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ON THE S-EQUIVALENCE OF SOME GENERAL SETS OF MATRICES

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ABSTRACT. To help classify the set of square matrices over a ring R under the relation of S-equivalence there is defined a module A_V together with a pairing on its torsion submodule, which is referred to as the Seifert system of V. It is shown that if R is a field, or R is a PID and det $(tV - V')$ has content 1, then the Seifert system characterizes an S-equivalence class. Furthermore, over a field S-equivalence is reducible to the notion of congruence.

1. Introduction. Two square matrices over a ring R are called S-equivalent if one can be derived from the other by a sequence of the following operations (or their inverses);

(1.1) Congruences, i.e., replacing V by PVP', with P unimodular over R,

(1.2) Row and column enlargements, i.e., replacing V by,

$$\text{(i)} \begin{bmatrix} 0 & 0 & 0 \\ 1 & a & b \\ 0 & c & V \end{bmatrix} \quad \text{or} \quad \text{(ii)} \begin{bmatrix} 0 & 1 & 0 \\ 0 & a & b \\ 0 & c & V \end{bmatrix}$$

To help classify matrices under this relation, we define a module A_V over the ring $R[t, t^{-1}]$, together with a pairing on its torsion submodule, which will be an invariant of the S-equivalence class of V. We refer to this as the Seifert system for V.

The geometric aspects of the study of S-equivalence have principally been developed in the work of Levine [5, 6, 7]. If $K \subseteq S^{2n+1}$ is an odd dimensional knot, then any Seifert surface for K determines an integral matrix, called a Seifert matrix. S-equivalence can in this case be interpreted as the matrix theoretic analogue of adding or subtracting handles to these surfaces. S-equivalence actually characterizes the so-called simple embeddings (see Kearton [3]). The module A_V then corresponds to the integral homology of the universal abelian cover of $S^{2n+1} - K$, whose pairing is defined geometrically in Blanchfield [1].

Seifert matrices for knots can algebraically be characterized by the condition $\det(V - eV') = \pm 1$, where e is either $+1$ or -1. These matrices have been classified algebraically by Trotter [10, 11]. The results of

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this paper are generalizations of theorems of his. The present treatment has several advantages, though. It applies also to Seifert matrices for links in S^3 (see Keef [4]). In addition, the methods used here are considerably more elementary, although the general outline remains much the same. Finally, in the knot case it can be shown that multiplication by 1-t is an automorphism of A_V, a fact which has a central position in previous studies. The theorems in this paper will be proven without reference to this map, which is not in general either one-to-one or onto.

2. The Seifert system. We will assume all rings are integral domains. If R is a ring, we let R^m denote the set of all m by 1 matrices (column vectors) over R, and let R' be the field of fractions of R. We write RC for the group ring over R of the infinite cyclic group generated by t, written multiplicitively. So $RC \cong R[t, t^{-1}]$, the ring of Laurent polynomials over R. Clearly $RC' = R'(t)$. Let $\overline{}$ denote the conjugation on RC which interchanges t and t^{-1}.

If V is a square matrix over R, we let $D(V) = \det(tV - V')$ and $E(V) = tV - V'$. It is easy to verify that $D(V)$ is, up to multiplication by units of RC, an invariant of the S-equivalence class of V. The relation $E(V) = -t\overline{E(V)}'$ is also easily checked.

In order to construct some algebraic invariants of an S-equivalence class, we begin with some general considerations. Suppose S is a ring with a conjugation $\overline{}$, and $u \in S$ is a unit. A matrix M over S will be called u-Hermitian if $u\bar{M}' = M$. Clearly $E(V)$ is $(-t)$-Hermitian over RC. We let A_M denote the module S^m/MS^m, and TA_M denote its submodule of S-torsion. We define a pairing on TA_M, which takes its values in S'/S as follows: if $x, y \in S^m$ project into TA_M, then there exists $a, b \in S^m$ satisfying $Ma = rx$, $Mb = sy$. Let $[x, y] = \bar{b}'Ma/r\bar{s}u \in S'$. To show this is independent of the choices of a, b, r and s, we note that,

$$(2.1) \qquad \begin{aligned} \bar{b}'x/\bar{s}u &= \bar{b}'Ma/r\bar{s}u = \bar{b}'\bar{M}'a/r\bar{s} \\ &= \overline{Mb}'a/r\bar{s} = \bar{y}'a/r \end{aligned}$$

Observe further that if $x = Ma$ (so $r = 1$) or $y = Mb$ (so $s = 1$), then $[x, y] \in S$. This implies that we may view [,] as a pairing on TA_M with values in S'/S.

