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A CLASSICAL BANACH SPACE: l_∞/c_0

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ABSTRACT. This is an expository paper in which we study some of the structural and geometric properties of the Banach space l_∞/c_0 using its identification with $C(\beta\mathrm{N}\backslash\mathrm{N})$. In particular, it is noted that although l_∞/c_0 is not a dual space, its unit ball has an abundance of extreme points. Also, its smooth points are classified and its complemented subspaces are studied.

1. Introduction. The Banach space l_∞/c_0 certainly falls into the category of a "classical Banach space". Not only has it been around since the time of Banach's original monograph (1932) **[2]**, but it is also classical in the sense of Lacey **[17]** or Lindenstrauss and Tzafriri **[18]** since it is congruent (isometrically isomorphic) to the space $C(\beta\mathrm{N}\backslash\mathrm{N})$. However, many of its interesting properties have not been as widely circulated as those of some of the other classical Banach spaces. In this paper, which is of an expository nature, it is our intention to begin to rectify this situation.

We will begin with some definitions. l_∞ is the linear space of bounded sequences of real numbers and c_0 is the subspace of sequences which converge to zero. Both of these spaces, when provided with the supremum norm, $\|x\| = \sup|x_n|$ where $x = \{x_n\}_{n\geqq 1}$, are Banach spaces. The quotient space l_∞/c_0 is the usual linear space of cosets $\hat{x} = x + c_0$, $x \in l_\infty$. When provided with the quotient norm $\|\hat{x}\| = \inf\{\|x - y\| : y \in c_0\}$, $\hat{x} = x + c_0$, it is a Banach space.

Although much is known about quotient spaces in general, this is not the appropriate method for studying l_∞/c_0. Indeed, much more is known about $C(T)$, the space of continuous real valued functions on the compact, Háusdorff space T, and we shall see shortly that l_∞/c_0 is congruent to $C(\beta\mathrm{N}\backslash\mathrm{N})$.

Recall that if T is a Tychonoff space, (i.e., completely regular and Hausdorff) then its Stone-Čech compactification βT can be described as

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follows. Let $C(T)$ be the Banach space of all bounded, continuous, real-valued functions on T with the norm given by $\|f\| = \sup\{|f(t)|: t \in T\}$ and let $B(C(T)^*)$ denote the closed unit ball of the dual space $C(T)^*$. Then by identifying $t \in T$ with the evaluation function $\phi_t \in B(C(T)^*)$, where $\phi_t(f) = f(t)$ for $f \in C(T)$, we can depict βT as the weak*-closure of $\{\phi_t: t \in T\}$ in $B(C(T)^*)$ and T can be considered as a dense subset of βT. Further, every $f \in C(T)$ has a uniquely determined norm-preserving extension $\hat{f} \in C(\beta T)$.

Also recall that a topological space T is said to be *zero-dimensional* if it has a base of open-and-closed sets and T is *totally disconnected* if the only connected subsets of T are the singletons. If T is a compact, Hausdorff space, then each of the above are equivalent to the *zero-set separation property*, i.e., any two disjoint zero-sets in T can be separated by disjoint open-and-closed sets. (A *zero-set* is a set of the form $\{t \in T: f(t) = 0\}$ where $f \in C(T)$.) Finally, T is said to be *extremally disconnected* if the closure of every open set is open. We note that for T compact, Hausdorff, this property implies each of the above.

For the definition or explanation of other terms or notions used in this paper, the reader is referred to one of the books **[9]**, **[10]**, **[20]** or **[23]**.

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2. Properties of βN and βN\N. Let N be the set of positive integers. We will have occasion to use the following properties of the Stone-Čech compactification βN and its closed subspace βN\N.

(2.1) Each point of N is isolated in βN and N is an open, dense subset of βN.

(2.2) βN\N is perfect, i.e., it has no isolated points.

(2.3) βN is totally disconnected and the open-and-closed subsets of βN are of the form $\mathrm{cl}_{\beta N} A$, where $A \subseteqq$ N.

