sin t_3}{\sin t_3 - \sin t_1}\right).$$

Letting $k_j = [(1 - z)^2/z][ze^{it_j}/(1 - ze^{it_j})^2]$, $j = 1, 2, 3$, we would obtain $(1 - z)^2 p_0'(z) = 2\lambda_1 k_1 + 2\lambda_2 k_2 + 2\lambda_3 k_3$. Substituting in for λ_3 and λ_1 would yield

$$\begin{aligned}(1 - z)^2 p_0'(z) &= 2\lambda_2\left[(k_1 - k_3)\left(\frac{\sin t_2 - \sin t_3}{\sin t_3 - \sin t_1}\right) + (k_2 - k_3)\right] \\ &\quad + \frac{2k_1 \sin t_3 - 2k_3 \sin t_1}{\sin t_3 - \sin t_1} \\ &= A\lambda_2 + B,\end{aligned}$$

where A and B are complex constants.

We now prove that $A \neq 0$. If A were 0 then letting $s = (\sin t_2 - \sin t_3)/(\sin t_3 - \sin t_1)$ we would have

$$k_3 = \frac{s}{1 + s} k_1 + \frac{1}{1 + s} k_2,$$

that is, k_3 would lie on the line through k_1 and k_2. We note that k_1, k_2 and k_3 lie on the curve $h(e^{it})$, $0 \leqq t \leqq 2\pi$, where $h(e^{it}) = [(1 - z)^2/z] \cdot [ze^{it}/(1 - ze^{it})^2]$. However, $h(e^{it}), 0 \leqq t \leqq 2\pi$, is simply the fixed $(1 - z)^2/z$ scalar multiple of the image of $|\zeta| = r$ under the Köbe function $\zeta(1 - \zeta)^{-2}$. The Köbe function maps all circles $|\zeta| = r \leqq 2 - \sqrt{3}$ onto convex analytic curves containing no straight line segments. Thus k_3 can not lie on the line through k_1 and k_2. Consequently A is non-zero.

Since A is non-zero the image of (0, 1) under the map $A\lambda + B$ would be a straight line segment containing the point $(1 - z)^2 p_0'(z)$ in its interior. By continuity we could vary λ to obtain a p_1 in P_1 such that $|(1 - z)^2 \cdot p_1'(z)| > |(1 - z)^2 p_0'(z)|$ contradicting the extremal property of p_0. (Note that although the a_1 of $p_1(z) = 1 + 2a_1 z + \cdots$ may not equal the a of $p_0(z)$, nevertheless, by continuity a_1 will be real and positive.)

Letting $k(z) = z/(1 - z)^2$ we have shown for any z in $|z| \leqq 2 - \sqrt{3}$ that if $p_2(z)$, $p_2'(o) = 2a > 0$, maximizes $|zp'(z)/k(z)|$ over P_1 then

$$p_2(z) = \lambda\frac{(1 + e^{it_1}z)}{(1 - e^{it_1}z)} + (1 - \lambda)\frac{(1 + e^{it_2}z)}{(1 + e^{it_2} z)}, \qquad 0 \leqq \lambda \leqq 1,$$

and therefore proving (2) reduces to showing that

$$|\lambda\, k(e^{it_1}z) + (1 - \lambda)\, k(e^{it}z)| \leqq |k(z)|,\ |z| \leqq r_0,$$

for all $0 \leqq \lambda \leqq 1$ and all t_1, t_2 in $[0, 2\pi]$ with $\lambda e^{it_1} + (1 - \lambda)e^{it_2} = a$. Letting $\psi_2 = -(t_1 + t_2)/2$, $\psi_1 = (t_1 - t_2)/2$ and $z = \xi \exp(i\psi)$, we can rewrite the above inequality as

$$|\lambda k(e^{i\psi_1}\xi) + (1 - \lambda)\, k(e^{-i\psi_1}\xi)\,| \leqq |k(e^{i\psi_2}\xi)|$$

for all $0 \leqq \lambda \leqq 1$ and all ψ_1, ψ_2 in $[0, 2\pi]$ with $\lambda e^{i\psi_1} + (1 - \lambda)\, e^{-i\psi_1} = ae^{i\psi_2}$. But