DEFINITION 2.2. By the Seifert system of M we will mean the module A_M together with this pairing on TA_M. The Seifert systems of M and N are isomorphic if there is an isomorphism of A_M and A_N which restricts to an isometry of their S-torsion submodules.

We next consider the behavior of the Seifert system under a change of ground rings. Suppose S and S_0 are rings with conjugations, and $f: S \to S_0$ is a homomorphism which preserves these conjugations. If M is a u-Hermitian matrix over S, then $f(M)$ (the matrix obtained by mapping all M's entries into S_0) is clearly $f(u)$-Hermitian over S_0. Furthermore, f induces a map $F: A_M \to A_{f(M)}$ by mapping $x \in S^m$ to $f(x) \in S_0^m$. If f is also injective, then it determines a map $f: S'/S \to S_0'/S_0$. Using this map an easy verification shows that the pairing [,] is preserved by F.

One particularly nice situation occurs when M is non-singular. In this case $TA_M = A_M$, and the following gives us a useful expression for [,].

LEMMA 2.3 *If M is a non-singular u-Hermitian matrix, then* $[x, y] = \bar{y}'M^{-1}x$.

PROOF. Suppose $Ma = rx$. Therefore $a/r = M^{-1}x \in S'^m$, and so by (2.1), $[x, y] = \bar{y}'a/r = \bar{y}'M^{-1}x$.

If V is a square matrix over a ring R, we denote $A_{E(V)}$ by the simpler A_V, which we refer to as the Seifert system determined by V.

PROPOSITION 2.4. *If V and W are S-equivalent matrices over R, then the Seifert systems determined by V and W are isomorphic.*

PROOF. Since the argument varies only in minor detail from that in Trotter [11, p. 177–179], we will be content with an outline. It is easily verified that if Q is a unimodular matrix over RC, then $E(V)$ and $QE(V)\bar{Q}'$ give isomorphic Seifert systems. So if P is a unimodular matrix over R, and $W = PVP'$ then $E(W) = PE(V)\bar{P}'$, so their Seifert systems are isomorphic. If W is the row enlargement of 1.2 i, and,

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tc - b' & 0 & I \end{bmatrix}$$

then it is easy to verify that $E(V)$ and $QE(W)\bar{Q}'$, and hence $E(V)$ and $E(W)$, determine isomorphic Seifert systems. A similar argument applies to column enlargements.

The remainder of the paper will be an investigation of the converse of (2.4). Specifically, it will be shown that if R is a field, or if R is a PID and the content of $D(V)$ (i.e., the gcd of its coefficients) is a unit in R, then the Seifert system completely characterizes the S-equivalence class of V.

3. S-equivalence with field coefficients. Throughout this section we let F be a field. S-equivalence over F will be shown to be equivalent to the notion of congruence.

PROPOSITION 3.1. *Any square matrix over F is S-equivalent to a matrix of the form,*

$$\begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix} \text{ with } W \text{ non-singular.}$$

Matrices in this form will be called *reduced.*

PROOF. The following can actually be viewed as an algorithm for putting a matrix into reduced form. Let V be a square matrix over F. If V is non-singular we are done. If not, there exists a non-singular matrix P such that the top row of PV is identically zero. So PVP' has the form,

$$\begin{bmatrix} 0 & 0 \\ a & V_0 \end{bmatrix}$$

Note if $a = 0$, then V is also congruent to,

$$\begin{bmatrix} V_0 & 0 \\ 0 & 0 \end{bmatrix}$$

and we can start our process over with $V = V_0$. If $a \neq 0$, then V is congruent to a matrix of the form,

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & d & b \\ 0 & c & V_1 \end{bmatrix}$$

and once again we can start our process over with $V = V_1$. Continuing as long as possible yields the result.

