

## REMARKS ON HYPONORMAL TRIGONOMETRIC TOEPLITZ OPERATORS

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ABSTRACT. Presented in this note are a characterization of hyponormal trigonometric Toeplitz operators of degree two and a characterization of operators (in the class of hyponormal trigonometric Toeplitz operators) whose self-commutators are of rank one.

For  $\phi$  in  $L^\infty$  of the unit circle, the Toeplitz operator  $T_\phi$  with symbol  $\phi$  is the operator on  $H^2$  of the unit disk defined by the equation  $T_\phi(f) = P(\phi f)$  where  $P$  is the orthogonal projection onto  $H^2$ . A trigonometric Toeplitz operator of degree  $n$  is a Toeplitz operator  $T_\phi$  with  $\phi(z) = \sum_{k=-n}^n a_k z^k$ , where  $a_n$  or  $a_{-n} \neq 0$ . An operator  $T$  is called hyponormal if its self-commutator,  $T^*T - TT^*$ , is a positive operator.

It is obvious that, given  $\phi(z) = a_1 z + a_{-1} \bar{z}$ ,  $T_\phi$  is a hyponormal operator if and only if  $|a_1| - |a_{-1}| \geq 0$ . A less obvious fact is that, given  $\phi(z) = \sum_{k=-n}^n a_k z^k$ , the self-commutator of the hyponormal trigonometric Toeplitz operator  $T_\phi$  of degree  $n$  is of rank zero (in other words,  $T_\phi$  is normal if and only if  $|a_n| = |a_{-n}| (>0)$ ; this fact was first observed by Ito and Wong [4, remark 4]. The purpose of this note is to present the following extensions which, we hope, may shed some light on the problem of characterizing hyponormal Toeplitz operators [1, problem 1], and on the problem of identifying operators (in the class of trigonometric Toeplitz operators) the ranks of whose self-commutators are given. These extensions also provide us with means to construct easily hyponormal operators that are not subnormal (for other types of examples, see: Halmos [3, problems 160 and 164]; Putnam [5, p. 60]; Fan and Stampfli [2]).

**THEOREM 1.** *Let  $\phi(z) = \sum_{k=-2}^2 a_k z^k$ , where  $a_2$  or  $a_{-2} \neq 0$ . The trigonometric Toeplitz operator  $T_\phi$  of degree two is hyponormal if and only if  $|a_2|^2 - |a_{-2}|^2 \geq |a_2 \bar{a}_{-1} - \bar{a}_{-2} a_1|$ .*

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**THEOREM 2.** *Let  $\phi(z) = \sum_{k=-n}^n a_k z^k$ , where  $a_n$  or  $a_{-n} \neq 0$ . The self-commutator of the hyponormal trigonometric Toeplitz operator  $T_\phi$  of degree  $n (\geq 2)$  is of rank one if and only if*

$$(1) \quad |a_n|^2 - |a_{-n}|^2 = |a_n \bar{a}_{-n+1} - \bar{a}_{-n} a_{n-1}| > 0.$$

Before proceeding to the proofs of the theorems, we would like to state some notations and some facts about the self-commutator of  $T_\phi$ ; the proofs of these facts are omitted, since they involve only standard computations. The matrix representation (with respect to the natural basis  $\{1, z, z^2, \dots\}$ ) of the self-commutator  $C_\phi (= T_\phi^* T_\phi - T T_\phi^*)$  of a trigonometric Toeplitz operator  $T_\phi$  of degree  $n$  is of the form  $M_\phi \oplus 0$  in which the  $n \times n$  hermitian matrix  $M_\phi$ , whose  $(i, j)$ -entry is equal to  $\sum_{k=i}^n (\bar{a}_k a_{k+j-i} - a_{-k} \bar{a}_{-k-j+i})$  for  $i \geq j \geq 1$ , is the compression of  $C_\phi$  to  $V\{1, z, \dots, z^{n-1}\}$  and  $0$  is the zero operator on  $V\{z^n, z^{n+1}, \dots\}$ . Thus, of course,  $T_\phi$  is hyponormal if and only if  $M_\phi$  is positive, and  $C_\phi$  is of rank one if and only if  $M_\phi$  is. Moreover, let  $\psi(z) = \sum_{k=1}^{n-1} a_{k+1} z^k + \sum_{k=-1}^{-n+1} a_{k-1} z^k$ ; then,  $M_\psi$  is precisely the compression of  $M_\phi$  to  $V\{z, z^2, \dots, z^{n-1}\}$ .