(2.4) βN\N is totally disconnected and each nonempty open-and-closed subset of βN\N is of the form $\mathrm{cl}_{\beta N} A$\N, where $A \subseteqq$ N is infinite. Further, each such open-and-closed set is homeomorphic to βN\N.

(2.5) βN and βN\N each have a base consisting of c (the power of the continuum) open-and-closed sets and card(βN) = card(βN\N) = 2^c.

(2.6) βN is extremally disconnected.

(2.7) βN\N is not extremally disconnected; in fact, the closure of the union of any strictly increasing sequence of open-and-closed sets in βN\N is never open.

(2.8) βN\N contains a homeomorphic copy of βN.

Properties 2.1 to 2.8 of βN and βN\N can be found in Gillman and Jerison **[10]** or in Walker **[23]**.

The congruence of l_∞ and l_∞/c_0 with $C(\beta\mathbf{N})$ and $C(\beta\mathbf{N}\backslash\mathbf{N})$, respectively, is of central importance in the sequel. These relationships are briefly described below.

(2.9) If $\mathbf{N}$ is given the discrete topology, then $l_\infty = C(\mathbf{N})$, and the restriction mapping $R: C(\beta\mathbf{N}) \to C(\mathbf{N})$, defined by $R(\hat{f}) = \hat{f}|_{\mathbf{N}}$ for $\hat{f} \in C(\beta\mathbf{N})$, is a linear isometry of $C(\beta\mathbf{N})$ onto $C(\mathbf{N})$. Thus, l_∞ is isometrically isomorphic to $C(\beta\mathbf{N})$. This is denoted by $l_\infty \equiv C(\beta\mathbf{N})$.

(2.10) Let I be the closed ideal in $C(\beta\mathbf{N})$ consisting of functions which vanish on $\beta\mathbf{N}\backslash\mathbf{N}$, that is, $I = \{\hat{f} \in C(\beta\mathbf{N}) : \hat{f}(t) = 0 \text{ for all } t \in \beta\mathbf{N}\backslash\mathbf{N}\}$. Further, let $c_0(\mathbf{N})$ be the functions in $C(\mathbf{N})$ which vanish at infinity, that is, $c_0(\mathbf{N}) = \{f \in C(\mathbf{N}) : \text{for each } \varepsilon > 0, \{t \in \mathbf{N} : |f(t)| > \varepsilon\} \text{ is finite}\}$. Then $c_0(\mathbf{N}) = c_0$ and the restriction of R to I defines a linear isometry from I onto $c_0(\mathbf{N})$. Thus, $c_0 \equiv I$.

(2.11) The mapping $\sigma: C(\beta\mathbf{N})/I \to C(\beta\mathbf{N}\backslash\mathbf{N})$ given by $\sigma(\hat{f} + I) = \hat{f}|_{\beta\mathbf{N}\backslash\mathbf{N}}$, for $\hat{f} \in C(\beta\mathbf{N})$, defines a linear isometry from $C(\beta\mathbf{N})/I$ onto $C(\beta\mathbf{N}\backslash\mathbf{N})$. Thus we get, as mentioned above, that $l_\infty/c_0 \equiv C(\beta\mathbf{N}\backslash\mathbf{N})$.

The results in 2.9 to 2.11 follow, more or less, directly from the definitions and the Tietze extension theorem.

3. l_∞/c_0 is not a dual space. One nearly immediate consequence of the identification of l_∞/c_0 with $C(\beta\mathbf{N}\backslash\mathbf{N})$ is that l_∞/c_0 is not a dual space. Grothendieck [15] has shown that, for a compact Hausdorff space T, T must be hyperstonian (see definition below) in order for $C(T)$ to be congruent to a dual space. However, $\beta\mathbf{N}\backslash\mathbf{N}$ is not hyperstonian since it is not extremally disconnected.

In order to state Grothendieck's theorem we need some definitions. Let T be a compact Hausdorff space, $\mathscr{B}$ be the σ-algebra of Borel subsets of T and let rca$(T, \mathscr{B})$ denote the Banach space of regular, countably additive Borel measures μ on T with bounded variation. The norm is given by the variation of μ on T. That is, $\|\mu\| = |\mu|(T) = \sup \sum_{i=1}^{n} |\mu(A_i)|$ where the supremum is taken over all finite partitions $\{A_1, \ldots, A_n\}$ of T.