The reduced form can be used to analyse the algebraic structure of A_V, Clearly if V is the reduced matrix in (3.1), then $A_V \cong A_W \oplus FC^k$ (where k is the number of rows and columns of zeros). To help determine the structure of A_W, we let $R_c^m \subseteq RC^m$ denote the R submodule consisting of vectors whose entries are elements of R.

LEMMA 3.2. *If M and N are unimodular matrices over R of the same size, then $RC^m/(tM - N)RC^m$ is isomorphic as an R module to R^m, where the t automorphism is given by multiplication by NM^{-1}.*

PROOF. Clearly, mapping the standard basis for RC^m to the standard basis for R^m produces a map f: $RC^m \to R^m$, which is clearly an R-linear isomorphism when restricted to R_c^m. Furthermore, if $s \in R_c^m$, then $f((tM - N)s) = NM^{-1}Ms - Ns = 0$, and so since R_c^m generates RC^m as an RC module, we can define, $\bar{f}$: $RC^m/(tM - N)RC^m \to R^m$. Clearly $\bar{f}$ is an isomorphism if we can show that RC^m splits as an R module into $(tM - N)RC^m \oplus R_c^m$. Since f is identically zero on the first summand and is an isomorphism on the second, their intersection is zero. An easy computation shows that $(tM - N)RC^m + R_c^m$ is an RC submodule of RC^m, and since it contains R_c^m, which generates RC^m, the proof is complete.

We call attention specifically to the fact contained in the proof of (3.2) that $R_c^m \subseteq RC^m$ is mapped isomorphically onto $RC^m/(tM - N)RC^m$ under the natural map. The standard basis for R_c^m therefore gives a basis for this module which we will often use (without specifically mentioning it) to determine matrix representations of bilinear forms or linear functions.

COROLLARY 3.3. *If V is a non-singular matrix over F, then A_V is isomorphic as an F vector space to F^m, where the t-automorphism corresponds to multiplication by $V'V^{-1}$.*

COROLLAY 3.4. *If V is a square matrix over F, then $D(V) \neq 0$ if and only if $\dim_F A_V$ is finite if and only if V is S-equivalent to a non-singular matrix. In this case, $\deg(D(V)) = \dim_F A_V$, which equals the size of any non-singular matrix S-equivalent to V.*

THEOREM 3.5. *Let*

$$V_0 = \begin{bmatrix} W_0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } V_1 = \begin{bmatrix} W_1 & 0 \\ 0 & 0 \end{bmatrix}$$

be matrices in reduced form. The following are then equivalent.

(1) *V_0 is S-equivalent to V_1.*

(2) *V_0 is congruent to V_1.*

Further, if these two matrices have the same size, then (1) *and* (2) *are equivalent to*

(3) *W_0 is S-equivalent to W_1, and*

(4) *W_0 is congruent to W_1.*

PROOF. Clearly (4) implies (1), (2) and (3). Furthermore (2) and (3) imply (1). We now show (1) implies (4), which will conclude the proof. Note if

$$V = \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix}$$

is in reduced form, then clearly A_W is isometric to TA_V. By (2.4) there is an isometry of TA_{V_0} and TA_{V_1}, so the proof will be complete once it is shown that an isometry of A_{W_0} and A_{W_1} implies that W_0 and W_1 are congruent.

Using the assumed isometry identify A_{W_0} and A_{W_1}, and call the resulting module A. Note the two interpretations of A give two representations of A as a set of column vectors using (3.2). Observe further that if $i = 0$ or 1, $D(W_i) = \det(W_i)\det(tI - W_i'W_i^{-1})$. This implies that up to a constant $D(W_i)$ is the characteristic polynomial of the automorphism of A given by multiplication by t, which is independent of the basis used to compute it. We call this common polynomial $D(A)$.

