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A CLASS OF NEGATIVE-AMPHICHEIRAL KNOTS AND THEIR ALEXANDER POLYNOMIALS

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1. Introduction. A knot K is *amphicheiral* provided there is an orientation-reversing autohomeomorphism ρ of space ($S^3 = \mathbf{R}^3 \cup \{\infty\}$) mapping K to itself; if ρ is a piecewise-linear involution, then K is *strongly amphicheiral.* According to R. H. Crowell and R. H. Fox [7, p. 9], "It is not hard to show that the figure-eight knot is amphicheiral. The experimental approach is best...". An accompanying figure then convinces the reader, in six views, that the standard projection of this knot can be deformed to its mirror image.

The motivation for the present study is the simple observation that the strong amphicheirality of the figure-eight knot is obvious from the single view of this knot appearing in Figure 1(a), since a half-rotation (about a normal to the plane of projection at the origin $\circ$) takes the knot to its mirror image.

Figure 1

Unfortunately, while the invertibility of this knot is obvious from its standard projection (as given in knot tables, for example [22, p. 391], where it is denoted 4_1), about a half-dozen views would now be needed to visulize a deformation which takes the given projection of 4_1 to itself while reversing the orientation of (that is, the direction of an arrow on) the knot.

The strong amphicheirality of the other knots in Figure 1 likewise follows from their invariance under the extension α of the reflection of $\mathbf{R}^3$ in the origin to S^3; that is, under the composition of the half-rotation mentioned above with the reflection in the plane of projection.

Figure 1(b) and (c) are actually 12-crossing projections of the same 10-crossing knot 10_{31} (see Figure 2). In one projection the knot passes through the origin (as well as the point ∞) so that α reverses orientation on the knot, thereby showing 10_{31} to be strongly negative-amphicheiral; while in the other projection α preserves orientation, showing that 10_{31} is a strongly positive-amphicheiral knot as well.