**PROOF OF THEOREM 1.** The proof can be concluded by basic arguments on positivity of matrices, with the aid of the following identities:

$$\begin{aligned} \det M_\phi &= (|a_2|^2 - |a_{-2}|^2) \sum_{k=1}^2 (|a_k|^2 - |a_{-k}|^2) \\ &\quad - |\bar{a}_2 a_1 - a_{-2} \bar{a}_{-1}|^2 \\ &= (|a_2|^2 - |a_{-2}|^2)^2 + |a_2 a_1|^2 - |a_2 a_{-1}|^2 \\ (2) \quad &\quad - |a_{-2} a_1|^2 + |a_{-2} a_{-1}|^2 - |a_2 a_1|^2 - |a_{-2} a_{-1}|^2 \\ &\quad + \bar{a}_2 a_1 \bar{a}_{-2} a_{-1} + a_2 \bar{a}_1 a_{-2} \bar{a}_{-1} \\ &= (|a_2|^2 - |a_{-2}|^2)^2 - |a_2 \bar{a}_{-1} - \bar{a}_{-2} a_1|^2. \end{aligned}$$

**PROOF OF THEOREM 2.** Necessity: Observe the compression of  $M_\phi$  to  $V\{z^{n-2}, z^{n-1}\}$  is precisely  $M_\psi$ , where  $\psi(z) = a_n z^2 + a_{n-1} z + a_{-n+1} \bar{z} + a_{-n} \bar{z}^2$ . Since  $M_\phi$  is of rank one, the rank of  $M_\psi$  is less than or equal to one, but, since  $|a_n| \neq |a_{-n}|$  (if not so, then  $T_\phi$  would be a normal operator), it is one. Hence, (1) follows, replacing  $M_\phi$  by  $M_\psi$  in (2).

Sufficiency: We need the following lemma, which may be of independent interest.

**LEMMA.** *Let  $A$  be a positive operator acting on  $H$  ( $\dim H \geq 3$ ),  $e_0$  a unit vector in  $H$ , and  $\{e_1, e_2, \dots\}$  an orthonormal basis for  $M$ , the orthogonal complement of  $e_0$  with respect to  $H$ . If these two conditions are satisfied: (I) The rank of compression of  $A$  to  $M$  is one, and (II) There exists a  $k (\geq 1)$  such that  $(Ae_k, e_k) \neq 0$  and  $(Ae_0, e_0) \cdot (Ae_k, e_k) = |(Ae_k, e_0)|^2$ , then  $A$  is of rank one.*

**PROOF.** Since the compression of  $A$  to  $M$  is positive and of rank one,

$A$  can be written as

$$\begin{bmatrix} x_0 & x_1 & x_2 & \dots \\ \bar{x}_1 & |\alpha_1|^2 & \bar{\alpha}_1\alpha_2 & \dots \\ \bar{x}_2 & \alpha_1\bar{\alpha}_2 & |\alpha_2|^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

with respect to  $\{e_0, e_1, e_2, \dots\}$ , where  $x_n = (Ae_n, e_0)$  for  $n \geq 0$  and  $\alpha_m\bar{\alpha}_n = (Ae_m, e_n)$  for  $m, n \geq 1$ . So, to complete the proof of this lemma, we shall show that there exists an  $\alpha$  such that  $x_0 = |\alpha|^2$  and  $x_n = \bar{\alpha}\alpha_n$  for  $n \geq 1$ . To this end, let  $B$  be the compression of  $A$  to  $V\{e_0, e_k, e_n\}$ ; then,