$$\lambda k(e^{i\psi_1}\xi) + (1 - \lambda)\, k(e^{-i\psi_1}\xi = \xi\left[\frac{\lambda e^{i\psi_1}}{(1-e^{i\psi_1}\xi)^2} + \frac{(1-\lambda)e^{-i\psi_1}}{(1-e^{-i\psi_1}\xi)^2}\right]$$
$$= \frac{\xi[ae^{i\psi_2} + \xi^2 e^{-i\psi_2} - 2\xi]}{(1 - e^{i\psi_1}\xi - e^{-i\psi_1}\xi + \xi^2)^2}.$$

Thus it suffices to show

$$\max_{|\xi|\leqq r_0}\left|\frac{\xi(ae^{i\psi_2} + \xi^2 e^{-i\psi_2} - 2\xi)}{(1 - e^{i\psi_1}\xi - e^{-i\psi_1}\xi + \xi^2)^2\, k(e^{i\psi_2}\xi)}\right| \leqq 1,$$

a quantity which depends only on the independent variagles a and ψ_2. Since the maximum is taken on the boundary we let $\xi = r_0 e^{i\theta}$, $r_0 = 2 - \sqrt{3}$, $\psi_2 = \psi$ and square the above expression to obtain

$$\frac{|ae^{i\psi} - 2\xi + (2a\cos\psi - ae^{i\psi})\xi^2|^2[1 + r_0^2 - 2r_0\cos(\psi + \theta)]^2}{[1 + r_0^4 + 2r_0^2\cos 2\theta + 4a^2r_0^2\cos^2\psi - 4r_0 a\cos(\psi + \theta)]^2}$$

which, upon noting that $1 + r_0^2 = 4r_0$, $1 + r_0^4 = 4r_0 - 2r_0^2 + 4r_0^3$, becomes, after a fairly long computation,

$$\frac{[1 + a^2(3 + \cos^2(\theta - \psi)) - 4a\cos(\theta - \psi)]\,[2 - \cos(\theta + \psi)]^2}{[a^2\cos^2\psi + 3 + \cos^2\theta - 4a\cos\theta\cos\psi]^2}. \tag{7}$$

Since the denominator of (7) is $(a\cos\psi - 2\cos\theta)^2 + 3(1 - \cos^2\theta)$, we see that it never vanishes. Therefore, the quantity in (7) being $\leqq 1$ is equivalent upon cross multiplication to

$$\begin{aligned} h(a) &= (-\cos^4\psi)a^4 + (8\cos^3\psi\cos\theta)a^3 \\ &\quad + [RP - 2(\cos^2\psi)Q - 16\cos^2\theta\cos^2\psi]a^2 \\ &\quad + [8Q\cos\theta\cos\psi - 4P\cos(\theta - \psi)]a + [P - Q^2] \\ &= A_4a^4 + A_3a^3 + A_2a^2 + A_1a + A_0 \leqq 0 \end{aligned} \tag{8}$$

where $R \equiv 3 + \cos^2(\theta - \psi)$, $P \equiv (2 - \cos(\theta + \psi))^2$, $Q \equiv 3 + \cos^2\theta$, and $M \equiv \cos^2\psi + 3 + \cos^2\theta - 4\cos\theta\cos\psi$.

Factoring out the P and expanding M^2 we see that

$$\begin{aligned} h(1) &= -\cos^4\psi + 8\cos^3\psi\cos\theta + RP - 2Q\cos^2\psi - 16\cos^2\theta\cos^2\psi \\ &\quad + 8Q\cos\theta\cos\psi - 4P\cos(\theta - \psi) + P - Q^2 \\ &= -M^2 + P(2 - \cos(\theta - \psi))^2. \end{aligned}$$