The norm closed proper cone in rca$(T, \mathscr{B})$ consisting of positive normal measures is denoted $N^+(T, \mathscr{B})$. Recall that μ is *normal* if $\mu(B) = 0$ for each Borel set B of the first category in T. Further, let $N(T, \mathscr{B}) = N^+(T, \mathscr{B}) - N^+(T, \mathscr{B})$ be the closed ideal in rca$(T, \mathscr{B})$ generated by $N^+(T, \mathscr{B})$. The *support* of $\mu \in$ rca$(T, \mathscr{B})$ is the set $S(\mu) = \bigcap \{F \subseteqq T : F \text{ is closed}, |\mu|(F) = |\mu|(T)\}$. Finally, we say that the compact Hausdorff space T is *hyperstonian* if T is extremally disconnected and $\bigcup \{S(\mu) : \mu \in N^+(T, \mathscr{B})\}$ is dense in T. We can now state the result mentioned above.

THEOREM 3.1. (GROTHENDIECK). *If T is a compact Hausdorff space, X is a Banach space, $L: C(T) \to X^*$ is an isometric isomorphism of $C(T)$ onto X^* and $J: X \to X^{**}$ is the canonical embedding, then*

(i) *T is hyperstonian and*
(ii) *$L^* \circ J$ is an isometric isomorphism of X onto $N(T, \mathscr{B})$.*

The converse of this theorem also obtains and is due to Dixmier **[8]**. In particular, Dixmier shows that if T is hyperstonian then $N(T, \mathscr{B})^*$ is congruent to $C(T)$. A statement of this theorem and proofs for it and Theorem 3.1 can be found in Bade **[1]**, Lacey **[17]** or Peressini **[19]**.

4. Geometry of $\mathbf{B}(l_\infty/c_0)$.

(a) Extreme points of $\mathbf{B}(l_\infty/c_0)$. Even though l_∞/c_0 is not a dual space, the ball $B(l_\infty/c_0)$ has an abundance of extreme points. In fact, if ext(A) denotes the extreme points of the set A, we have that $B(l_\infty/c_0) = \overline{\text{co}}(\text{ext}[B(l_\infty/c_0)])$. This follows from a theorem of Bade (cf. **[12]**, **[13]**) since $\beta\mathrm{N}\backslash\mathrm{N}$ is totally disconnected (see 2.3). However, we will give below a different proof of this fact.

First, we show that each extreme point of $B(l_\infty/c_0)$ is the image, under the quotient mapping, of an extreme point in $B(l_\infty)$.

THEOREM 4.1. *If $q\colon l_\infty \to l_\infty/c_0$ is the quotient map, then* $\text{ext}[B(l_\infty/c_0)] = q(\text{ext}[B(l_\infty)])$.

PROOF. Recall that $l_\infty \equiv C(\beta\mathrm{N})$ and $l_\infty/c_0 \equiv C(\beta\mathrm{N})/I \equiv C(\beta\mathrm{N}\backslash\mathrm{N})$ where $I = \{f \in C(\beta\mathrm{N})\colon f(t) = 0 \text{ for all } t \in \beta\mathrm{N}\backslash\mathrm{N}\}$. Thus, it suffices to show that $\text{ext}[B(C(\beta\mathrm{N}\backslash\mathrm{N}))] = \pi(\text{ext}[B(C(\beta\mathrm{N}))])$ where $\pi\colon C(\beta\mathrm{N}) \to C(\beta\mathrm{N}\backslash\mathrm{N})$ is the quotient map given by $\pi(f) = f|_{\beta\mathrm{N}\backslash\mathrm{N}}$, for $f \in C(\beta\mathrm{N})$. Also recall that $f \in \text{ext}[B(C(T))]$, T compact Hausdorff, if and only if $|f(t)| = 1$ for all $t \in T$.