$$\det B = -|x_k\alpha_n|^2 + x_k\bar{x}_n\bar{\alpha}_k\alpha_n + x_n\bar{x}_k\alpha_k\bar{\alpha}_n - |x_n\alpha_k|^2 = -|x_k\alpha_n - x_n\alpha_k|^2.$$

Since  $\det B \geq 0$ , this implies  $x_k\alpha_n = x_n\alpha_k$ . Put  $\alpha = x_k/\alpha_k$  (recall  $|\alpha_k|^2 = (Ae_k, e_k) \neq 0$ ). We have  $x_n = \alpha\alpha_n$  for  $n \geq 1$ ; this, along with the condition (II), implies  $x_0 = |\alpha|^2$ . So that proof of this lemma is complete.

Observe that  $x_n = \alpha\alpha_n$  holds even without the condition (II).

Now, we are to finish the rest of the proof of theorem 2 by induction on the degree of  $T_\phi$ .

Suppose  $T_\phi$  is of degree two. It is clear that (1), plus (2), implies  $M_\phi$  is of rank one.

Assuming, when  $T_\phi$  is of degree  $n = k$ , (1) is a sufficient condition for  $M_\phi$  being of rank one, we want to show, when  $T_\phi$  is of degree  $n = k + 1$ , (1) still is true.

Since the compression of  $M_\phi$  to  $V\{z, \dots, z^k\}$  is precisely  $M_\phi$  where  $\psi(z) = \sum_{j=1}^k a_{j+1}z^j + \sum_{j=-1}^{-k} a_{j-1}z^j$ , we have that, by the induction hypothesis,  $M_\phi$  satisfies the condition (I) of the lemma—that is, the rank of  $M_\phi$  is one. Hence, to complete the proof, we must show that  $M_\phi$  also satisfies the condition (II)—that is,

$$(3) \quad (|a_{k+1}|^2 - |a_{-k-1}|^2) \sum_{j=1}^{k+1} (|a_j|^2 - |a_{-j}|^2) = |\bar{a}_{k+1}a_1 - a_{-k-1}\bar{a}_{-1}|^2.$$

Since the left hand side of this equation is equal to

$$\begin{aligned} & (|a_{k+1}|^2 - |a_{-k-1}|^2) \sum_{j=2}^{k+1} (|a_j|^2 - |a_{-j}|^2) \\ & + (|a_{k+1}|^2 - |a_{-k-1}|^2) (|a_1|^2 - |a_{-1}|^2), \end{aligned}$$

and since the first term of the above expression is equal to  $|\bar{a}_{k+1}a_2 - a_{-k-1}\bar{a}_{-2}|^2$  (due to the induction hypothesis that the rank of  $M_\phi$  is one, whence the determinant of the compression of  $M_\phi$  to  $V\{z, z^k\}$  is zero), (3) is equivalent to

$$(4) \quad (|a_{k+1}|^2 - |a_{-k-1}|^2) (|a_1|^2 - |a_{-1}|^2) = |\bar{a}_{k+1}a_1 - a_{-k-1}\bar{a}_{-1}|^2 - |\bar{a}_{k+1}a_2 - a_{-k-1}\bar{a}_{-2}|^2.$$

To establish this equation, let  $c = (\bar{a}_{k+1}a_1 - a_{-k-1}\bar{a}_{-1})/(|a_{k+1}|^2 - |a_{-k-1}|^2)$ . Then,

$$(5) \quad \begin{aligned} \bar{a}_{k+1}a_1 - a_{-k-1}\bar{a}_{-1} &= c(|a_{k+1}|^2 - |a_{-k-1}|^2), \\ \bar{a}_ka_1 - a_{-k}\bar{a}_{-1} + \bar{a}_{k+1}a_2 - a_{-k-1}\bar{a}_{-2} &= c(a_{k+1}\bar{a}_k - \bar{a}_{-k-1}a_{-k}). \end{aligned}$$