Let $a \in F$. By the theory of partial fractions $F(t)$ is isomorphic as an F vector space to $F[t, t^{-1}, (t - a)^{-1}] \oplus L_a$, where L_a can be described as the set of all rational polynomial expressions h/g such that $\deg(h) < \deg(g)$, $g(0) \neq 0$, $g(a) \neq 0$. Define an F linear map f_a: $F(t)/FC \to F$ by setting it equal to zero on $F[t, t^{-1}, (t - a)^{-1}]$, and letting it equal $h(a)/g(a)$ for $h/g \in L_a$.

Let A^* be the dual space of F linear functionals on A. Note maps h: $A \to A^*$ correspond to bilinear forms on A, and maps h: $A^* \to A$ correspond to bilinear forms on A^* (since A^{**} can be identified with A).

Suppose $a \in F$ satisfies $D(A)(a) \neq 0$. We define a bilinear form on A with values in F by combining the pairing [,] into $F(t)/FC$ with the map f_a into F, i.e., $(x, y)_a = f_a([x, y])$. If $W = W_0$ or W_1, and we consider the vector representation of A given by (3.2), we claim $(,)_a$ has matrix $(aW - W')^{-1}$. To see this observe that if $b, c \in F_C^m$ represent $x, y \in A$, then by (2. 3), $[x, y] = \bar{b}'E(W)^{-1}c \in F(t)/FC$.

Note $E(W)^{-1} = \operatorname{adj}(E(W))/D(W)$. This implies that all the entries of $E(W)^{-1}$ are in L_a, since $D(W)(a) = D(A)(a) \neq 0$ by supposition, $D(W)(0) = \det(-W') \neq 0$ since W' is non-singular, and $D(W)$ has a larger degree than any entry of $\operatorname{adj}(E(W))$. Since f_a when restricted to L_a is merely substitution by a, we have, $f_a([x, y]) = b'(aW - W')^{-1}c$ which establishes the claim.

Suppose there actually exists a pair of distinct non-roots of $D(A)$, $r, s \in F$. We then have a pair of bilinear forms on A, and hence a pair of adjoint maps h_r, h_s: $A \to A^*$. Let g: $A^* \to A$ be given by $(r - s)^{-1}(h_r^{-1} - h_s^{-1})$. So g has matrix representation $(r - s)^{-1}((rW - W') - (sW - W')) = W$. g, in turn defines a bilinear form on A^* which also has matrix W.

We summarize this construction by noting that there is a bilinear form on A^* completely determined by the pairing [,], and with respect to one basis it has matrix W_0 and with respect to another basis it has matrix W_1, and so W_0 and W_1 are congruent.

Assume now that $D(A)$ does not have two non-roots. Embed F in a field F' where $D(A)$ does have two non-roots (say by adjoining an indeterminant). Consider the diagram:

$$\begin{array}{ccc} A^* & \xrightarrow{\quad g \quad} & A \\ \downarrow & & \downarrow \\ (F'A)^* & \xrightarrow{\quad g' \quad} & F'A \end{array}$$

By the naturality of the Seifert system under an extention of the ground ring, g' exists as above. In fact, since g' has matrix W, all of whose entries are in F, g' can easily be seen to restrict to g as shown. g once again determines a congruence class of matrices to which W_0 and W_1 must both belong, which therefore completes the proof.

Note that we only used S-equivalence in the above proof to establish an isometry between A_{W_0} and A_{W_1}. Since the number of zero rows and columns in a reduced matrix V equals the rank of A_V/TA_V as an FC module, we have actually shown the following result.

THEOREM 3.6. *If V_0 and V_1 are square matrices over a field, then they are S-equivalent if and only if their Seifert systems are isomorphic.*

We single out one fact established in the proof of (3.5).

COROLLARY 3.7. *If V is a non-singular matrix over a field F, then there exists an F linear map $g: A_V^* \to A_V$, whose matrix with respect to the basis for A_V given by the isomorphism $F_c^m \subseteq FC^m \to A_V$ and its dual basis in A_V^* is V. Furthermore, g does not depend on the way A_V is presented as the Seifert system of some matrix.*

4. S-equivalence of knot-like matrices. Throughout this section we assume R is a PID. If V is a square matrix over R, then if we view it as a matrix over R', its Seifert system is given by the R' vector space $R'A_V$, for which all the results of the previous section apply.

