

## NON-UNIQUENESS FOR NONLINEAR BOUNDARY-VALUE PROBLEMS

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In this paper, we consider the equation

$$(1) \quad A(u) = f$$

where  $A$  is a  $C^2$ -Fredholm mapping of index zero between the Banach spaces  $X$  and  $Y$ . We prove that, if there is a continuous map  $u \rightarrow T(u)$  of  $X$  into  $B(X, Y)$  such that, as  $u$  varies, a real eigenvalue of the linear eigenvalue problem

$$(2) \quad A'(u)h = \lambda T(u)h$$

crosses zero, then there is an  $f_0$  in  $Y$  such that (1) has more than one solution for  $f = f_0$ . (We will formulate our theorem more precisely in §1 and explain our notation.) In fact, our proof shows that there is a  $u_0$  in  $X$  such that  $A$  is not one-to-one when restricted to any neighbourhood of  $u_0$ . Our theorem seems to be the first result of its type in the literature. Some special cases were mentioned (without proof) by the author in [7], [8] and [9]. Note that many uniqueness results have been discussed in the literature. These essentially assume that  $A'(u)$  is invertible for  $u \in X$  and some extra assumptions. We discuss how our results, when combined with simple uniqueness theorems, give an almost complete classification of when non-uniqueness occurs for certain weakly nonlinear equations. We also discuss very briefly several other applications. One of these answers a question of Berestycki and Lions [4].

Our proof of non-uniqueness depends essentially upon the existence of an eigenvalue which is real on a set  $U$  and changes sign on  $U$ . Thus we need conditions which ensure that certain eigenvalues stay real. Complex eigenvalues seem to cause extra difficulties which we have not overcome. We discuss this in more detail at the end of §1. Note that our non-uniqueness appears to be new (and non-trivial) for the finite-dimensional case.

In §1, we prove our main result and, in §2, we discuss some applications.

I should like to thank the referee for suggesting that the statement of the main result could be simplified.

**1. The main result.** We prove a general abstract non-uniqueness theorem. Assume that  $X$  and  $Y$  are Banach spaces and that  $A: \mathcal{D} \rightarrow Y$  is a  $C^2$ -mapping of a connected open subset  $\mathcal{D}$  of  $X$  into  $Y$  such that  $A'(x)$  is Fredholm of index zero for  $x \in X$ . Here  $A'(x)$  denotes the Fréchet derivative of  $A$  at  $x$ . Let  $S = \{x \in \mathcal{D}: N(A'(x)) \neq \{0\}\}$  and  $S_i = \{x \in S: \dim N(A'(x)) \leq i\}$ . As usual  $N(B)$  and  $R(B)$  denote the kernel and range of  $B$  respectively. We need the following well-known lemma.

LEMMA 1. (cp. Abraham and Robbins [1], Theorem 1.7). *If  $x_0 \in \mathcal{D}$ , there exist local  $C^2$ -diffeomorphisms  $\phi$  and  $\psi$  defined in neighbourhoods of 0 and  $A(x_0)$  respectively (and both mapping into  $X$ ) such that*

- (i)  $\phi(0) = x_0$ ,  $\phi'(0) = I$  and  $\psi(A(x_0)) = 0$ ,
- (ii)  $\psi'(Ax_0)$  maps  $R(A'(x_0))$  onto a closed complement  $M$  to  $N(A'(x_0))$ , and
- (iii) for  $x$  near 0,  $(\psi A \phi)(x) = u + g(u, v)$ . Here  $x = u + v$  with  $v \in N(A'(x_0))$  and  $u \in M$  and  $g$  is a  $C^2$ -map of a neighbourhood of zero in  $M \oplus N(A'(x_0)) (= X)$  into  $N(A'(x_0))$  such that  $g(0, 0) = 0$  and  $g'(0, 0) = 0$ .

The above result is a slight variant of the result in [1] but is readily seen to be true by examining the proof in [1]. It follows easily from this representation and the invariance of domain theorem (in finite dimensions) that, if  $A$  is one-to-one in a neighbourhood of  $x_0$ , then  $A$  is a homeomorphism when restricted to some neighbourhood of  $x_0$  (i.e., a local homeomorphism). This result also follows from the Smale degree (cp., Tromba [17, Theorem 4]).

We also need the following well known lemma.

LEMMA 2. (i)  $S_1$  is an open subset of  $S$ .

- (ii) Suppose  $x \in S$  and  $\varepsilon > 0$ . There is a  $\delta > 0$  such that, if  $\|y - x\| \leq \delta$ ,  $z \in N(A'(y))$  and  $\|z\| = 1$ , then  $\|z - z_0\| \leq \varepsilon$  where  $z_0 \in N(A'(x))$  and  $\|z_0\| = 1$ .

The first part follows easily from Goldberg [12, Theorem V.1.2]. The second part is an upper semicontinuity result on the kernels of linear operators. It is not stated in [11] but follows easily from the proof of Theorem V.1.2(i) there.

The next lemma is the key to the proof of our main result. We say that  $S$  locally disconnects at  $x_0$  if, for every sufficiently small neighbourhood  $W$  of  $x_0$ ,  $W \cap S$  is not connected.

LEMMA 3. Assume that  $x_0 \in \bar{S} \subseteq S$ , that  $A$  is a local homeomorphism near  $x_0$ , that  $\bar{S}$  locally disconnects at  $x_0$ , that  $V$  is a neighbourhood of  $x_0$  and that  $\bar{S}$  is a closed subset of  $S$ . Then there exists  $x_1 \in S_1 \cap V \cap \bar{S}$  such that  $\bar{S}$  locally disconnects at  $x_1$ .

