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ON FUNCTIONS WITH WEIERSTRASS BOUNDARY POINTS EVERYWHERE

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In this paper we answer a conjecture of the second author by proving that any meromorphic function in the unit disc with radial limits on at most a set of measure zero must have Weierstrass points at each point of $|z| = 1$. Given any tangential set D and any closed proper subset E of $|z| = 1$, an analytic function f is constructed such that a) if E has measure zero, then f has no finite or infinite radial limits and the restricted cluster set $C_{D(\theta)}(f, \exp(i\theta))$ is uniformly bounded for all $\exp(i\theta)$ in E, b) if E has capacity zero then f has no finite or infinite radial limits on the complement of E but $C_{D(\theta)}(f, \exp(i\theta)) = \{0\}$ for all $\exp(i\theta)$ in E.

Introduction. While answering some open questions due to J. L. Doob [3] the second author introduced the class $k(0)$, the set of Bloch functions which have no finite radial limits at any point of the unit disc. The second author conjectured [5] that for any f in $k(0)$ and for any θ in $[0, 2\pi]$ the cluster set of f at $e^{i\theta}$ is the entire complex plane, that is, that every point of $|z| = 1$ is a Weierstrass point.

In this paper we prove that any meromorphic function with radial limits on at most a set of measure zero must have Weierstrass points at every point of $|z| = 1$. This answers the conjecture in the affirmative for a class of functions much more general than $k(0)$. This result is a corollary to the more general theorem 1 in which we prove that the presence of a dense set of Weierstrass points implies that every point of $|z| = 1$ is a Weierstrass point which extends the claims of the conjecture to functions which may have radial limits on sets whose measure is greater than zero.

In theorems 3 and 4 we show how to construct analytic functions which have no finite or infinite radial limits but for which the restricted cluster set is bounded or degenerate. The technique makes use of Mergelyan's theorem, Blaschke products, and a construction due to A. Lohwater and G. Piranian.

DEFINITIONS. If f is an arbitrary complex valued function (not necessarily even continuous) defined in $|z| < 1$, then the cluster set of f at the

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boundary point $\exp(i\theta)$ is $C(f, \exp(i\theta))$, the set of complex numbers w for which there is a sequence $\{z_n\}$ in $|z| < 1$ with $z_n \to \exp(i\theta)$ and $f(z_n) \to w$.

If E is a subset of $|z| < 1$, then the cluster set of f at the point $\exp(i\theta)$ restricted to E is $C_E(f, \exp(i\theta))$, the set of w for which there is a sequence $\{z\}_n$ in E with $z_n \to \exp(i\theta)$ and $f(z_n) \to w$. If $C_E(f, \exp(i\theta))$ is the entire complex plane for every Stolz angle E at $\exp(i\theta)$, then $\exp(i\theta)$ is said to be a Plessner point. If $C_E(f, \exp(i\theta))$ is a single point for every symmetric Stolz angle E at $\exp(i\theta)$, then $\exp(i\theta)$ is said to be a Fatou point. If $C(f, \exp(i\theta))$ is the entire complex plane, then $\exp(i\theta)$ is said to be a Weierstrass point [**2**, p. 149].

Functions with Weierstrass points everywhere on $|z| = 1$.

THEOREM 1. *If $f(z)$ is meromorphic in $|z| < 1$ and has a dense set of Weierstrass points, then every point of $|z| = 1$ is a Weierstrass point.*

PROOF. Let $\exp(i\theta)$ be an arbitrary point of $|z| = 1$ and let v be an arbitrary complex number. Since f has a dense set of Weierstrass points we can choose points $\exp(i\theta_n)$ and z_n such that

$$|\exp(i\theta_n) - \exp(i\theta)| < 1/n,$$
$$|z_n - \exp(i\theta_n)| < 1/n,$$
$$|f(z_n) - v| < 1/n.$$

Thus $z_n \to \exp(i\theta)$ and $f(z_n) \to v$. This proves that the cluster set of f at $\exp(i\theta)$ is the entire complex plane and concludes the proof of theorem 1.

COROLLARY 2. *If $f(z)$ is meromorphic in $|z| < 1$ and the set of points of $|z| = 1$ at which f has finite or infinite radial limits has measure zero, then every point of $|z| = 1$ is a Weierstrass point.*

PROOF. Plessner's theorem [**2**, Theorem 8.2] says that for an arbitrary meromorphic function in $|z| < 1$, almost all points of $|z| = 1$ are either Plessner points or Fatou points. Since every Plessner point is a Weierstrass point, if a function has radial limits on at most a set of measure zero, then by Plessner's theorem, the set of Plessner (and hence Weierstrass) points is of measure 2π and therefore dense.