Since $M = (2 - \cos(\theta - \psi))(2 - \cos(\theta + \psi))$, we conclude that $h(1) = 0$. Thus $h(a) = (1 - a)(H(a)$ where

$$\begin{aligned} H(a) &= [(\cos^4\psi)a^3 + (\cos^4\psi - 8\cos^3\psi\cos\theta)a^2 \\ &\quad + (\cos^4\psi - 8\cos^3\psi\cos\theta - RP + 2Q\cos^2\psi \\ &\quad + 16\cos^2\theta\cos^2\psi)a + P - Q^2] \\ &= (\cos^4\psi)(a^3 + B_2a^2 + B_1a + B_0) = (\cos^4\psi)h_1(a). \end{aligned}$$

It suffices to show $H(a) \leqq 0$. Note that

$$\begin{aligned} H(0) &= P - Q^2 = (2 - \cos(\theta + \psi))^2 - (3 + \cos^2\theta)^2 \\ &= (5 - \cos(\theta + \psi) + \cos^2\theta)(-1 - \cos(\theta - \psi) - \cos^2\theta) \leqq 0 \end{aligned}$$

while

$$\begin{aligned} H(1) &= 3\cos^4\psi - 16\cos^3\psi\cos\theta - RP + 2Q\cos^2\psi + 16\cos^2\psi\cos^2\theta \\ &\quad + P - Q^2 = 2[\cos^4\psi - 4\cos^3\psi\cos\theta + P - Q^2 - 2P\cos(\theta - \psi) \\ &\quad + 4Q\cos\theta\cos\psi] + [\cos^4\psi - 8\cos^3\psi\cos\theta - P + Q^2 \\ &\quad + 4P\cos(\theta - \psi) - 8Q\cos\theta\cos\psi - RP + 2Q\cos^2\psi \\ &\quad + 16\cos^2\theta\cos^2\psi]. \end{aligned}$$

The term in the last set of square brackets is $M^2 - P(2 - \cos(\theta - \psi))^2 \equiv 0$ exactly as before. Note that we can rewrite what is left as

$$\begin{aligned} &-\cos^4\psi + 4\cos^3\psi\cos\theta + 2P\cos(\theta - \psi) - P + Q^2 - 4Q\cos\theta\cos\psi \\ &= -(1 - \sin^2\psi)^2 + 4(1 - \sin^2\psi)\cos\psi\cos\theta + (2\cos\theta\cos\psi + \sin\theta\sin\psi - 1) \\ &\quad \cdot(2 - \cos\theta\cos\psi + \sin\theta\sin\psi)^2 + (4 - \sin^2\theta)^2 - 4(4 - \sin^2\theta)\cos\theta\cos\psi \\ &= 15 + 2\sin^2\psi - \sin^4\psi - 8\sin^2\theta + \sin^4\theta - 12\cos\theta\cos\psi + 4\sin^2\theta\cos\theta\cos\psi \\ &\quad - 4\sin^2\psi\cos\psi\cos\theta + (2\cos\theta\cos\psi + 2\sin\theta\sin\psi - 1)\cdot(5 - \sin^2\theta \\ &\quad - \sin^2\psi + 2\sin^2\theta\sin^2\psi - 4\cos\theta\cos\psi + 4\sin\theta\sin\psi - 2\cos\theta\cos\psi\sin\theta\sin\psi) \\ &= 10 + 3\sin^2\psi - \sin^4\psi - 7\sin^2\theta + \sin^4\theta + 2\cos\theta\cos\psi \\ &\quad + 2\sin^2\theta\cos\psi\cos\theta - 6\sin^2\psi\cos\theta\cos\psi - 4\cos^2\theta\cos^2\psi\sin\theta\sin\psi \\ &\quad - 8\cos^2\theta\cos^2\psi + 6\sin\theta\sin\psi - 2\sin^3\theta\sin\psi - 2\sin^3\psi\sin\theta \\ &\quad + 4\sin^3\theta\sin^3\psi + 6\sin^2\theta\sin^2\psi + 2\cos\theta\cos\psi\sin\theta\sin\psi = 2 + 11\sin^2\psi \\ &\quad - \sin^4\psi + \sin^4\theta + (2\cos\theta\cos\psi)\cdot(1 + \sin^2\theta - 3\sin^2\psi + \sin\theta\sin\psi) \\ &\quad + 2\sin\theta\sin\psi(1 - \sin\theta\sin\psi + \sin^2\theta + \sin^2\psi) + [8 - 8\sin^2\psi - 8\sin^2\theta \\ &\quad - 8\cos^2\theta\cos^2\psi + 8\sin^2\theta\sin^2\psi] + [4\sin\theta\sin\psi - 4\sin^3\theta\sin\psi \\ &\quad - 4\sin^3\psi\sin\theta - 4\cos^2\theta\cos^2\psi\sin\theta\sin\psi + 4\sin^3\theta\sin^3\psi]. \end{aligned}$$