PROOF. Without loss of generality, we may assume that  $x_0 = 0$ ,  $A(x_0) =$

0,  $A|_V$  is a homeomorphism,  $V$  is connected and  $V \setminus \tilde{S}$  is not connected. By Lemma 1 and by (nonlinear)  $C^2$ -changes of coordinates in  $X$  and  $Y$ , we see that we may assume that  $A(x) = u + g(u, v)$  for  $x$  near 0. Here  $u, v$  and  $g$  are as in Lemma 1. A simple calculation shows that, if  $x = u + v$ , then  $N(A'(x)) = N(g'_2(u, v))$  (where  $g'_2$  denotes the partial derivative with respect to the second variable). Thus  $\dim N(A'(x)) = \dim N(g'_2(u, v))$ .

The remainder of the proof is by a series of steps.

*Step 1.*  $V \setminus (\tilde{S} \setminus S_1)$  is connected in the case where  $X$  is finite-dimensional. We first apply Proposition 4 in Church [5] with  $k = n - 2$  (where  $n = \dim X$ ). This implies that  $\dim A(\tilde{S} \setminus S_1) \leq n - 2$ . Thus  $\dim A(\tilde{S} \setminus S_1) \leq n - 2$ . By Engelking [11, Theorem 1.8.13], it follows that  $A(V) \setminus A(\tilde{S} \setminus S_1)$  is connected. (Note that in order to use Proposition 4 in [5] we need to assume  $A$  is  $C^2$  rather than  $C^1$ .) Since  $A|_V$  is a homeomorphism, it follows that  $V \setminus (\tilde{S} \setminus S_1) = A^{-1}(A(V) \setminus A(\tilde{S} \setminus S_1))$  is connected.

*Step 2.* We prove that  $V \setminus (\tilde{S} \setminus S_1)$  is connected in the general case if  $V$  is sufficiently small. First note that, by shrinking  $V$ , we may assume  $V = V_1 \times V_2$ , where  $V_1$  and  $V_2$  are connected and open in  $M$  and  $N(A'(0))$  respectively (with the notation of Lemma 1). By our earlier comments, we may assume that  $A(x) = u + g(u, v)$ . Let

$$A_u = \{v \in V_2 : \dim N(g'_2(u, v)) \geq 2\} = \{v \in V_2 : \dim N(A'(u + v)) \geq 2\}$$

and  $\tilde{A}_u = \{v : u + v \in \tilde{S}\} \cap A_u$ . By Step 1,  $V_2 \setminus \tilde{A}_u$  is connected. (We apply Step 1 to the map  $v \rightarrow g(u, v)$  of  $N(A'(0))$  into itself.) Now  $V \setminus (\tilde{S} \setminus S_1) = \bigcup_{u \in V_1} \{u\} \times (V_2 \setminus \tilde{A}_u)$ . If  $V \setminus (\tilde{S} \setminus S_1) = T_1 \cup T_2$  were a disconnection of  $V \setminus (\tilde{S} \setminus S_1)$ , then, since any connected subset of  $V \setminus (\tilde{S} \setminus S_1)$  must be wholly in  $T_1$  or  $T_2$ , it follows that  $T_1 = \bigcup_{u \in C} \{u\} \times (V_2 \setminus \tilde{A}_u)$ , where  $C \subseteq V_1$ . A similar formula holds for  $T_2$  (except that the union is over  $V_1 \setminus C$ ). By Lemma 2(i),  $V \setminus (\tilde{S} \setminus S_1)$  is open. It follows that  $T_1$  and  $T_2$  are open and thus  $C$  and  $V_1 \setminus C$  are open. Since this contradicts the connectedness of  $V_1$ ,  $V \setminus (\tilde{S} \setminus S_1)$  is connected as required.

*Step 3.* Since  $V \setminus \tilde{S}$  is not connected,  $V \setminus \tilde{S} = T_1 \cup T_2$  where  $T_1$  and  $T_2$  give a disconnection. Since  $\tilde{S}$  is closed,  $T_1$  and  $T_2$  are open. Assume by way of contradiction that, for all  $x$  in  $S_1 \cap \tilde{S}$ , there are arbitrarily small neighbourhoods  $W_x$  of  $x$  such that  $W_x \setminus \tilde{S}$  is connected. It follows that  $W_x \setminus \tilde{S}$  is wholly in  $T_1$  or  $T_2$ . Since  $S_1$  is open in  $S$ , we may choose  $W_x$  such that  $W_x \cap (\tilde{S} \setminus S_1) = \emptyset$ . For  $i = 1, 2$ , let

$$\tilde{T}_i = T_i \bigcup_{x \in A_i} W_x,$$

where  $A_i = \{x \in S_1 \cap \tilde{S} : W_x \setminus \tilde{S} \subseteq T_i\}$ . The  $\tilde{T}_i$  are open and disjoint and  $V \setminus (\tilde{S} \setminus S_1) = \tilde{T}_1 \cup \tilde{T}_2$ . Since this contradicts the connectedness of  $V \setminus (\tilde{S} \setminus S_1)$ , (proved in Step 2), it follows that there exists  $x \in S_1 \cap \tilde{S}$  such that  $\tilde{S}$  locally disconnects at  $x$ . This completes the proof.

We have proved a slightly more general form of Lemma 3 than we really need here because we feel it may be useful elsewhere. Lemma 3 usually enables us, in trying to prove that  $A$  is not a local homeomorphism, to reduce to the case where  $A'(x)$  has a one-dimensional kernel. Let  $B(X, Y)$  denote the space of continuous linear maps from  $X$  to  $Y$ .