Suppose $\hat{p} \in \text{ext}[B(C(\beta\mathrm{N}\backslash\mathrm{N}))]$. Then $|\hat{p}(t)| = 1$ for all $t \in \beta\mathrm{N}\backslash\mathrm{N}$. For each $t \in \beta\mathrm{N}\backslash\mathrm{N}$ there is an open neighbourhood V_t of t in $\beta\mathrm{N}\backslash\mathrm{N}$ such that $\hat{p}|_{V_t}$ is constant. Now there is an open neighbourhood $\hat{V}_t$ of t in $\beta\mathrm{N}$ such that $V_t = \hat{V}_t \cap (\beta\mathrm{N}\backslash\mathrm{N})$. Since $\beta\mathrm{N} = \bigcup\{\hat{V}_t\colon t \in \beta\mathrm{N}\backslash\mathrm{N}\} \cup \{\{n\}\colon n \in \mathrm{N}\}$ is compact, there is a finite family $\{\hat{V}_{t_1}, \ldots, \hat{V}_{t_k}, \{n_1\}, \ldots, \{n_j\}\}$ which covers $\beta\mathrm{N}$. We may assume that $n_i \notin \hat{V}_{t_l}$, for any i, l.

Now, define $p\colon \beta\mathrm{N} \to \mathbf{R}$ as follows:

$$p(t) = \begin{cases} \hat{p}|_{V_{t_i}}, \text{ for } t \in \hat{V}_{t_i},\ i = 1, \ldots, k, \\ \quad 1, \text{ for } t = n_i,\ i = 1, \ldots, j. \end{cases}$$

p is continuous; for, if $t_0 \in \mathrm{N}$, then $\{t_0\}$ is an open neighbourhood of t_0. If $t_0 \in \beta\mathrm{N}\backslash\mathrm{N}$, then $t_0 \in V_{t_i} \subseteqq \hat{V}_{t_i}$ for some i, $1 \leqq i \leqq k$. Suppose that the net $t_\delta \to t_0$ in $\beta\mathrm{N}$. Then (t_δ) is eventually in $\hat{V}_{t_i}$ and eventually

$$p(t_\delta) = \hat{p}|_{V_{t_i}}(t_\delta) = \hat{p}|_{V_{t_i}}(t_0) = p(t_0).$$

So $p(t_\delta) \to p(t_0)$ and p is continuous at t_0.

Since $p \in C(\beta\mathrm{N})$ and $|p(t)| = 1$ for all $t \in \beta\mathrm{N}$, then $p \in \text{ext}[B(C(\beta\mathrm{N}))]$.

Further, it is easily seen that $\hat{p} = p|_{\beta N \setminus N} = \pi(p)$. Therefore, $\text{ext}[B(C(\beta N \setminus N))] \subseteqq \pi(\text{ext}[B(C(\beta N))])$.

Clearly the reverse containment holds and the theorem is proven.

A closed subspace M of a real normed linear space X is *proximinal* in X if for each $x \in X$ there is $y \in M$ such that $\|x - y\| = \inf\{\|x - z\|: z \in M\}$.

THEOREM 4.2. *c_0 is proximinal in l_∞.*

PROOF. It suffices to show that $I = \{f \in C(\beta N): f(t) = 0 \text{ for all } t \in \beta N \setminus N\}$ is proximinal in $C(\beta N)$.

Let $f \in C(\beta N)$ and $F = f|_{\beta N \setminus N} \in C(\beta N \setminus N)$. By Tietze's extension theorem (see, for example, Dunford and Schwartz [9]) there is an $h \in C(\beta N)$ such that $h|_{\beta N \setminus N} = F = f|_{\beta N \setminus N}$ and $\|h\| = \|F\| = \sup_{t \in \beta N \setminus N} |h(t)|$. Now $f - h \in I$ and $\|f + I\| = \|h + I\| = \inf\{\|h - g\|: \ g \in I\} \leqq \|h\| = \|h|_{\beta N \setminus N}\| = \|f|_{\beta N \setminus N}\|$. On the other hand, for any $g \in I$, we have $\|h - g\| \geqq \|(h - g)|_{\beta N \setminus N}\| = \|h|_{\beta N \setminus N}\| = \|f|_{\beta N \setminus N}\|$. Thus, $\|f|_{\beta N \setminus N}\| = \|f + I\|$.