Since each of the terms in square brackets is identically zero, we can conclude $H(1)$ is nonpositive upon noting the expansion of the following nonnegative expression.

$$3\sin^2\phi+(\sin\phi+\sin^3\theta)^2+(1+\cos^2\theta)\sin^4\theta+2(1-\cos\theta\cos\phi)\sin^2\phi$$
$$+2[1-\cos(\theta+\phi)]\sin^2\phi+[1+\cos(\theta-\phi)]^2+(\cos\theta-\cos\phi)^2\sin^2\phi$$
$$+(\cos\theta+\cos\phi)^2\sin^2\theta+2\sin^2\phi\cos^2\theta = 1+10\sin^2\phi+\cos^2\theta\sin^2\phi$$
$$+2\sin^4\theta+\cos^2\theta\sin^2\theta+[\cos^2\theta\sin^2\phi+\cos^2\phi\sin^2\theta+\cos^2\phi\cos^2\theta$$
$$+\sin^2\phi\sin^2\theta]-2\sin^2\phi\sin^2\theta+2\sin\phi\sin^3\theta+2\sin\theta\sin^3\phi$$
$$+2\sin\theta\sin\phi-6\cos\theta\cos\phi\sin^2\phi+2\sin\theta\sin\phi\cos\theta\cos\phi$$
$$+2\cos\theta\cos\phi\sin^2\theta+2\cos\theta\cos\phi. = 2+11\sin^2\phi-\sin^4\phi$$
$$+\sin^4\theta+\sin^2\theta+2\sin\theta\sin\phi(1+\sin^2\phi+\sin^2\theta-\sin\phi\sin\theta)$$
$$+2\cos\theta\cos\phi(1+\sin^2\theta-3\sin^2\phi+\sin\theta\sin\phi)$$

where in the second equality we observe that $\sin^4\theta + \sin^2\theta\cos^2\theta = \sin^2\theta$, while the term in brackets is identically 1.

Recall $h_1(a) = a^3 + B_2a^2 + B_1a + B_0$ where

$$B_2 = 1 - 8\sec\phi\cos\theta$$
$$B_1 = 1 - 8\sec\phi\cos\theta - RP\sec^4\phi + 2Q\sec^2\phi + 16\cos^2\theta\sec^2\phi$$

and $B_0 = \sec^4\phi(P - Q^2)$.

Now, $h_1(0) = B_0 = \sec^4\phi[(2 - \cos(\theta + \phi))^2 - (3 + \cos^2\theta)^2]$. So $B_0 \leqq 0$

$$\text{if and only if}\quad (2-\cos(\theta+\phi))^2 \leqq (3+\cos^2\theta)^2$$
$$\text{if and only if}\quad 2-\cos(\theta+\phi) \leqq 3+\cos^2\theta$$
$$\text{if and only if}\quad -\cos(\theta+\phi) \leqq 1+\cos^2\theta$$

which certainly holds as $-\cos(\theta + \phi) \leqq 1 \leqq 1 + \cos^2\theta$. Hence

$$h_1(0) \leqq 0.$$

Now, we will assume that $h_1(1) \leqq 0$. This will be proved later. Then, from the properties of a cubic, h_1 will have 3 roots, r_1, r_2, and r_3, with $r_1 \geqq 1$. Since $r_1 + r_2 + r_3 = -B_2 = 8\sec\phi\cos\theta - 1$, we consider two cases.