LEMMA 4. Assume that  $\tilde{u} \in S_1$  and that there is a continuous mapping  $x \rightarrow T(x)$  of a neighbourhood of  $u$  into  $B(X, Y)$  such that

- (i)  $f(T(\tilde{u})h) \neq 0$  where  $h$  spans  $N(A'(\tilde{u}))$  and  $f$  spans  $N(A'(\tilde{u})^*)$ ,
- (ii) for all  $x$  near  $\tilde{u}$ , the equation  $A'(x)y = \lambda T(x)y$  has a unique real eigenvalue  $\lambda(x)$  near 0, and
- (iii)  $\lambda(x)$  takes both positive and negative values in every neighbourhood of  $\tilde{u}$ .

Then  $A$  is not a local homeomorphism in any neighbourhood of  $\tilde{u}$ .

REMARK. A minor variant of Lemma 1.3 in Crandall and Rabinowitz [6] shows that assumption (ii) is a consequence of assumption (i) and that  $\lambda(x)$  depends continuously on  $x$ .

PROOF. As before, we may assume that  $\tilde{u} = 0$  and  $A(0) = 0$ . With the notation of Lemma 1,  $A = \phi \circ G \circ \phi$ , where  $G(u, v) = u + g(u, v)$ . Note that  $\phi'(x)$  and  $\phi'(x)$  are invertible for  $x$  small. Now  $A'(x) = \phi' \circ G' \circ \phi'(x)$ . (Note that we have omitted the arguments for  $\phi'$  and  $G'$  to improve readability. We will not need them. The only point to note is that, since  $\phi(0) = 0$ ,  $\phi'(0) = 0$ , and  $G(0) = 0$ , they will be small.) Thus the equation  $A'(x)y = \lambda T(x)y$  becomes

$$(3) \quad G'(\phi(x))w = \lambda \tilde{T}(x)w,$$

where  $w = \phi'(x)y$  and  $\tilde{T}(x) = (\phi')^{-1}T(x)(\phi'(x))^{-1}$  (for the appropriate argument of  $\phi'$ ). Note that  $\tilde{T}(0) = \phi'(0)^{-1}T(0)$  and that  $w = y$  if  $x = 0$ . By our condition that  $f(T(0)h) \neq 0$ , it follows that  $\tilde{f}(\tilde{T}(0)h) \neq 0$ , where  $\tilde{f}$  spans  $N(G'(0)^*)$ . (Thus  $\tilde{f}$  annihilates  $R(G'(0))$  and, by the definition of  $G$ ,  $\tilde{f}(h) \neq 0$ .) We choose  $\tilde{f}$  such that  $\tilde{f}(h) = 1$ . By Lemma 2(ii), one easily sees that  $\tilde{f}(w) \neq 0$  if  $x$  is small and  $w \neq 0$  (since  $w$  is near  $h$ ). Thus  $w = h + z$  where  $z \in M$ . (Since  $\tilde{f}(w) \neq 0$ , we can multiply  $w$  by a constant to ensure that  $\tilde{f}(w) = 1$ .) Define  $P$  by  $Px = x - \tilde{f}(x)h$ . Since  $G'(\phi(x))(h + z) = z + g'_1(\phi(x))z + g'_2(\phi(x))h$  (by the formulae for  $G$ ), equation (3) is equivalent to the pair of equations

$$(4) \quad z = \lambda P \tilde{T}(x)(h + z)$$

and

$$(5) \quad \tilde{f}(g'_1(\phi(x))z + g'_2(\phi(x))h) = \lambda \tilde{f} \tilde{T}(x)(h + z).$$

Now equation (4) has a unique solution  $z = \lambda R(x, \lambda)h$  in  $M$  for small

$\lambda$ , where  $R(x, \lambda)$  is the inverse of  $(I - \lambda P \tilde{T}(x))|_M$ . Note that  $R$  is continuous and  $R(x, 0) = I$ . Thus (5) becomes

$$(6) \quad \tilde{f}(g'_2(\phi(x))h) = \lambda \tilde{f}(\tilde{T}(x)(h + \lambda R(x, \lambda)h) - g'_1(\phi(x))R(x, \lambda)h).$$

Since  $R$  is continuous,  $g'_1(0, 0) = 0$  and  $\tilde{f}(\tilde{T}(x)h) \neq 0$ , we see that the right-hand side of (6) is  $\lambda W(x, \lambda)$  where  $W$  has the same sign as  $\tilde{f}(\tilde{T}(x)h)$  if  $(x, \lambda)$  is small. Since  $\lambda(x)$  is a solution of (6), and since  $\lambda(x)$  changes sign arbitrarily close to zero, it follows that  $\tilde{f}(g'_2(\phi(x))h)$  changes sign arbitrarily close to zero. Since  $\phi$  is a local homeomorphism near 0, it follows that  $\tilde{f}(g'_2(u, v)h)$  changes sign arbitrarily close to zero. (Here we have reverted to ordered pair notation.)