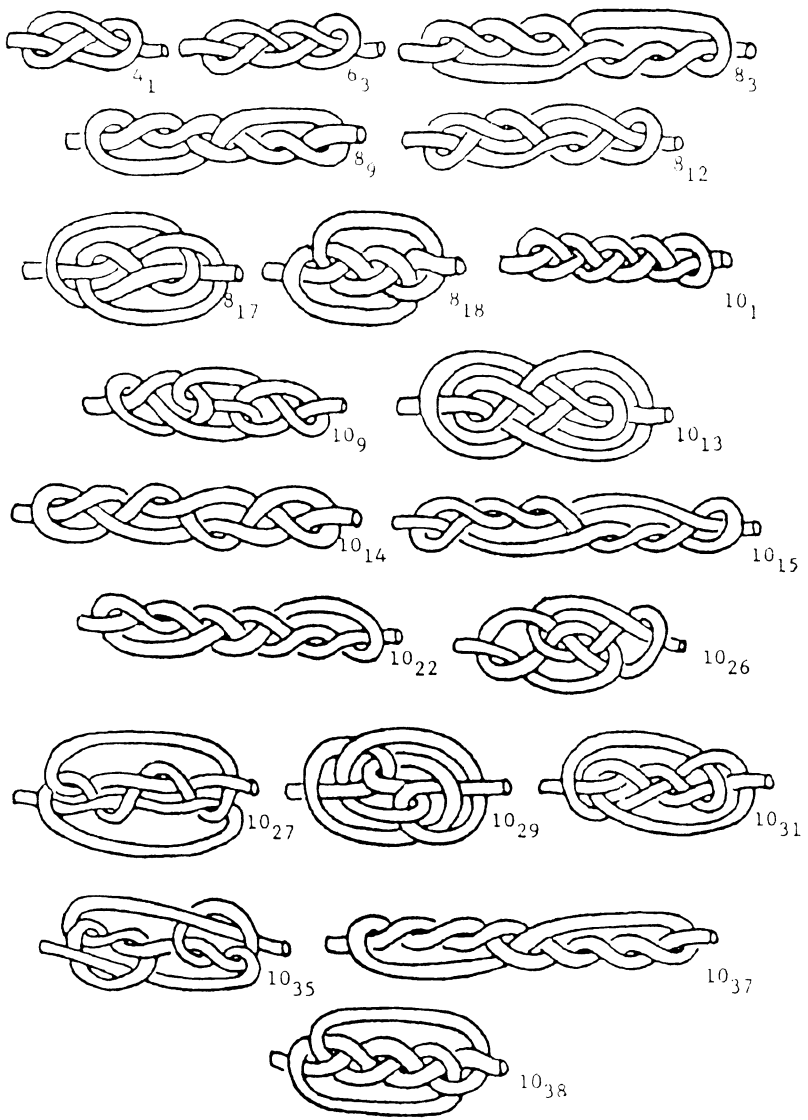

Figure 2

Knots, such as 10_{31}, which are both positive- and negative-amphicheiral are clearly invertible. H. Trotter exhibited a 15-crossing knot K, which is neither invertible nor amphicheirai, which he used to construct composite knots which are either negative-amphicheiral or positive-amphicheiral, but not invertible [**29**]. These knots can be described as the completions, by th map α, of the positive x_1-axis on the one hand, and of a semi-circle "centered" at the origin on the other, into which has first been tied a knot of type K. W. Whitten has exhibited both positive-amphicheiral and negative-amphicheiral prime knots which are non-invertible [**32**], while F. Bonahon and L. Siebenmann have shown many of the classical prime negative-amphicheiral knots, including 8_{17} (see [**8**, Prob. 10]), to be non-invertible [**4**].

Being strongly amphicheiral is not as special for amphicheiral knot types as might first appear. The fact (as indicated in Figure 2) that each of the 20 amphicheiral knot types on 10 or fewer crossings (see [**20**]) is strongly negative-amphicheiral leads to the following

CONJECTURE 1. [**16**, Prob. 1.5]. A knot is amphicherial if and only if it is strongly amphicheiral. (A candidate for a counterexample to this conjecture is Whitten's negative-amphicheiral knot $\mathscr{K}_{(+)}$ [**32**].)

REMARK. Subsequent to the writing of this paper, both R. Hartley [**11**] and W. Whitten [**33**] have determined infinite collections of both positive- and negative-amphicheiral counterexamples to Conjecture 1.

Loosely related is the following conjecture.

CONJECTURE 2. A minimal crossing projection of an amphicheiral knot has an even number of crossings.

Attention will now be restricted to strongly negative-amphicheiral knots K which, like the knots 4_1 and 10_{31} considered above, have centrally symmetric embeddings of the form indicated in Figure 3(a), where it can be assumed (possibly following a half-rotation of K about the horizontal axis) that the left ray emerges at the top of the projection so that a knot $\mathscr{K}$ and a 2-component link $\mathscr{L}$ are formed as indicated in Figure 3(b) and (c).

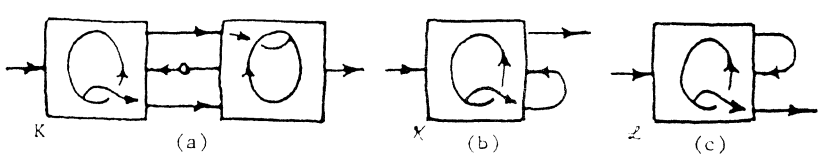

Figure 3

The knot K will be called an *α-completion* of the knot $\mathscr{K}$, and $\mathscr{L}$ the link *associated with this completion.*

This ad hoc terminology should cause no confusion, although it certainly fails to anticipate that other strongly negative-amphicheiral knot constructions can be obtained by varying the (odd) number of strings in the central portion of Figure 3(a) (such as that yielding the knot 8_{18}, as given in Figure 2) or that strongly positive-amphicherial knots can also be constructed in this manner (such as the knot 10_{31}, given in Figure 1(c), which can be completed from half of its projection so as to be invariant under reflection in a point.

The Alexander polynomial of K will be expressed in terms of those of $\mathscr{K}$ and $\mathscr{L}$, each of which has half the number of crossings, much as Y. Hashizume and F. Hosokawa had done [**13**] in determining the Alexander polynomial of the symmetric skew knot unions of S. Kinoshita and H. Terasaka [**15**].