DEFINITION 4.1. A matrix V over R is called knot-like if the content of $D(V)$ (i.e., the gcd of its coefficients) is a unit in R.

Any Seifert matrix for a knot is knot-like over the integers. This can be seen by the relation $D(V)(e) = \pm 1$ where e is $+1$ or -1, which is true for these matrices (see Trotter [11]).

PROPOSITION 4.2. *V is knot-like if and only if A_V is a torsion free R module of finite rank.*

PROOF. By (3.4), $\operatorname{rank}(A_V) = \dim_{R'}(R'A_V)$ is finite if and only if $D(V) \neq 0$. So if $D(V) \neq 0$ we have an exact sequence,

$$0 \longrightarrow RC^m \xrightarrow{E(V)} RC^m \longrightarrow A_V \longrightarrow 0.$$

If we tensor this with R_p ($=R/pR$, where $p \in R$ is a prime) we get

$$0 \longrightarrow \operatorname{Tor}_R(R_p, A_V) \longrightarrow R_pC^m \xrightarrow{E(V)} R_pC^m \longrightarrow R_p \otimes A_V \longrightarrow 0.$$

So A_V has no p-torsion if and only if $E(V)$ is non-singular over R_pC if and only if $p \nmid D(V)$. Letting p vary over all primes in R gives the result.

PROPOSITION 4.3. *If V is a square matrix over R, and the content of $D(V)$ is square-free (e.g., if V is knot-like), then V is S-equivalent to a non-singular matrix.*

PROOF. Assume V is singular. Then V is congruent to a matrix V_0 whose top row is zero. If $p \in R$ divides the first column of V_0, then clearly $p^2|D(V)$, which cannot happen. So V_0 is in turn congruent to a matrix V_1 which can be row reduced. Continuing as long as possible yields the result.

COROLLARY 4.4. *V is S-equivalent to a unmodular matrix if and only if A is a finitely generated free R module.*

PROOF. If V is S-equivalent to a unimodular matrix, then by (3.2), A_V has the stated form. Conversely suppose $A_V \cong R^m$. By (4.2) V is knot-like, so by (4.3) we may assume it is non-singular. If $p \in R$ is a prime, then $R_p \otimes A_V \cong R_p^m$, so by (2.4) (with $F = R_p$), V is non-singular mod p, i.e. $p \nmid \det(V)$. Letting p vary over all primes gives the result.

We are heading towards the following result on knot-like matrices.

THEOREM 4.5. *Two knot-like matrices over a PID are S-equivalent if and only if their Seifert systems are isomorphic.*

Before we can enter into its proof we will need some auxiliary concepts and Lemmas.

Assume M is a finite dimensional R' vector space. A free R module $N \subseteqq M$ is called a lattice if $R'N = M$. Let $N^* \subseteqq M^*$ be the set of all $f \in M^*$ satisfying $f(N) \subseteqq R$. N^* is called the dual lattice of N. If $\{a_i\}$ is a basis for N over R (which clearly also must be a basis for M over R'), then the dual basis $\{a_i^*\}$ for M^* must clearly also be a basis for N^* over R.

Suppose $g: M^* \to M$ is some fixed homomorphism. We call a lattice $N \subseteqq M$ integral if $g(N^*) \subseteqq N$. An integral lattice N determines a congruence class of matrices over R as follows: if $\{a_i\}$ and $\{a_i^*\}$ are dual basis for N and N^* respectively, then the matrix for g with respect to these basis has all of its entries in R, since $g(N^*) \subseteqq N$, and is clearly well defined up to a congruence over R. We call a representative of this congruence class "the" matrix generated by N and denote it by V_N. The ambiguity in this terminology will be offset by the fact that a basis for N will usually be implied.