We now show that this contradicts our assumption that  $A$  (and thus  $G$ ) is a homeomorphism on some small neighbourhood  $V$  of zero. Without loss of generality,  $V = V_1 \times V_2$  where  $V_1$  is open in  $M$ ,  $V_2$  is open in  $\text{span}\{h\}$  and  $V_1$  and  $V_2$  are connected. Since  $G$  is a local homeomorphism, it follows from the formula for  $G$  that, for each  $q$  in  $V_1$ , the map  $v \rightarrow g(q, v)$  is a homeomorphism of  $V_2$  onto its image. Let  $V_2 = \{\alpha h : \alpha \in (\varepsilon_1, \varepsilon_2)\}$ , where  $\varepsilon_1 < 0 < \varepsilon_2$ . Then, we have that, for each  $q \in V_1$ , the map  $\alpha \rightarrow \tilde{f}(g(q, \alpha h))$  is one-to-one on  $(\varepsilon_1, \varepsilon_2)$ . Thus it is strictly increasing on  $(\varepsilon_1, \varepsilon_2)$  or strictly decreasing on  $(\varepsilon_1, \varepsilon_2)$ . Since  $V_1$  is connected, a simple connectedness argument shows that the map must be strictly increasing on  $(\varepsilon_1, \varepsilon_2)$  for all  $q \in V_1$  or strictly decreasing on  $(\varepsilon_1, \varepsilon_2)$  for all  $q \in V_1$ . We consider the former case. The latter is similar. Since the map  $\alpha \rightarrow \tilde{f}(g(q, \alpha h))$  is increasing on  $(\varepsilon_1, \varepsilon_2)$  for all  $q \in V_1$ , it follows by differentiating with respect to  $\alpha$  that  $\tilde{f}g'_2(q, \alpha h)h \geq 0$  if  $q \in V_1$  and  $\alpha \in (\varepsilon_1, \varepsilon_2)$ . Since this contradicts the result of the previous paragraph, we have completed the proof.

The relationship between the sign of  $g'_2(q, v)$  and  $\lambda((q, v))$  was suggested by the work in [6]. Our proof differs from the proof of the related result (Theorem 1.16 in [6]) by the use of Lemma 1. Results similar to Lemma 4 can probably also be established by degree theory. Finally, the condition that  $f(T(\tilde{u})h) \neq 0$  can probably be weakened.

We now obtain our main result.

**THEOREM 1.** *Assume that there exists a continuous mapping  $x \rightarrow T(x)$  of  $\mathcal{D}$  into  $\mathcal{B}(X, Y)$  such that*

- (i) *the equation  $A'(x)y = \lambda T(x)y$  has a real eigenvalue  $\lambda(x)$  for  $x \in \mathcal{D}$  and  $\lambda(x)$  depends continuously on  $x$ ,*
  - (ii)  *$\lambda(x)$  changes sign on  $\mathcal{D}$ , and*
  - (iii)  *$f(T(x)h) \neq 0$  if  $x \in S_1$ , if  $h$  spans  $N(A'(x))$  and if  $f$  spans  $N(A'(x)^*)$ .*
- Then  $A$  is not a local homeomorphism on  $\mathcal{D}$ .*

**PROOF.** Assume by way of contradiction that  $A$  is a local homeomorphism. Since  $\lambda(x)$  changes sign in  $\mathcal{D}$ ,  $\tilde{\mathcal{D}} = \{x \in \mathcal{D} : \lambda(x) = 0\}$  disconnects

$\mathcal{D}$ . If  $\tilde{\mathcal{D}}$  has non-empty interior one can easily use the rank theorem (cp. Tromba [17, Theorem 1]) to show that  $A$  is not locally one-to-one (or even locally finite to one). If  $\tilde{\mathcal{D}}$  has empty interior, let  $\mathcal{D}^* = \{z \in \tilde{\mathcal{D}}: \lambda(x) \text{ takes positive and negative values in every neighbourhood of } z\}$ . Note that  $\mathcal{D}^*$  is closed. A simple connectedness argument like that in the proof of Step 3 in Lemma 3 shows that  $\mathcal{D}^*$  disconnects  $\mathcal{D}$ . Since we could replace  $\mathcal{D}$  by any open subset,  $\mathcal{D}^*$  locally disconnects at any point  $y$  of  $\mathcal{D}^*$ . By Lemma 3, it follows that there exists  $z \in S_1 \cap \mathcal{D}^*$  near  $y$  such that  $\mathcal{D}^*$  locally disconnects at  $z$ . If we apply Lemma 4 with  $u = z$ , we find that  $A$  is not a local homeomorphism. (Here we are using the remark after the statement of Lemma 4 and that, since  $z \in \mathcal{D}^*$ ,  $\lambda(x)$  changes sign near  $z$ .) This completes the proof.

REMARK 1. Note that, by our basic assumptions,  $A'(x)$  is Fredholm of index zero for  $x \in X$ . We do not know if (iii) can be removed. The assumptions are more difficult to verify than is immediately apparent. For example, if there exist complex  $\lambda$ 's in the spectrum  $\sigma(A(x), T(x))$  of the pair  $(A'(x), T(x))$  which bifurcate from real points of the spectrum as  $x$  varies, then it is difficult to find a  $\lambda(x)$  defined on  $\mathcal{D}$  which depends continuously on  $x$ . (Here  $\sigma(A(x), T(x)) = \{\lambda \in \mathbb{C}: \tilde{A}'(x) - \lambda \tilde{T}(x) \text{ is not invertible}\}$ , where  $\tilde{A}'(x)$  denotes the natural extension of  $A'(x)$  to the complexification of  $X$  and  $\tilde{T}(x)$  is defined similarly.) One case where this difficulty can often be avoided is when we can use the theory of positive operators and take  $\lambda(x)$  to be the smallest real eigenvalue of  $A'(x)y = \lambda T(x)y$ . Even if the spectrum is always real it is often not easy to choose a  $\lambda(x)$  which one can keep control of if eigenvalues are not simple (or more generally their multiplicities change as  $x$  varies). Note that we need to keep some control of  $\lambda(x)$  to prove it changes sign. One can sometimes use the theory of oscillation kernels to prove that all eigenvalues are simple and then this difficulty disappears. (In this case, it suffices to assume that  $A$  is  $C^1$  rather than  $C^2$ .)