Flipping the right-hand side of K, as illustrated in Figure 3(a), about a horizontal axis (at the expense of introducing three additional crossings in the projection) allows the Alexander polynomial Δ_K of K to be computed from those of $\mathscr{K}$ and $\mathscr{L}$ by J. W. Alexander's original method [**1**], since the geometric symmetry of K is then captured in the matrix having Δ_K as its determinant (see [**31**] for details). However, J. H. Conway's tangle theoretic methods [**6**] will provide a less onerous proof of this relation. Here, the *Alexander polynomial* $\Delta_{\mathscr{L}}$ of the 2-component link $\mathscr{L}$ will be that originally defined by Alexander [**1**, p. 301]. In terms of the 2-variable polynomial $\Delta_{\mathscr{L}}(x_1, x_2)$ which R. H. Fox defined for such links. [**8**, p. 131], $\Delta_{\mathscr{L}}(x)$ would be $(x-1)\Delta_{\mathscr{L}}(x, x)$.

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2. The main result.

THEOREM. *If a negative-amphicheiral knot K is the α-completion of a knot $\mathscr{K}$, then its Alexander polynomial is given by*

$$\Delta_K(x) = \Delta_{\mathscr{K}}^2(x) - x^{-1}\Delta_{\mathscr{L}}^2(x),$$

where $\Delta_{\mathscr{K}}(x)$ and $\Delta_{\mathscr{L}}(x)$ are the Alexander polynomials of $\mathscr{K}$ and the associated link $\mathscr{L}$.

PROOF. Those familiar with the tangle theoretic approach of Conway [**6**] will recognize the α-completion K of a knot $\mathscr{K}$ to be the numerator of

the sum of a tangle $\mathscr{Q}$, which has $\mathscr{K}$ as its numerator, and the mirror image of the tangle obtained by rotating $\mathscr{Q}$ by $-\pi/2$ radians in the plane of projection, as indicated in Figure 4(a) and (b). Figure 4(c) shows the knot K^* of Figure 6 from this point of view.

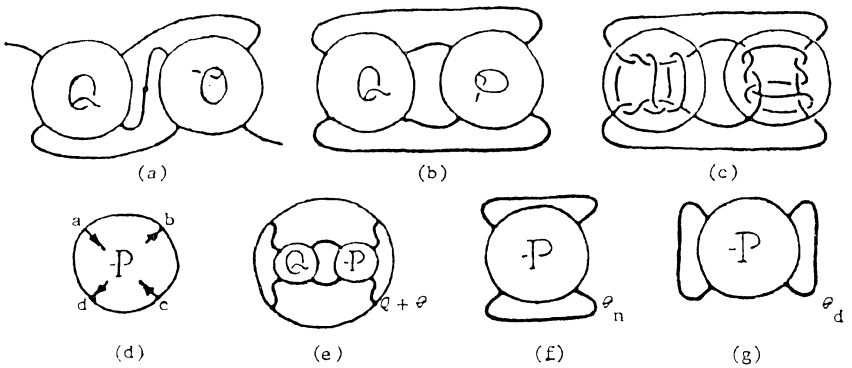

Figure 4

A tangle $\mathscr{P}$ is a standard three-ball spanned by a pair of oriented smooth arcs meeting the boundary (transversely) in four standard points, indicated in Figure 4(d), so that the strings run from a to d and from c to b (as in the left tangle of Figure 4(c)) or from a to b and c to d (as in the right tangle).

The sum of tangles $\mathscr{Q}$ and $\mathscr{P}$ is indicated in Figure 4(e); while the completions, in S^3, of the tangle $\mathscr{P}$ by standard arcs, indicated in Figure 4(f) and (g), yield its numerator $\mathscr{P}_n$ and its denominator $\mathscr{P}_d$, one of which will be a knot and the other a 2-component link.