Theorem 1 is stronger than Corollary 2 since there are analytic functions with a dense set of Weierstrass points which have radial limits almost everywhere [**8**, p. 229].

Of course Theorem 1 and Corollary 2 can be localized if we know that there is a non-degenerate arc I of $|z| = 1$ on which the set of points of

I at which f has a radial limit is of measure zero or the set of Weierstrass point of I is dense.

Restricted cluster sets. We shall say that D is a tangential set if D is a subset of $|z| < 1$ and the closure of D intersects $|z| = 1$ only at $z = 1$. For example, D could be the radius from 0 to 1, a Stolz angle with vertex at $z = 1$, or a disc internally tangent to $|z| = 1$ at $z = 1$. Let $D(\theta)$ denote the set obtained from D by a rotation about the origin through an angle θ so that $D(\theta) \cap \{|z| = 1\}$ is the point $\exp(i\theta)$. If $\exp(i\theta)$ is an arbitrary point of the unit circle, then *the restricted cluster set of f with respect to $D(\theta)$* at the point $\exp(i\theta)$ is the set of points w for which there is a sequence of points z_n in $D(\theta)$ tending to $\exp(i\theta)$ with $f(z_n) \to w$. The extended cluster set of f with respect to $D(\theta)$ is denoted by $C_{D(\theta)}(f, \exp(i\theta))$.

A tangential set D can approach $|z| = 1$ so fast that there is no disc which is internally tangent to $|z| = 1$ at 1 which contains D. For example, let D_n be the part of the disc $|z - 1/(2n)| < 1 - 1/(2n)$ which lies inside the sector $|\arg z| < 1/n$. Then $D = \bigcup_{n=1}^{\infty} D_n$ is a simply connected domain internally tangent to $|z| = 1$ at $z = 1$ which lies in no disc internally tangent to $|z| = 1$ at $z = 1$.

If D is a tangential set, then we can always enlarge D, if necessary, by taking the union of D and its conjugate, and then taking the convex hull. The resulting set will contain D, be simply connected, be symmetric with respect to the real axis, be tangential and will also contain part of a radius to $z = 1$. For the theorems of this paper there is no loss of generality in assuming that tangential sets are simply connected and contain part of the radius to $z = 1$.

We shall show that for any tangential set D there is an analytic function f which has no (finite or infinite) radial limits (and hence $C(f, \exp(i\theta)) = \mathbf{C}$ everywhere) but for which $C_D(f, 1)$ is a bounded set.

THEOREM 3. *For any tangential set D there is an analytic function f which has no finite or infinite radial limit but for which $C_D(f, 1)$ is bounded.*

PROOF. Let $\{J_n\}$ be the sequence of circular arcs defined by $J_n = \{|z| = 1 - 1/(n+2)\colon \operatorname{dist}(z, D) \geqq 1/n\}$ and let $\{\delta_n\}$ be the sequence of discs $|z| < 1 - 2/(n+2)$. Each J_n is disjoint from $\bar{D}$. Furthermore, $k_n = J_{2n} \cup J_{2n-1} \cup \bar{\delta}_n \cup \bar{D}$ is compact. Finally no k_n separates the plane.

Mergelyan's theorem [7, p. 423] guarantees for every $\varepsilon > 0$ and for every compact set K whose complement is connected, and for every complex function continuous on K and analytic in the interior of K, the existence of a polynomial $P(z)$ such that $|F(z) - P(z)| < \varepsilon$ for all z in K.

By Mergelyan's theorem there is a polynomial $p_1(z)$ such that

$$|p_1(z)| < 1/2 \qquad \text{for } z \text{ in } J_1 \cup \bar{\delta}_1 \cup \bar{D},$$
$$|p_1(z) - 1| < 1/2 \text{ for } z \text{ in } J_2.$$

We use $p_1(z)$ to inductively define a sequence of complex functions F_n, $n \geqq 2$, on the compact sets k_n:

$$F_n(z) = \begin{cases} 0 \quad \text{for } z \text{ in } \bar{\delta}_n \cup \bar{D}, \\ \sum_{k=1}^{n-1} (-1)^{k+1} p_k(z) \text{ for } z \text{ in } J_{2n-1}, \\ \sum_{k=1}^{n-1} (-1)^{k+1} p_k(z) - 1 \text{ for } z \text{ in } J_{2n}. \end{cases}$$