REMARK 2. There is one case where this last difficulty can be avoided. Assume that there exist an interval  $(a, b) \subseteq R$  with  $0 \in (a, b)$  and a positive integer  $k$  such that  $\sigma(A(x), T(x)) \cap (a, b)$  consists of eigenvalues  $\lambda_1(x) \leq \lambda_2(x) \leq \dots \leq \lambda_k(x)$  together with a subset of  $[\lambda_k(x), b)$ . In addition, assume that  $\lambda_i(x)$ ,  $i = 1, \dots, k$ , depend continuously on  $x$ . We then take  $\lambda = \lambda_k(x)$  and see if it changes sign. To achieve the above assumptions, we have the same trouble as before if complex spectrum bifurcates from real spectrum. To achieve the continuous dependence of  $\lambda_i(x)$  upon  $x$ , we must count eigenvalues with suitable multiplicities. Our examples in §2 will be of this type. The essential idea in the method in this remark is to examine the number of eigenvalues in  $(a, 0)$ : in particular, the number of negative eigenvalues when  $a = -\infty$ . There seems to be a fundamental

difficulty in applying this idea if complex eigenvalues bifurcate from real eigenvalues. It is possible that there is a double eigenvalue  $\tilde{\lambda}(x_0)$  of  $A'(x_0)w - \lambda T(x_0)w = 0$  where  $\tilde{\lambda}(x_0) > 0$  which branches to two complex eigenvalues which continue to exist and stay non-real until eventually they again collapse to a double real eigenvalue  $\tilde{\lambda}(x_1)$  with  $\tilde{\lambda}(x_1) < 0$ . This change can happen with  $A$  always being a local homeomorphism, even though the number of real eigenvalues less than zero has changed. Thus our ideas seem no longer to work. It seems difficult to decide when this behaviour can occur. For this reason the last result mentioned in Remark 3 after Theorem 5 in [9] does not seem to follow from the present work. (It is proved by a more direct argument.)

One question that sometimes arises is the following. Is  $A$  a local homeomorphism if  $S$  is a rather small set (and thus certainly does not locally disconnect at any  $x$ )? In applications, this case is usually easy to dispose of directly but we mention several general results. If  $X$  is infinite-dimensional,  $S$  is locally compact, and  $A$  is proper, then a result in [15] implies that  $A$  is a homeomorphism. If  $S$  is zero-dimensional in the sense of [11],  $A$  is  $C^3$  and  $\dim X \geq 3$ , then it can be shown that  $A$  is a local homeomorphism. (The proof uses a number of Church's ideas.)

Finally note that we always have that  $A$  is a local homeomorphism if  $A'(x)$  is invertible for  $x \in X$  and that  $A$  is globally one-to-one if, for each  $u, v \in X$ , there is a continuous invertible linear operator  $H(u, v)$  such that  $A(u) - A(v) = H(u, v)(u - v)$ .

**2. Applications.** In this section, we give a few simple applications of the results in §1. We assume that  $\Omega$  is a subset of  $\mathbf{R}^n$  of finite measure,  $L$  is an unbounded closed linear operator on  $L^\infty(\Omega)$  with non-empty resolvent set and compact resolvent. We also assume that there is a  $C \in \mathbf{R}$  such that  $\langle Lu, u \rangle \geq C\langle u, u \rangle$  for  $u \in \mathcal{D}(L)$  where  $\langle \cdot, \cdot \rangle$  is the usual scalar product on  $L^2(\Omega)$ . For simplicity we also assume that  $L$  extends to a self-adjoint operator  $\tilde{L}$  on  $L^2(\Omega)$  with compact resolvent and that if  $g \in L^\infty(\Omega)$ , every solution of  $\tilde{L}u = gu$  is in  $\mathcal{D}(L)$ . (These assumptions are in fact consequences of our earlier ones.) Let  $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \dots$  denote the distinct eigenvalues of  $L$  and assume that  $g: \mathbf{R} \rightarrow \mathbf{R}$  is  $C^2$  and that there exist  $y_1, y_2 \in \mathbf{R}$  and  $\tilde{\lambda}_i$  such that  $g'(y_1) < \tilde{\lambda}_i < g'(y_2)$ . We firstly assume that  $g'$  is bounded on  $\mathbf{R}$ . We prove that the map  $u \rightarrow Lu - g(u)$  is not locally one-to-one as a map of  $\mathcal{D}(L)$  (with the graph norm) to  $L^\infty(\Omega)$ . We apply Theorem 1. If  $u \in L^\infty(\Omega)$ , let  $\lambda_1(u) \leq \lambda_2(u) \leq \dots$  denote the eigenvalues counting multiplicity of  $Lh - g'(u)h = \lambda h$ . These exist and depend continuously on  $u$ . (These results are most easily established by proving the results in  $L^2(\Omega)$  first.) If  $u(x) = k$  on  $\Omega$ , one easily sees that  $\lambda_i(u) = \lambda_i - g'(k)$ , where  $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$  are the eigenvalues of  $L$  counting multiplicity. Choose  $j$  such that  $\lambda_j = \tilde{\lambda}_i$ . Then  $\lambda_j(\bar{y}_k) = \tilde{\lambda}_i - g'(\bar{y}_k)$ , where  $\bar{y}_k$  is the constant func-