Let $g_0 = f - h$, then $g_0 \in I$ and $\|f - g_0\| = \|h\| = \|f|_{\beta N \setminus N}\| = \|f + I\|$; so $\|f - g_0\| = \inf\{\|f - g\| : g \in I\}$. Thus I is proximinal in $C(\beta N)$.

Note that the above proof relies on nothing more than the fact that $\beta N \setminus N$ is a closed subspace of the compact Hausdorff space βN. In fact the above theorem is a special case of a result due to Blatter and Seever, and Holmes and Ward (cf. [21, Theorem 2.16]).

Finally, to obtain the result mentioned at the beginning of this section, we need a result due to Godini [11].

THEOREM 4.3. (GODINI). *If X is a real normed linear space, $M \subseteqq X$ is a closed subspace and $q: X \to X/M$ is the quotient map, then the following are equivalent:*

(i) *$q(B(X)) = B(X/M)$.*

(ii) *$q(B(X))$ is closed in X/M.*

(iii) *M is proximinal in X.*

We now get an immediate result of the above theorems.

COROLLARY 4.4. $B(l_\infty/c_0) = \overline{\text{co}}(\text{ext}[B(l_\infty/c_0)])$.

PROOF. $B(l_\infty/c_0) = q(B(l_\infty)) = q(\overline{\text{co}}(\text{ext}[B(l_\infty)])) = \overline{\text{co}}(q(\text{ext}[B(l_\infty)])) = \overline{\text{co}}(\text{ext}[B(l_\infty/c_0)])$.

Although the unit ball in l_∞/c_0 has an abundance of extreme points in its unit ball, Bourgain [3] has shown recently that l_∞/c_0 does not admit an equivalent strictly convex norm.

(b) Smooth points of $B(l_\infty/c_0)$. As we have seen above, the quotient map $q: l_\infty \to l_\infty/c_0$ has great respect for extreme points of $B(l_\infty)$. However, it has no regard whatsoever for smooth points. In fact, the ball $B(l_\infty/c_0)$ has no smooth points (see Theorem 4.5 below). So l_∞/c_0 gives a nice example of a Banach space whose norm is nowhere Gateaux differentiable without having to resort to any renorming theorems.

Recall that the smooth points of the unit ball of $C(T)$, T compact Hausdorff, are precisely the functions $f \in C(T)$ with $\|f\| = \sup\{|f(t)|: t \in T\} = 1$ which peak at some $t_0 \in T$, that is, for which there exists $t_0 \in T$ such that $|f(t_0)| = 1 > |f(t)|$ for all $t \in T$, $t \neq t_0$. Further, the points of Fréchet differentiability of the supremum norm on $C(T)$ are precisely those functions which peak at an isolated point of T (Cox and Nadler **[4]**, Sundaresan **[22]**).

THEOREM 4.5. *$B(l_\infty/c_0)$ has no smooth points.*

PROOF. Again we use the identification of l_∞/c_0 with $C(\beta\mathbf{N}\backslash\mathbf{N})$. Let $f \in C(\beta\mathbf{N}\backslash\mathbf{N})$, $\|f\| = 1$, and set $A = \{t \in \beta\mathbf{N}\backslash\mathbf{N}: |f(t)| = \|f\| = 1\}$. $A \neq \varnothing$ and, since $A = \bigcap_{n=1}^{\infty}\{t \in \beta\mathbf{N}\backslash\mathbf{N}: |f(t)| > \|f\| - 1/n\}$, it is a G_δ subset of $\beta\mathbf{N}\backslash\mathbf{N}$. Therefore A has nonempty interior (see Walker **[23**, p. 78]) and, hence, contains an open-and-closed subset of $\beta\mathbf{N}\backslash\mathbf{N}$. So $\mathrm{card}(A) \geqq 2$ since $\beta\mathbf{N}\backslash\mathbf{N}$ has no isolated points. Thus, f is not a point of smoothness of $B(C(\beta\mathbf{N}\backslash\mathbf{N}))$ and the theorem is proven.

