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MORSE FUNCTIONS AND SUBMANIFOLDS OF HYPERBOLIC SPACE

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ABSTRACT. We study the hypersurface of the Hyperbolic space H^{2m+1}. In H^{2m+1}, there are Morse functions. If we assume that these Morse functions have index 0, m or $2m$ at all these critical points, then we can determine the hypersurface.

1. Introduction. Let M be a differentiable manifold of class C^∞. By a Morse function f of M, we mean a differentiable function on M having only non-degenerate critical points.

In [5], Nomizu and Rodriguez showed the following result of a geometric nature analogous to Reeb's Theorem. If M ($\dim M = n \geqq 2$) is a connected, complete Riemannian manifold isometrically immersed in R^{n+p} such that every Morse function of the form L_p has index 0 or n at any of its critical points, then M is embedded as a Euclidean subspace or a Euclidean n-sphere. Here $L_p(x) = (d(x, p))^2$, $p \in R^{n+p}$, $x \in M$ and d is the Euclidean distance function (see also [4]).

Cecil [1] characterized the metric spheres in hyperbolic space H^m in terms of hyperbolic distance functions L_p. In [2], Cecil and Ryan studied umbilic submanifolds in a hyperbolic space through the introduction of new classes of Morse functions, L_π (directed distance from a hyperplane) and L_h (directed distance from a horosphere). They proved the following theorem.

THEOREM A. *Let M^n, ($n \geqq 2$), be a connected, complete Riemannian manifold isometrically immersed in H^m. Every Morse function of the form L_p or L_π has index* 0 *or n at all its critical points if and only if M^n is embedded as a sphere, horosphere or equidistant hypersurface in a totally geodesic $H^{n+1} \subset H^m$.*

In this paper, we shall study more general submanifolds in a hyperbolic space using Morse functions. For background material and notation, we refer the reader to [2].

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2. Theorems.

THEOREM 1. *Let M be a connected, complete Riemannian manifold isometrically immersed in H^{2m+1} $(m > 1)$ as a hypersurface with constant mean curvature. Every Morse function of the form L_p, L_π or L_h has index 0, m or $2m$ at all its critical points if and only if M is embedded as a sphere, horosphere or equidistant hypersurface or as a standard product $S^m(c_1) \times H^m(c_2)$ (see* [**6**, p. 252]).

THEOREM 2. *Let M be as in Theorem* 1. *If every Morse fundtion of the form L_p, L_π or L_h has exactly two critical points, then M is a sphere or horosphere.*

These theorems are immediate consequences of the following Lemma.

LEMMA. *Under the assumption of Theorem* 1, *the second fundamental form has at most two distinct eigenvalues at each point.*

PROOF. Let $x \in M$ and ξ be a field of unit normal vectors. Let a be the eigenvalue of A^ξ with largest absolute value. If $a = 0$, then all of the eigenvalues are equal as desired. If not, we may assume that $a > 0$.

We assume that $a > 1$. Take t_1 such that $b < \coth t_1 < a$, where b is the next largest positive eigenvalue, if any. If a is the only positive eigenvalue, just consider $\coth t_1 < a$. Then for $p = (\cosh t_1)x + (\sinh t_1)\xi$, L_p has x as a non-degenerate critical point. The index at x is equal to the multiplicity, say k, of the eigenvalue a. Then we have $k = 2m$ or m. If $k = 2m$, then a is an eigenvalue of A^ξ with multiplicity $2m$, so that x is umbilic.

Suppose that $k = m$. Then the following two subcases should be discussed.

(i) There exists a positive eigenvalue of A^ξ other than a.

(ii) The opposite of (i).

First we consider (i). Assume that b is the next largest positive eigenvalue of A^ξ and that $b > 1$. Then we take $t_2 > 0$ such that $a > \coth t_1 > b > \cot t_2 > c$, where c is the third largest positive eigenvalue, if any (if a and b are the only positive eigenvalues, just consider $a > \coth t_1 > b > \coth t_2$). Then for

$$p = (\cosh t_2)x + (\sinh t_2)\xi,$$

L_p has non-degenerate critical point at x with index m. Thus multiplicity of b is m.

If $b \leqq 1$, we use the L_π function. We take t_2 such that

$$a > \coth t_1 > 1 \geqq b > \tanh t_2 > c$$

where c is the third largest positive eigenvalue if any (if a and b are the only positive eigenvalues, just consider $a > \coth\ t_1 > 1 \geqq b > \tanh\ t_2$). Then for

$$\sigma = (\sinh t_2)x + (\cosh t_2)\xi, \qquad \sigma \in \Sigma^{2m+1}, \qquad \pi = \Omega(\sigma, 0),$$

L_π has non-degenerate critical point at x with index m. Thus the multiplicity of b is m.

(ii) If there exist non-zero eigenvalues of A^ξ other than a, then let b be the smallest eigenvalue of A^ξ. Note that $-b$ is the largest positive eigenvalue of $A^{-\xi}$. We take $t_2 > 0$ such that $-b > \coth t_2 > -c$, where c is the next smallest eigenvalue of A^ξ such that $-c > 1$ (resp. $-c < 1$) if any (if b is the only negative eigenvalue of A^ξ, just consider $-b > \coth$ t_2). By the same argument as above, for

$$p = (\cosh t_2)x + (\sinh t_2)\xi, \text{ (resp. } \sigma = (\sinh t_2)x + (\cosh t_2)\xi),$$

L_p (resp. L_π) has x as a non-degenerate critical point of index m. Thus the multiplicity of b is also m. Therefore A^ξ has at most two distinct eigenvalues at each point.

If $a \leqq 1$, we consider the distance function L_π, and we have the same conclusion.

This completes the proof of the lemma.

As $m > 1$ and the mean curvature of M is constant, the eigenvalues of the second fundamental form are constant. Hence from **[6]** we have the conclusion of the Theorems.

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