tion whose value is  $y_k$  on  $\Omega$ . Thus, by our assumptions on  $y_1$ ,  $y_2$  and  $\tilde{\lambda}_i$ ,  $\lambda_j(u)$  changes sign on  $L^\infty(\Omega)$ . We need to show that  $\lambda_j(u)$  changes sign on  $\mathcal{D}(L)$ . To do this, we use a density argument. Choose  $u_n \in \mathcal{D}(\tilde{L})$  such that  $u_n \rightarrow \bar{y}_1$  in  $L^2(\Omega)$ . (Remember that, since  $\tilde{L}$  is self-adjoint,  $\mathcal{D}(\tilde{L})$  is dense in  $L^2(\Omega)$ .) By choosing a subsequence, we may assume that  $u_n \rightarrow \bar{y}_1$  pointwise a.e. on  $\Omega$ . We may assume without loss of generality that  $\tilde{L}$  is invertible (otherwise, we replace  $L$  by  $L - cI$ ). Now  $\tilde{L}u_n \in L^2(\Omega)$ . If we approximate this by  $f_n \in L^\infty(\Omega)$  and replace  $u_n$  by  $(\tilde{L})^{-1}f_n$ , we may assume without loss of generality that  $u_n \in \mathcal{D}(L)$ . It follows easily by the dominated convergence theorem that  $g'(u_n)V \rightarrow g'(\bar{y}_1)V$  weakly in  $L^2(\Omega)$  as  $n \rightarrow \infty$  for each  $V \in L^2(\Omega)$ . Here we use that  $g'$  is bounded on  $\mathbf{R}$ . Since  $L$  has compact resolvent, it follows that  $A(u_n)V \rightarrow A(\bar{y}_1)V$  strongly in  $L^2(\Omega)$ , where  $A(u) = (\tilde{L})^{-1}g'(u)$ . It now follows easily by using the theory of collective compactness (cp. [3]) that  $\lambda_j(u_n) \rightarrow \lambda_j(\bar{y}_1)$  as  $n \rightarrow \infty$ . Since  $u_n \in \mathcal{D}(L)$  and since we could use a similar argument for  $\bar{y}_2$ , it follows that  $\lambda_j(u)$  changes sign on  $\mathcal{D}(L)$  as required. We now can apply Theorem 1 with  $X = \mathcal{D}(L)$  (with the graph norm),  $\mathcal{D} = X$ ,  $Y = L^\infty(\Omega)$ ,  $\lambda = \lambda_j$  and  $T(u)$  the natural injection of  $\mathcal{D}(L)$  into  $L^\infty(\Omega)$ . The remaining conditions are easy to check. The only point to note is that because of the self-adjointness properties, we may take  $f = h$  in (iii) in the statement of Theorem 1. Thus we have proved our claim.

We now consider the case where  $g'$  is not bounded on  $\mathbf{R}$ . The above result still holds if we assume that there exist  $u_n^i \in \mathcal{D}(L)$  such that  $|u_n^i(x)| \leq K$  for all  $n$  and all  $x \in \Omega$  and  $u_n^i(x) \rightarrow y_i$  as  $n \rightarrow \infty$  a.e. on  $\Omega$ . The proof needs only minor modifications. For example, this condition holds if  $\Omega$  is the closure of its interior, if  $\partial\Omega$  has zero measure and if  $C_0^\infty(\text{int } \Omega) \subseteq \mathcal{D}(L)$ , where  $C_0^\infty(\text{int } \Omega)$  denotes the  $C^\infty$ -functions of compact support in  $\text{int } \Omega$ . This last condition nearly always holds for elliptic operators.

Our methods can also be used to handle the case where  $g$  also depends on  $x \in \Omega$ . We sketch this briefly. We assume that  $g: \bar{\Omega} \times \mathbf{R} \rightarrow \mathbf{R}$  is continuous and that the first and second partial derivatives with respect to the second variable exist continuously (as a function of both variables). Let  $h_+(x) = \sup_{y \in \mathbf{R}} g'_2(x, y)$  and  $h_-(x) = \inf_{y \in \mathbf{R}} g'_2(x, y)$ . To avoid awkward technicalities we assume either that (i)  $|g'_2(x, y)|$  is bounded on  $\bar{\Omega} \times [0, \infty)$  or (ii)  $|g'_2(x, y)| \rightarrow \infty$  as  $y \rightarrow \infty$  uniformly for  $x \in \bar{\Omega}$  (and  $\Omega$  is connected). We also assume a similar condition for  $y \leq 0$ . (These conditions could certainly be weakened but it is unclear if they can be entirely removed.) Let us consider the case where  $g'_2(x, y)$  is bounded on  $\mathbf{R}$ . (The other cases are similar except that we must add the same extra condition on  $\mathcal{D}(L)$  as before.) It is not difficult to construct measurable finite-valued functions  $u_n$  such that  $g'_2(x, u_n(x)) \rightarrow h_+(x)$  on  $\Omega$  as  $n \rightarrow \infty$ . By truncation, we can assume  $u_n \in L^2(\Omega)$ , provided that we only require convergence a.e. We can then argue as before to get  $u_n \in \mathcal{D}(L)$ . We can now repeat