We note that the above proof shows that the points of smoothness on the ball $B(l_\infty) \equiv B(C(\beta\mathbf{N}))$ are actually points of Fréchet differentiability of the norm of l_∞. For, if $f \in B(C(\beta\mathbf{N}))$, $\|f\| = 1$, peaks at $t_0 \in \beta\mathbf{N}$, then $t_0 \in \mathbf{N}$ and is an isolated point of $\beta\mathbf{N}$.

Finally, we note that it is known that l_∞/c_0 cannot be renormed with a smooth norm. One way to see this is to observe that l_∞/c_0 is a Grothendieck space, that is, weak and weak* sequential convergence coincide in its dual. (l_∞/c_0 is the continuous linear image of the Grothendieck space l_∞.) Then couple this with the result of Johnson (see **[7**, p. 215]) which states that a smooth Grothendieck space is reflexive.

5. Complemented subspaces of l_∞/c_0. A closed subspace M of a Banach space X is said to be complemented if X can be written as a direct sum of M and a closed subspace N of X. In this case the projection $P: X \to M$ of X onto M along N is continuous. M and N are called *complementary subspaces*; we denote this by writing $X = M \oplus N$. In this section some complemented subspaces of l_∞/c_0 will be identified.

We begin by exploiting a theorem of Rosenthal's to show that each infinite dimensional complemented subspace of l_∞/c_0 contains an isometric isomorphic copy of l_∞. Rosenthal's result (see **[7**, p. 156]) is that if T is extremally disconnected and a Banach space X contains no copy of l_∞, then every bounded linear operator $L: C(T) \to X$ is weakly compact. Recall that a bounded linear operator $L: X \to Y$, X and Y Banach spaces, is *weakly compact* (respectively, *compact*) if the image $L(B(X))$ of the unit ball in X is relatively weakly (respectively, norm) sequentially compact in Y.

THEOREM 5.1. *If M is an infinite dimensional complemented subspace of l_∞/c_0, then M contains a subspace congruent to l_∞.*

PROOF. Let $P: l_\infty/c_0 \to M$ be the continuous projection of l_∞/c_0 onto M along its complement and $q: l_\infty \to l_\infty/c_0$ be the quotient map. By Rosenthal's theorem, assuming there is no copy of l_∞ contained in M, $Q = P \circ q$ is weakly compact since l_∞ is congruent to $C(\beta N)$ and βN is extremally disconnected.

So $Q(B(l_\infty))$ is relatively weakly compact and, by Theorem 4.3, $P(B(l_\infty/c_0)) = Q(B(l_\infty))$ is also weakly compact. Thus, P is a weakly compact operator and, since l_∞/c_0 is congruent to $C(\beta N\backslash N)$, $P^2 = P$ is compact (see [9, p. 494]). Hence M is finite dimensional.

An immediate consequence of this theorem is that l_∞/c_0 has no infinite dimensional complemented subspaces which are separable or reflexive.

Next, we show that l_∞/c_0 is congruent to its square $l_\infty/c_0 \times l_\infty/c_0$ with an appropriate norm. From this we see that $l_\infty/c_0 = M \oplus N$ where M and N are each closed subspaces congruent to l_∞/c_0.

THEOREM 5.2. *If $l_\infty/c_0 \times l_\infty/c_0$ is given the norm $\|(x, y)\|_0 = \max(\|x\|, \|y\|)$, $x, y \in l_\infty/c_0$, then there exists an isometric isomorphism L of $l_\infty/c_0 \times l_\infty/c_0$ onto l_∞/c_0.*

PROOF. Let A and B be nonempty disjoint open-and-closed subsets of $\beta N\backslash N$ with $A \cup B = \beta N\backslash N$. By (2.4), there are homeomorphisms $\phi: A \to \beta N\backslash N$ and $\psi: B \to \beta N\backslash N$. For $f, g \in C(\beta N\backslash N)$, define $L(f, g) = h$ where

$$h(t) = \begin{cases} f(\phi(t)), & t \in A \\ g(\psi(t)), & t \in B. \end{cases}$$