Assume V is a non-singular knot-like matrix over R. By (3.7) there is a homomorphism $g: (R'A_V)^* \to R'A_V$. Furthermore, if we consider the lattice $N_V \subseteqq R'A_V$ given by the image of the maps $R_c^m \subseteqq R_c'^m \cong R'A_V$, we note that N_V is integral and generates the matrix V. The above discussion now makes the following obvious.

PROPOSITION. 4.6. *Suppose V and W are non-singular knot-like matrices whose Seifert systems are isomorphic. We identify A_V and A_W using this isomorphism and call the resulting module A. If $N_V = N_W$, then V and W are congruent.*

4.4 and 4.6. now imply the following result.

COROLLARY 4.7. *If V and W are unimodular matrices over R, then they are S-equivalent if and only if they are congruent.*

The strategy of the proof of (4.5) will be to identify A_V and A_W as above, then to augment N_V and N_W in some reasonable fashion until they agree and then invoke (4.6). So we assume we have an RC module $A \subseteqq R'A$ and a homomorphism $g\colon R'A^* \to R'A$. We call an integral lattice $N \subseteqq A$ *admissible* if and only if it generates A as an RC module, and V_N in knot-like.

Suppose N, $N' \subseteqq R'A$ are lattices. We choose bases $\{a_i\}$ and $\{b_i\}$ for N and N', and let $d(N, N')$ equal the determinant of a change of basis matrix from $\{b_i\}$ to $\{a_i\}$. Note $d(N, N')$ is only determined up to multiplication by units of R. If $p \in R$ is a prime, and $R_{(p)}$ is the local ring at p, then $R_{(p)}N$ and $R_{(p)}N'$ can be viewed as lattices over $R_{(p)}$. If $o(\ )$ is the valuation determined by p, then $o(d(R_{(p)}N, R_{(p)}N')) = o(d(N, N'))$.

LEMMA 4.8. *Let N be an admissible lattice. An integral lattice N' which generates A as an RC module is admissible if and only if $d(N, N')$ is a unit of R.*

PROOF. Let P be a change of basis matrix from a basis for N' to one for N. Clearly $D(V_{N'}) = \det(P)^2 D(V_N)$, so the content of $D(V_{N'})$ is 1 if and only if $\det(P)$ is a unit.

The augmentation of our admissible lattices is based on the following operation called *transferral of factors*.

LEMMA 4.9. *If a and k are relatively prime in R, then the matrices,*

$$V_0 = \begin{bmatrix} x & a & q' \\ kw & k^2y & ks' \\ p & kr & V \end{bmatrix} \text{ and } V_1 = \begin{bmatrix} k^2x & a & kq' \\ kw & y & s' \\ kp & r & V \end{bmatrix}$$

are S-equivalent.

This is proven in Trotter [11]. In going from V_0 to V_1 we say we are transferring a factor from the second row to the first column. We would like to relate this to our admissible lattices. Suppose $N \subseteqq R'A$ is an integral lattice which has as a basis $\{c_i\}$, which generates the matrix V_0 above. If we let N' be the lattice generated by $k^{-1}c_1, kc_2, c_3, \ldots, c_m$, then clearly $V_1 = V_{N'}$.

LEMMA 4.10. *N is admissible if and only if N' is,*

PROOF. By (4.8) V_0 is knot-like if and only if V_1 is. The lemma therefore reduces to showing that $A_{V_0} = A$ if and only if $A_{V_1} = A$. We show

$N' \subseteqq A_{V_0}$, $N \subseteqq A_{V_1}$ being handled similarly. A_{V_0} is presented by $E(V_0)$, and examining its second column (and using $(a, k) = 1$), we see that $tc_1 \in kA_{V_0}$, and so $k^{-1}c_1 \in A_{V_0}$, which implies the result.

PROOF OF 4.5. We assume V and W are non-singular knot-like matrices, and N_V, $N_W \subseteqq A$. Let $d = d(N_V, N_V \cap N_W)$. Since $N_V \cap N_W \subseteqq N_V$, d is an element of R. Note $d(N_W, N_W \cap N_V) = d(N_W, N_V) \cdot d(N_V, N_V \cap N_W) = d$ by (4.8). We induct on the sum of the exponents in a prime factorization of d. Clearly if d is a unit $N_V = N_W$, and so by (4.6) V and W are congruent. So assume p is a prime which divides d.