the earlier argument to find that  $\lambda_i(u)$  changes sign on  $\mathcal{D}(L)$  (and thus there is nonuniqueness) provided there is an  $i$  such that  $\lambda'_i(h_+) < 0$  and  $\lambda'_i(h_-) > 0$ . Here  $\lambda'_i(v)$  is the  $i$ -th eigenvalue counting multiplicity of  $Lh - vh = \lambda h$ . This result is also true in the other cases if we interpret  $\lambda'_i(v) = \infty$  for all  $i$  if  $v(x) = -\infty$  on  $\Omega$  and  $\lambda'_i(v) = -\infty$  if  $v(x) = +\infty$  on  $\Omega$ . On the other hand, if there is an  $i$  such that  $\lambda'_i(h_+) < 0$ ,  $\lambda'_{i+1}(h_+) > 0$ ,  $\lambda'_i(h_-) < 0$ , it is fairly easy to show that our equation always has at most one solution. (Here for simplicity, we are assuming that  $h_+$  and  $h_-$  are finite on  $\Omega$ . The statement needs to be modified slightly in the other cases. For example, if  $h_+(x) = \infty$  on  $\Omega$ , we never have uniqueness while if  $h_-(x) = -\infty$  on  $\Omega$ , we have uniqueness if  $\lambda'_1(h_+) > 0$ .) The above uniqueness result is proved by showing that, if  $Lu_1 - g(u_1) = Lu_2 - g(u_2)$ , then  $\lambda'_j(a) = 0$  for some  $j$  where

$$a(x) = (u_1(x) - u_2(x))^{-1}(g(x, u_1(x)) - g(x, u_2(x))).$$

Since  $h_-(x) \leq a(x) \leq h_+(x)$  on  $\Omega$ , we obtain a contradiction by a simple use of the minimax characterization of eigenvalues (cp. [16]). Similar ideas can usually be used to decide whether uniqueness holds in cases where  $\lambda'_j(h_+) = 0$  or  $\lambda'_j(h_-) = 0$  for some  $j$ . If  $g$  is independent of  $x$ , it is easy to check that the above condition to ensure uniqueness simply becomes that there is an  $i$  such that  $\tilde{\lambda}_{i-1} < g'(y) < \tilde{\lambda}_i$  on  $\mathbf{R}$ , where  $\tilde{\lambda}_0 = -\infty$ .)

One final comment on this example. The complications are caused by the fact that, in interesting cases  $g$  need not be  $C^2$  (or even defined) as a map of  $\mathcal{D}(\tilde{L})$  into  $L^2$ . Thus we cannot use only  $L^2$  but must also use other spaces. It is possible to make our assumptions on  $\tilde{L}$  rather than  $L$ . We assume that  $\tilde{L}$  is self-adjoint, bounded below and with compact resolvent on  $L^2(\Omega)$  and that  $(\tilde{L} - \alpha I)^{-1}L^\infty(\Omega) \subseteq L^\infty(\Omega)$  for all  $\alpha \in L^\infty(\Omega)$ . This case was announced at the end of [10].

As a second application, we take  $L$  to be a second order elliptic self-adjoint differential operator with reasonably smooth coefficients on a smooth bounded domain  $\Omega$  with Dirichlet boundary condition (as in [8]) and we assume that  $g: \mathbf{R} \rightarrow \mathbf{R}$  is  $C^2$  such that  $y^{-1}g(y) \rightarrow \nu < \tilde{\lambda}_1$  as  $y \rightarrow -\infty$  and  $y^{-1}g(y) \rightarrow \mu > \tilde{\lambda}_1$  as  $y \rightarrow \infty$  (where  $\mu$  and  $\nu$  may be infinite). Here  $\tilde{\lambda}_i$  were defined earlier. In general, we use the notation of the previous example. The domain of  $L$  is  $\{u \in W^{1,2}(\Omega): Lu \in L^\infty(\Omega)\}$ . We prove that, if  $\mu > \tilde{\lambda}_2$  (or more generally,  $g'(y) > \tilde{\lambda}_2$  for some  $y$ ), then there is an  $f$  in  $L^\infty(\Omega)$  such that  $Lu = g(u) - f$  has at least three solutions. This shows that the very nice results of Ambrosetti and Prodi [2] do not all generalize to the case where  $\mu > \tilde{\lambda}_2$ . (We proved in [8] that most of their results do generalize.) Note that one easily checks that in this case  $C_0^\infty(\Omega) \subseteq \mathcal{D}(L)$ . Since  $\nu < \tilde{\lambda}_2$  and  $\mu > \tilde{\lambda}_2$ , we can argue as in the previous example to check that  $\lambda_2(u)$  changes sign on  $\mathcal{D}(L)$ . (The only points to

note are that  $\lambda_1 < \lambda_2$  and  $\lambda_1(u) < \lambda_2(u)$ .) Define  $H: \mathcal{D}(L) \rightarrow L^\infty(\Omega)$  by  $H(u) = g(u) - Lu$ . We can apply the argument in the proof of Theorem 1 to deduce that there exists  $V \in \mathcal{D}(L)$  such that  $\lambda_2(V) = 0$ , and  $H$  is not a local homeomorphism in any neighbourhood of  $V$ . Thus there exist  $u_1$  and  $u_2$  near  $V$  in  $\mathcal{D}(L)$  and  $f_0 \in L^\infty(\Omega)$  such that  $H(u_1) = f_0$  and  $H(u_2) = f_0$ . Since  $\lambda_1(V) < \lambda_2(V) = 0$  and  $u_1$  and  $u_2$  are near  $V$ ,  $\lambda_1(u_1) < 0$  and  $\lambda_1(u_2) < 0$ . However, by Theorem 4(1) in [8], there exists  $u_3 \in \mathcal{D}(L)$  such that  $H(u_3) = f_0$  and  $\lambda_1(u_3) \geq 0$ . Thus  $u_3 \neq u_1$  and  $u_3 \neq u_2$ . Hence  $H(u) = f_0$  has at least three solutions, as required. This answers a question of Berestycki and Lions [4, Remark II.2]. In fact, if  $H$  is proper (as a map of  $\mathcal{D}(L)$  with the graph norm into  $L^\infty(\Omega)$ ), then  $f_0$  can be chosen such that  $H(u) = f_0$  has at least four solutions. We sketch the proof of this and leave out some tedious details. If the singular set has non-empty interior, the result follows from the rank theorem. Otherwise, we can argue as in the proof of Theorem 1 and find  $V$  such that  $\lambda_2(V) = 0$ ,  $L - g'(V)I$  has a one-dimensional kernel (and thus  $\lambda_2(V) < \lambda_3(V)$ ) and  $\lambda_2(u)$  takes positive and negative values in every neighbourhood of  $V$ . By perturbing  $V$ , we can find  $V_1$  such that  $\lambda_2(V_1) < 0 < \lambda_3(V_1)$  and  $f_1 = H(V_1)$  is a regular value of  $H$ . Finally, if we apply Leray-Schauder degree theory to the equation