Now $h \in C(\beta N\backslash N)$, for suppose $t_\delta \to t$, t_δ, $t \in \beta N\backslash N$. If $t \in A$, then t_δ is eventually in A, so we may consider that $t_\delta \to t$ in A. Thus $\phi(t_\delta) \to \phi(t)$ and $h(t_\delta) = (f \circ \phi)(t_\delta) \to (f \circ \phi)(t) = h(t)$. Similarly, if $t \in B$, then $h(t_\delta) \to h(t)$. Therefore, $h \in C(\beta N\backslash N)$ and $L: C(\beta N\backslash N) \times C(\beta N\backslash N) \to C(\beta N\backslash N)$. L is easily seen to be linear.

L is surjective, for if $h \in C(\beta N\backslash N)$, letting $f = h \circ \phi^{-1}$ and $g = h \circ \psi^{-1}$ we have $L(f, g) = h$. Further, $\|L(f, g)\| = \sup\{|L(f, g)(t)|: t \in \beta N\backslash N\} = \max(\sup\{|f(\phi(t))|: t \in A\}, \sup\{|g(\psi(t))|: t \in B\}) = \max(\|f\|, \|g\|) = \|(f, g)\|_0$. Thus L is an isometric isomorphism of $C(\beta N\backslash N) \times C(\beta N\backslash N)$ onto $C(\beta N\backslash N)$.

COROLLARY 5.3. *$l_\infty/c_0 = M \oplus N$ where M and N are closed subspaces of l_∞/c_0 and both are isometrically isomorphic to l_∞/c_0.*

PROOF. Let L be the congruence from the previous theorem and let $M = L(l_\infty/c_0 \times \{0\})$ and $N = L(\{0\} \times l_\infty/c_0)$. Then M and N are congruent to $l_\infty/c_0 \times \{0\}$ and $\{0\} \times l_\infty/c_0$, respectively, and each of these are congruent to l_∞/c_0. Further, since both are closed subspaces, $l_\infty/c_0 \times \{0\}$ and $\{0\} \times l_\infty/c_0$ are complementary subspaces. Thus, $l_\infty/c_0 = L(l_\infty/c_0 \times l_\infty/c_0) = L(l_\infty/c_0 \times \{0\}) \oplus L(\{0\} \times l_\infty/c_0) = M \oplus N$ and $M \equiv N \equiv l_\infty/c_0$.

It turns out, however, that this is not the only way that a complemented subspace of l_∞/c_0 can be isomorphic to l_∞/c_0. To see this we need the fact that l_∞ is congruent to a closed subspace of l_∞/c_0. This follows from a result of Dean [**6**], but we give a different proof below. Also we use a result of Goodner [**14**], first proved by Dean [**5**], which states that if l_∞ is congruent to a subspace M of $C(T)$, T compact Hausdorff, then any complement of M in $C(T)$ is isomorphic to $C(T)$.

THEOREM 5.4. *$l_\infty/c_0 = M \oplus N$ where M and N are closed subspaces of l_∞/c_0, M is isometrically isomorphic to l_∞ and N is isomorphic to l_∞/c_0.*

PROOF. By (2.8), there is a $T \subseteqq \beta\mathbf{N}\backslash\mathbf{N}$ which is homeomorphic to $\beta\mathbf{N}$. By a result due to Parovičenko (see Walker [**23**, p. 81]), since the weight of T (least cardinal of a base for T) is at most $\aleph_1$, there exists a continuous surjection ϕ: $\beta\mathbf{N}\backslash\mathbf{N} \rightarrow T$.

Define L: $C(T) \rightarrow C(\beta\mathbf{N}\backslash\mathbf{N})$ by $L(f) = f \circ \phi$. Obviously, L is linear and $\|L(f)\| = \sup\{|f(\phi(t))|: t \in \beta\mathbf{N}\backslash\mathbf{N}\} = \sup\{|f(s)|: s \in T\} = \|f\|$. So, L is a linear isometry of $C(T)$ onto a closed subspace of $C(\beta\mathbf{N}\backslash\mathbf{N})$. Further, since $l_\infty \equiv C(\beta\mathbf{N}) \equiv C(T)$ is a P_1-space (Kelley [**16**]), i.e., it is complemented in any space containing it, then $M = L(C(T))$ is complemented in $C(\beta\mathbf{N}\backslash\mathbf{N})$. So $C(\beta\mathbf{N}\backslash\mathbf{N}) = M \oplus N$, where $M \equiv l_\infty$ and it follows from the result of Goodner mentioned above that N is isomorphic to $C(\beta\mathbf{N}\backslash\mathbf{N}) \equiv l_\infty/c_0$.