The second equation of (5) holds because of the observation following the lemma, which states that the ratio of the  $(1, k)$ -entry of  $M_\phi$  (the quantity on the left hand side of this equation) and the  $(k + 1, k)$ -entry of  $M_\phi$  (the second factor on the other side of this same equation) is equal to  $c$ . We want to solve (5) for  $a_1$  and  $a_{-1}$ . Setting  $d = (\bar{a}_{k+1}a_2 - a_{-k-1}\bar{a}_{-2})/(\bar{a}_{k+1}a_{-k} - a_{-k-1}\bar{a}_k)$ , we obtain  $a_1 = a_{k+1}c - a_{-k-1}d$ ,  $\bar{a}_{-1} = \bar{a}_{-k-1}c - \bar{a}_{k+1}d$ . Therefore,

$$(6) \quad \begin{aligned} |a_1|^2 - |a_{-1}|^2 &= |a_{k+1}c|^2 - a_{k+1}c\bar{a}_{-k-1}\bar{d} - \bar{a}_{k+1}\bar{c}a_{-k-1}d \\ &\quad + |a_{-k-1}d|^2 - |a_{-k-1}c|^2 + \bar{a}_{-k-1}ca_{k+1}\bar{d} \\ &\quad + a_{-k-1}\bar{c}\bar{a}_{k+1}d - |a_{k+1}d|^2 \\ &= (|a_{k+1}|^2 - |a_{-k-1}|^2) (|c|^2 - |d|^2) \\ &= (|a_{k+1}|^2 - |a_{-k-1}|^2) \\ &\quad \cdot [|\bar{a}_{k+1}a_1 - a_{-k-1}\bar{a}_{-1}|/(|a_{k+1}|^2 - |a_{-k-1}|^2)^2 \\ &\quad - |\bar{a}_{k+1}a_2 - a_{-k-1}\bar{a}_{-2}|^2/(|a_{k+1}|^2 - |a_{-k-1}|^2)^2]. \end{aligned}$$

The last equality is obtained simply by substitutions and by (1), which says  $|\bar{a}_{k+1}a_{-k} - a_{-k-1}\bar{a}_k| = |a_{k+1}|^2 - |a_{-k-1}|^2$ . It is clear that equation (6) implies equation (4). This completes the proof.

REMARKS. 1. The argument of  $\phi$  in theorem 1 can be replaced by an inner function, utilizing the fact that Toeplitz operators with inner functions as symbols are isometries, which are direct sums of unilateral shifts and unitary operators.

2. The argument of  $\phi$  in theorem 2 can be replaced by the linear fractional transformations  $\phi_a(z) = (z - a)/(1 - \bar{a}z)$ ,  $|a| < 1$ , since they are unitarily equivalent to the unilateral shift.

3. By deleting the word ‘‘hyponormal’’ from theorem 2, equation (1) still stands as a necessary condition, but no longer as a sufficient one. Indeed, let  $\phi(z) = z^3 + 2z^2 + \bar{z}^2$ ; it is obvious that  $|a_3|^2 - |a_{-3}|^2 = 1 = |a_3a_{-2}| - |a_{-3}a_2|$ , but

$$M_\phi = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4 & 2 \\ 0 & 2 & 1 \end{bmatrix},$$

which is not of rank one.

4. The results in this note seem to indicate that the characterization of hyponormal trigonometric Toeplitz operators of degree three should be very messy and that, if such characterization exists, then it should provide a clue to a related problem—characterization of hyponormal trigonometric Toeplitz operators whose self-commutators are of rank two.

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