CASE I. $B_2 \geqq -1$. Then $-B_2 \leqq 1$ and $1 \geqq -B_2 = r_1 + r_2 + r_3 \geqq 1 + r_2 + r_3$ and so $r_2 + r_3 \leqq 0$. Since $h_1(0) \leqq 0$ and $h_1(1) \leqq 0$, we conclude that h_1 has no roots in $(0, 1)$. Therefore $h_1(a) \leqq 0$ for $0 \leqq a \leqq 1$ and we are done.

CASE II. $B_2 < -1$. Assume that $r_2 \in (0,1)$. Then $r_2(r_2^2 + B_2r_2 + B_1) = -B_0$ and $r_2^2 + B_2r_2 + B_1 = -B_0/r_2 > -B_0$. Hence $0 < B_0 + B_1 - r_2 + r_2^2$ and thus

$$B_0 + B_1 > r_2(1 - r_2) > 0. \tag{9}$$

However, using (8) and the fact that $h(1) = 0$ we can solve for $B_0 + B_1$ to obtain $B_0 + B_1 = -(A_1 + 2A_0)/A_4 = (2\sec^4\phi)T$ where $T = P - Q^2 - 2P\cos(\theta - \phi) + 4Q\cos\theta\cos\phi$. Now, if we expand the expression for $-T$ and express most of the quantities in terms of $\sin\theta$ and $\sin\phi$, we obtain

$$-T = [3 + \sin^2\theta + 9\sin^2\phi + \sin^4\theta + 2\sin\theta\sin\phi(1 - \sin\theta\sin\phi + \sin^2\theta + \sin^2\phi)] + 2(\cos\theta\cos\phi)[-1 + \sin\theta\sin\phi + \sin^2\theta - \sin^2\phi].$$

Upon performing the same expansion of the nonnegative expression

$$4\sin^2\phi + (\sin\phi + \sin^3\theta)^2 + \sin^4\theta\cos^2\theta + (1 - \cos\theta\cos\phi)^2 + 2[1 - \cos(\theta + \phi)](1 + \sin^2\phi + \cos\theta\cos\phi) + \cos^2\theta\sin^2\phi + (1 + \sin\theta\cos\phi\sin^2\theta)$$

we see that they are the same. Hence $-T \geqq 0$ and this contradicts (9) so that B_2 cannot be < -1. Hence only Case I holds.

Upon proving $h_1(1) \leqq 0$, we will have $h_1(a) \leqq 0$ for $0 \leqq a \leqq 1$ as claimed. Accordingly we note that

$$\begin{aligned} h_1(1) &= 1 + B_2 + B_1 + B_0 \\ &= 2 - 8\sec\phi\cos\theta + B_1 + B_0 \\ &= 2\sec^4\phi[\cos^4\phi - 4\cos^3\phi\cos\theta + T]. \end{aligned}$$

Letting $S = \cos^4\phi - 4\cos^3\phi\cos\theta + T$ and expanding as before we see that

$$S = 2 + \sin^2\theta + 11\sin^2\phi + \sin^4\theta - \sin^4\phi + 2(1 - \sin\theta\sin\phi + \sin^2\theta + \sin^2\phi)\sin\theta\sin\phi + 2(1 + \sin\theta\sin\phi + \sin^2\theta - 3\sin^2\phi)\cos\theta\cos\phi.$$

Likewise, upon expanding the nonnegative expression

$$3\sin^2\phi + (\sin\phi + \sin^3\theta)^2 + (1 + \cos^2\theta)\sin^4\theta + 2(1 - \cos\theta\cos\phi)\sin^2\phi + 2[1 - \cos(\theta + \phi)]\sin^2\phi + [1 + \cos(\theta - \phi)]^2 + (\cos\theta - \cos\phi)^2\sin^2\phi + (\cos\theta + \cos\phi)^2\sin^2\theta + 2\sin^2\phi\cos^2\theta,$$

we see that $-S \geqq 0$ and hence $h_1(1) \leqq 0$ as we claimed.

The sharpness result of the lemma follows by choosing $w = z^2$.

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