The above results raise the interesting question of whether or not l_∞/c_0 is a primary space, that is, if $l_\infty/c_0 = M \oplus N$, must one of the summands be isomorphic to l_∞/c_0. Since $l_\infty/c_0 \equiv C(\beta\mathbf{N}\backslash\mathbf{N})$ is nonseparable, new techniques probably will be needed to solve this problem. Perhaps these will indicate a method for attacking the more difficult problem of deciding which nonseparable spaces $C(T)$ are primary.

REFERENCES

1. W.G. Bade, *The Banach Space C(S)*, Lecture Notes, Aarhus Universitat, **26** (1971).

2. S. Banach, *Théorie des Opérations Linéares*, Monografie Mat., Vol. **1**, Warsaw, 1932.

3. J. Bourgain, *l_∞/c_0 has no equivalent strictly convex norm*, Proc. Amer. Math. Soc. **78** (1980), 225–226.

4. S.H. Cox, Jr. and S.B. Nadler, Jr., *Supremum norm differentiability*, Ann. Soc. Math. Polonae **15** (1971), 127–131.

5. D.W. Dean, *Projections in certain continuous function spaces $C(H)$ and subspaces of $C(H)$ isomorphic with $C(H)$*, Can J. Math. **14** (1962), 385–401.

6. ———, *Subspaces of $C(H)$ which are direct factors of $C(H)$*, Proc. Amer. Math. Soc. **16** (1965), 237-242.

7. J. Diestel and J.J. Uhl, Jr., *Vector Measures*, Math. Surveys No **15**, Amer. Math. Soc., Providence, 1977.

8. J. Dixmier, *Sur certaines espaces considerés par M.H. Stone*, Summa Brasil. Math. **2** (1951), 151–182.

9. N. Dunford and J. Schwartz, *Linear Operators I*, Interscience, New York, 1958.

10. L. Gillman and M. Jerison, *Rings of Continuous Functions*, van Nostrand Reinhold, New York, 1960.

11. G. Godini, *Characterizations of proximinal linear subspaces in normed linear spaces*, Rev. Roumaine Math. Pures Appl. **18** (1973), 901–906.

12. D.B. Goodner, *Projections in normed linear spaces*, Trans. Amer. Math. Soc. **69** (1950), 89–108.

13. ———, *The closed convex hull of certain extreme points*, Proc. Amer. Math. Soc. **15** (1964), 254–258.

14. ———, *Subspaces of $C(S)$ isometric to m*, J. London Math. Soc. **3** (1971), 488–492.

15. A. Grothendieck, *Sur les applications lineare faiblement compactes d'espaces du type $C(K)$*, Can J. Math. **5** (1953), 129–173.

16. J.L. Kelley, *Banach spaces with the extension property*, Trans. Amer. Math. Soc. **72** (1952), 323–326.

17. H.E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Springer-Verlag, New York, 1974.

18. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces*, Springer-Verlag Lecture Notes **338** (1973).

19. A.L. Peressini, *Ordered Topological Vector Spaces*, Harper and Row, New York, 1967.

20. Z. Semadini, *Banach spaces of continuous functions*, vol. 1, Monografie Mat., Tom **55**, PWN, Warsaw, 1971.

21. I. Singer, *The Theory of Best Approximation and Functional Analysis*, SIAM, Philadephia, 1974.

22. K. Sundaresan, *Some geometric properties of the unit cell in the space $C(X; B)$*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **19** (1971), 1007–1012.

23. R.C. Walker, *The Stone-Čech Compactification*, Springer-Verlag, New York, 1974.

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