Conway defines, for any tangle $\mathscr{P}$, a formal fraction $D_{\mathscr{P}_n}(x)/D_{\mathscr{P}_d}(x)$, termed the polynomial fraction of $\mathscr{P}$, and then gives, as the polynomial fraction of $\mathscr{Q} + \mathscr{P}$, the formal

$$[D_{\mathscr{Q}_n}(x)D_{\mathscr{P}_d}(x) + D_{\mathscr{Q}_d}(x)D_{\mathscr{P}_n}(x)]/D_{\mathscr{Q}_d}(x)D_{\mathscr{P}_d}(x)$$

of those of $\mathscr{Q}$ and $\mathscr{P}$ (see [**6**, pp. 340 and 339] and [**9**, Proposition 14]). Also, if $\mathscr{K}'$ is the mirror image of a knot $\mathscr{K}$, then $D_{\mathscr{K}'}(x) = D_{\mathscr{K}}(x)$; while if $\mathscr{L}'$ is the mirror image of the 2-component link $\mathscr{L}$, then $D_{\mathscr{L}'}(x) = -D_{\mathscr{L}}(x)$ [**6**, pp. 337 and 340].

The polynomial fraction of the tangle obtained on deleting arcs from the top and bottom strings of the α-completion K of $\mathscr{K}$ in Figure 4(b) is thus

$$D_{\mathscr{K}}(x)/D_{\mathscr{L}}(x) + (-D_{\mathscr{L}}(x))/D_{\mathscr{K}}(x) = (D^2_{\mathscr{K}}(x) - D^2_{\mathscr{L}}(x))/D_{\mathscr{L}}(x)D_{\mathscr{K}}(x),$$

since $\mathscr{Q}_n = \mathscr{K}$ and $\mathscr{Q}_d = \mathscr{L}$, the associated link of this completion. Thus

$D_K(x) = D_{\mathscr{K}}^2(x) - D_{\mathscr{L}}^2(x)$. Since for any knot K, $D_K(x^{1/2})$ is the Alexander polynomial $\Delta_K(x)$ of K, while for any 2-component link L, $x^{1/2}D_L(x^{1/2})$ is $\Delta_L(x)$, the polynomial of Alexander referred to at the end of §1 [**6**, p. 338], it follows that $\Delta_K(x) = \Delta_{\mathscr{K}}^2(x) - x^{-1}\Delta_{\mathscr{L}}^2(x)$, which completes the proof.

The computation of $\Delta_K(x)$ is facilitated by the observation that $\Delta_{\mathscr{K}}(x)$ and $\Delta_{\mathscr{L}}(x) = (x - 1)\Delta_{\mathscr{L}}(x, x)$ (which, as often defined, are knot and link invariants only to within a factor of $\pm x^{\varepsilon}$) need not be determined by the methods of Conway or Alexander (compare with [**10**]), but can be obtained by any convenient method. To see this, note that each of $\Delta_K(x)$, $\Delta_{\mathscr{K}}(x)$ and $\Delta_{\mathscr{L}}(x, x) = (x - 1)^{-1}\Delta_{\mathscr{L}}(x)$ is a symmetric (finite Laurent) polynomial of even (reduced) degree, by [**27**, Theorem] and [**28**, Corollary 3 (note misprint in statement)], so that $\Delta_K(x)$ (normalized so $\Delta_K(1) = 1$) can be obtained as the difference of associates in $\mathbf{Z}[x, x^{-1}]$ of the symmetric polynomials $\Delta_{\mathscr{K}}^2(x)$ and $x^{-1}\Delta_{\mathscr{L}}^2(x) = (x - 2 + x^{-1})\Delta_{\mathscr{L}}^2(x, x)$ in but one way. An excellent resource for such computations is provided by the projections and Alexander polynomials (after Fox) of prime knots to ten crossings and prime 2-component links to nine crossings which J. Bailey has compiled in Appendix C of [**22**] from Conway's enumerations [**6**]; but note that the ordering of ten crossing knots in these tables is not an extension of that given by P. G. Tait for ten crossing alternating knots, as listed in [**6**] and used here.