The function $F_n(z)$ is continuous on k_n and analytic on the interior of k_n. Thus by Mergelyan's theorem we can find polynomials $p_n(z)$ such that:

$$|p_n(z)| < 1/2^n \text{ for } z \text{ in } \bar{\delta}_n \cup \bar{D},$$

(1) $$\left|\sum_{k=1}^{n} (-1)^{k+1} p_k(z)\right| < 1/2^n \text{ for } z \text{ in } J_{2n-1},$$

(2) $$\left|\sum_{k=1}^{n} (-1)^{k+1} p_k(z) - 1\right| < 1/2^n \text{ for } z \text{ in } J_{2n}.$$

Since the sequence of discs $\{\delta_n\}$ expands to $|z| < 1$ and $|p_n(z)| < 1/2^n$ on δ_n, it follows that $g(z) = \sum_{n=1}^{\infty}(-1)^{n+1}p_n(z)$ converges uniformly on any compact subset of $|z| < 1$ and defines an analytic function. Since each $|p_n(z)| < 1/2^n$ on $\bar{D}$, the function g is bounded by 1 in D. Consequently, the points of the cluster set of g at $z = 1$ with respect to D are also bounded by 1 in modulus.

Since D is a tangential set, a radius to $\exp(i\theta)$, $\theta \neq 0 \pmod{2\pi}$, must intersect infinitely many of the arcs J_{2n-1} on which (1) guarantees that g approaches 0 and infinitely many of the arcs J_{2n} on which (2) guarantees that g approaches 1. Therefore the only point at which g can have a radial limit is $z = 1$.

Let $B(z)$ be the Blaschke product with zeros at $z = 1 - \exp(-n)$, for $n = 1, 2, \ldots$. Then $B(z)$ has no radial limit at $z = 1$ [**1**, p. 12] and since the zeros accumulate only at $z = 1$ the function $B(z)$ is analytic on all of $|z| = 1$ except $z = 1$ [**4**, p. 68]. Let M be the limsup of $B(z)$ as z approaches 1 radially.

Set $f(z) = g(z) + 2B(z)/M$. The function $g(z)$ can not have an infinite radial limit at $z = 1$ since D contains a radial segment to $z = 1$ and g is bounded on D. Furthermore, whether g does or does not have a finite radial limit at $z = 1$ the function f can not have a radial limit at $z = 1$. Finally, f can not have a finite or infinite radial limit at any of the other points of $|z| = 1$. The restricted cluster set of f is clearly bounded at $z = 1$. This completes the proof of the theorem.

Using the technique of theorem 3 we can construct additional functions with no radial limits but whose restricted cluster sets are bounded.

THEOREM 4. *Let D be any tangential set and E be any closed proper subset of $|z| = 1$ and E' be the complement of E with respect to $|z| = 1$.*

(a) *There is an analytic function f with no finite or infinite limits on E' yet $C_{(D\theta)}(f, \exp(i\theta))$ is uniformly bounded for all $\exp(i\theta)$ in E.*

(b) *If E is of measure zero, then there is an analytic function f with no finite or infinite radial limits yet $C_{D(\theta)}(f, \exp(i\theta))$ is uniformly bounded for all $\exp(i\theta)$ in E.*

(c) *If E is of capacity zero, then there is an analytic function f with no finite or infinite radial limits on E' yet $C_{D(\theta)}(f, \exp(i\theta)) = \{0\}$ for all $\exp(i\theta)$ in E.*

PROOF. Let F be the union of the sets $D(\theta)$ taken over all $\exp(i\theta)$ in E A consideration of the components of the open set E' shows that F is not the open unit disc. If we follow the proof of theorem 3, replacing D by F in the definition of the sequence $\{J_n\}$ and in the application of Mergelyan's theorem, then we obtain a function $g(z)$ with no finite or infinite radial limits on E' yet with $C_{D(\theta)}(f, \exp(i\theta))$ uniformly bounded by 1 for all $\exp(i\theta)$ in E. This completes the proof of part (a).

A. J. Lohwater and G. Piranian [**2**, p. 27] showed for any closed set E of measure zero on $|z| = 1$ there is a bounded analytic function h such that the radial limit exists on E' and for each $\exp(i\theta)$ in E $\liminf_{r\to 1} |h(r \exp(i\theta))| = 0$, $\limsup_{r\to 1}|h(r \exp(i\theta))| = 1$. Clearly the function $f(z) = g(z) + 2h(z)$ satisfies the claims of part (b).

Finally, if E is a closed set of capacity zero then there is an analytic function $h(z)$ which tends to a finite radial limit everywhere on E' but which tends to 0 everywhere on E [**6**, p. 7]. Clearly the function $f(z) = g(z)h(z)$ satisfies the claims of part (c).

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