By the invariant factor theorem, we can select bases $b_1, \ldots, b_m$ and $c_1, \ldots, c_m$ for N_V and N_W respectively satisfying $c_i = r_i b_i$ for some $r_i \in R'$. Assume these are ordered so that $o(r_i) > 0$ for $i \leqq s$, $o(r_i) = 0$ for $s < i \leqq q$, and $o(r_i) < 0$ for $i > q$. We let S_V and S_W be the free $R_{(p)}$ modules generated by the b_i and c_i respectively for $i \leqq s$, U be the $R_{(p)}$ module generated by the b_i and c_i for $s < i \leqq q$, and T_V and T_W be the $R_{(p)}$ modules generated by the remaining b's and c's. So $S_W \subseteqq pS_V$ and $T_V \subseteqq pT_W$. Let S_W^*, S_V^*, U^*, T_W^* and T_V^* be the $R_{(p)}$ modules generated by the corresponding elements of the dual bases $\{b_i^*\}$ and $\{c_i^*\}$ for $R'A^*$ (e.g., S_V^* is generated by b_i^*, $i < s$). So $S_V^* \subseteqq pS_W^*$ and $T_W^* \subseteqq pT_V^*$.

Assume now that $s \geqq m - q$. If $m - q > s$, the same proof applies, reversing the roles of V and W.

Since $g(N_V^*) \subseteqq N_V$ and $g(N_W^*) \subseteqq N_W$, V mod p must have the form,

$$\begin{bmatrix} 0 & 0 & C \\ 0 & B & Z \\ D & X & Y \end{bmatrix}$$

where the three blocks of rows (respectively columns) correspond to S_V, U and T_V (respectively S_V^*, U^* and T_V^*). Observe the upper left corner must actually be divisible by p^2.

We claim that C and D are not both square and non-singular. Assume they are. $R_p \otimes A$ is then presented by the matrix

$$\begin{bmatrix} 0 & 0 & tC\text{-}D' \\ 0 & tB\text{-}B' & tZ\text{-}X' \\ tD\text{-}C' & tX\text{-}Z' & tY\text{-}Y \end{bmatrix}$$

The image of T_V under the natural map $R_{(p)}A \to R_p \otimes A$ is evidently isomorphic to the R_pC module presented by $tD\text{-}C'$ which is non zero by (3.2). However $T_V \subseteqq pT_W$ implies that it must in fact be zero.

We now assume D is not a square non-singular matrix. If C is the one which is not non-singular we apply a similar proof, switching rows and columns.

Since D is non-singular and $s \geqq m - q$, we may perform column operations on the first s columns of V so that the resulting matrix has a first column divisible by p. If we apply the corresponding row operations to V the result is a matrix V_1 congruent over R to V. The first row of V_1 cannot be divisible by p, since $p \nmid D(V)$, so clearly we can apply row and column operations to the last $m - q$ rows and columns of V_1 to produce a matrix V_2, congruent to V, whose first column is divisible by p, whose first diagonal entry is divisible by p^2 and whose first row has only one entry not divisible by p. Therefore we see that it is possible to transfor a factor out of a column of V corresponding to some element of S_V^* and into a row corresponding to some element of T_V.

Consider the lattice N' determined as in (4.10), where $V_{N'}$ is the result of transferring the fector. N' is evidently admissible, and it is easy to see that $o(d(N', N' \cap N_W)) < o(d)$, while all the other primes in $d(N', N' \cap N_W)$ and d occur to equal powers.

This completes the proof of (4.5).

The parallel between Seifert matrices for knots and knot-like matrices can be extended (see Keef [4]). For instance it can be shown that any pair of S-equivalent knot-like matrices whose determinants are a prime are in fact congruent. Further extensions are limited by the fact that 1-t is not an automorphism of A_V for a general knot-like matrix.

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