A conjecture of A. Kawauchi (namely, that for negative-amphicheiral knots $\Delta(x^2)$ is of the form $f(x)f(-x)$, where $f(-x^{-1}) = f(x)$ [**11**]) is verified for the special class of negative-amphicheiral knots under consideration on noting that

$$\Delta_K(x^2) = [\Delta_{\mathscr{K}}(x^2) + x^{-1}\Delta_{\mathscr{L}}(x^2)][\Delta_{\mathscr{K}}(x^2) - x^{-1}\Delta_{\mathscr{L}}(x^2)],$$

that $\Delta_{\mathscr{K}}(x)$ is symmetric in x and, by the symmetry of $\Delta_{\mathscr{L}}(x, x) = (x - 1)^{-1}\Delta_{\mathscr{L}}(x)$ in x, that

$$\begin{aligned}(-x^{-1})^{-1}\Delta_{\mathscr{L}}(x^{-2}) &= -x(x^{-2} - 1)\Delta_{\mathscr{L}}(x^{-2}, x^{-2}) \\ &= x^{-1}(x^2 - 1)\Delta_{\mathscr{L}}(x^2, x^2) = x^{-1}\Delta_{\mathscr{L}}(x^2).\end{aligned}$$

Appearing in a sequel to this paper [**30**] is a verification of this conjecture for strongly negative-amphicheiral knots, as well as of its converse; thereby answering a question raised by R. Hartley and Kawauchi in [**12**, Remark 3], where they settle the conjecture for the same class of knots and further show that the Alexander polynomial of a strongly positive-amphicheiral knot is the square of some Alexander polynomial.

3. Applications. There are knots on as few as eight crossings, such as the (3, 4) torus knot 8_{19} (see [**17**]), having no projection on which the over-

crossings and undercrossings alternate about the knot. An answer to the obvious question raised, on observing that each of the amphicheiral knots on ten or fewer crossings is an alternating knot (see Figure 2), is provided by the following example.

EXAMPLE 1. A prime non-alternating amphicheiral knot. K. Murasugi has shown [**17**, p. 188] that alternating knots have alternating Alexander polynomials in which no term is skipped. A search of the tables in [**22**, Appendix C] reveals a 2-component, 7-crossing link $\mathscr{L}$, shown in Figure 5, which has an Alexander polynomial $\Delta_{\mathscr{L}}(x_1, x_2) = x_1^3 x_2 + 1$ (after Fox) in which the difference of the degrees of its terms is so large that $\Delta_{\mathscr{L}}^2(x) = (x-1)^2 \Delta_{\mathscr{L}}^2(x, x)$ is a polynomial having fewer terms than its (reduced) degree. But then, since $\mathscr{L}$ has a projection for which the "associated knot" $\mathscr{K}$ is the unknot, the α-completion of $\mathscr{K}$ has an Alexander polynomial,

$$\Delta_K(x) = x^{10} - 2x^9 + x^8 + 2x^6 - 5x^5 + 2x^4 + x^2 - 2x + 1$$

which shows it to be non-alternating.

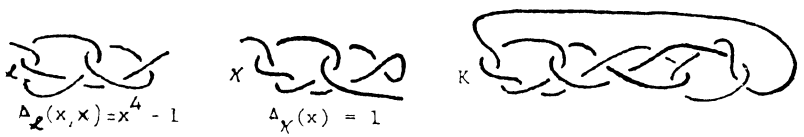

Figure 5

In this example, as well as in the following, it is crucial to demonstrate that the knot constructed is prime, since composite knots having these properties are easily constructed. The primality of these examples will be established following the presentation of the next example.

EXAMPLE 2. A prime amphicheiral knot having trivial Alexander polynomial. There is a projection of the Kinoshita-Terasaka 11-crossing knot $\mathscr{K}$ with trivial Alexander polynomial [**15**, p. 151] for which the associated 2-component link $\mathscr{L}$ splits into unknots, as indicated in Figure 6.

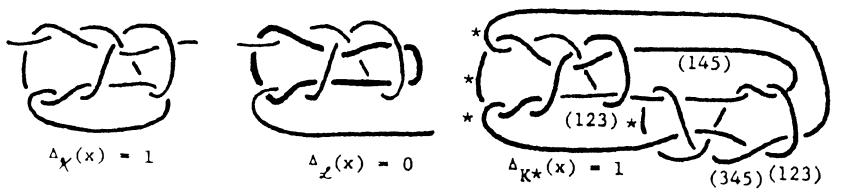

Figure 6

Since $\Delta_{\mathscr{L}}(x_1, x_2) = 0$; (by [**8**, Example 7] or [**22**, p. 416]), $\Delta_{\mathscr{L}}(x) = 0$ and it follows that the α-completion K^* of $\mathscr{K}$ has trivial Alexander polynomial.

The non-triviality of K^* is established by extending a representation which R. Riley gives of $\pi_1(S^3 - \mathscr{K})$ into the alternating group A_5 [**21**, p. 615] to $\pi_1(S^3 - K^*)$, as indicated in the given projection of K^*, where the images of a set of Wirtinger generators appear next to the corresponding overpasses.

Now if K^* were not prime, each of its factors would inherit its trivial Alexander polynomial [**8**, p. 144] and consequently have at least three bridges. To see this, note that each factor of K^* is non-alternating, since the genus of an alternating knot is half the degree of its Alexander polynomial [**18**, p. 294] and no non-alternating knot has as few as two bridges, by [**5**] or [**10**].

But then, since the bridge number of the composition of two knots is one less than the sum of the bridge numbers of the factors [**23**, Satz 7], it follows that the bridge number of K^* would be at least five. However, there is a direction in which the given projection of K^* has only four relative maxima (marked *), so the bridge number of this knot is at most four (see [**3**, p. 199]).

That the knot K of Example 1 is prime will follow, as with K^*, on showing that the assumption to the contrary implies that its bridge number is at least five; whereas Figure 5 shows it to have a 4-bridge presentation.

If K were composite, say $K = K' \sharp K''$ for non-trivial knots K' and K'',

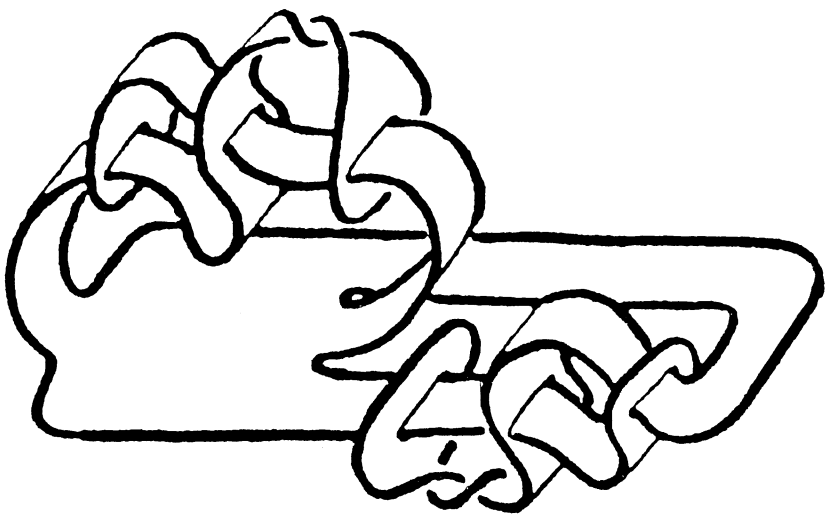

Figure 7

then, since $\Delta_K(x) = \Delta_{K'}(x) \cdot \Delta_{K''}(x)$ and the degree of any Alexander polynomial is even, one of the factors would have Alexander polynomial one while the other would have the same (non-alternating) Alexander polynomial as K. But then it would follow (as with the factors of K^* in Example 2) that each of K' and K'' is a non-alternating knot with bridge number at least three and hence that the bridge number of K would be at least five.

The 2-fold covering of S^3 branched over the knot K^* of Example 2 provided R. J. Stern with an example of an irreducible homology 3-sphere ($\neq S^3$) with an orientation reversing diffeomorphism (which happens to have period four) [**26**]. L. C. Siebenmann and the author show that this diffeomorphism will be an involution if K^* is replaced by any of an infinite class of positive-amphicheiral knots (including one with Alexander polynomial one) constructed for this purpose [**25**].

Now no knot with $\Delta(x) = 1$ is known to be non-slice [**16**, Prob. 1.36] and no slice knot is known to be non-ribbon ([**8**, Prob. 24] or [**16**, Prob. 1.33]). That K^*, indeed any α-completion of a ribbon knot, is a ribbon knot (hence a slice knot) is indicated in Figure 7.

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