$$(7) \quad u = L^{-1}(g(u) - f_1),$$

we know as in [8] that the sum of the indices of solutions is zero. As in [8], we may rewrite the equation such that  $g'(u(x)) > 0$  on  $\Omega$  if  $u(x)$  is a solution of (7). By a simple but tedious comparison argument, if  $L - g'(u)I$  is invertible, then  $\sup\{j: \lambda_j(u) < 0\}$  is equal to the number of eigenvalues (counting multiplicity) less than one of  $Lh = \lambda g'(u)h$ . Using this, Theorems 5.23 and 8.18 in Lloyd [14] and Theorem 4(1) in [8], we find that our equation has two solutions with index one. Thus there must be at least two other solutions. (Note that, since  $f_1$  is a regular value of  $H$ , it follows that  $I - L^{-1}g'(u_0)$  is invertible whenever  $u_0$  is a solution of (7).)

Finally, as in the first application, our methods can be used if  $g$  depends on  $x$ .

As our final application, we take  $L$  as in the second application and assume that  $\tilde{\lambda}_i \in \sigma(L)$ . Let  $\mathcal{N}$  and  $\mathcal{R}$  denote the kernel and range of  $L - \tilde{\lambda}_i I$  respectively and let  $I - P$  be the "orthogonal projection" of  $L^\infty(\Omega)$  into  $\mathcal{N}$ . (Thus  $\mathcal{R}(P) = \mathcal{R}$ .) We write elements of  $L^\infty(\Omega)$  as  $h + V$  where  $h \in \mathcal{N}$  and  $V \in \mathcal{R}$ . We assume  $g: \mathbf{R} \rightarrow \mathbf{R}$  is twice continuously differentiable and, for simplicity,  $y^{-1}g(y)$  is bounded on  $\mathbf{R}$ . If there is an  $\varepsilon > 0$  such that  $\tilde{\lambda}_{i-1} + \varepsilon \leq y^{-1}g(y) \leq \tilde{\lambda}_{i+1} - \varepsilon$  for  $y$  large (where

$\tilde{\lambda}_0 = -\infty$ ), then it is well known that for each  $f_0 \in \mathcal{R}$  and  $h \in \mathcal{N}$ , the equation

$$(8) \quad E_h(V) \equiv PH(h + V) = f_0$$

has a solution  $V$  in  $\mathcal{R} \cap \mathcal{D}(L)$ . This is most easily proved by first working in  $L^2(\Omega)$ . Note that  $H$  was defined in the second application. Equation (8) occurs in a good deal of the study of the equation  $H(u) = f$  (cp [7]). It is then interesting to ask if this solution is unique. If  $\tilde{\lambda}_{i-1} < g'(y) < \tilde{\lambda}_{i+1}$  on  $\mathbf{R}$ , it is easy to prove uniqueness for every  $h \in \mathcal{N}$  and  $f_0 \in \mathcal{R}$ . (Once again one works in  $L^2(\Omega)$ .) We prove that, if there exists  $y_1 \in \mathbf{R}$  such that  $g'(y_1) > \tilde{\lambda}_{i+1}$  or  $g'(y_1) < \tilde{\lambda}_{i-1}$ , then there exist  $h \in \mathcal{N}$  and  $f_0 \in \mathcal{R}$  such that (8) has more than one solution. We consider the former case. The other is similar. For each  $h \in \mathcal{N}$ , we consider  $E_h$  as a map of  $\mathcal{D}(L) \cap \mathcal{R}$  (with the graph norm) into  $\mathcal{R}$ . We try to apply Theorem 1 (for fixed  $h$ ). We set  $T(V) = I$ . Then our eigenvalue problem is

$$(9) \quad Ly - Pg'(h + V)y = \lambda y$$

(where  $y \in \mathcal{R} \cap \mathcal{D}(L)$ ). Much as before, one sees that this has a countable number of eigenvalues  $\tilde{\lambda}_1(h + V) \leq \tilde{\lambda}_2(h + V) \cdots$  (counting multiplicity) which depend continuously on  $h + V$ . (Note that this still holds if we only assume  $h + V \in L^\infty(\Omega)$ .) Let us assume by way of contradiction that (8) has a unique solution for all  $h \in \mathcal{N}$  and  $f_0 \in \mathcal{R}$ . By applying Theorem 1 to  $E_h$ , we see that, for fixed  $i$ ,  $\tilde{\lambda}_i(h + V)$  has fixed weak sign for all  $V \in \mathcal{D}(L) \cap \mathcal{R}$ . (By this we mean that  $\tilde{\lambda}_i(h + V)$  is non-negative on all of  $\mathcal{D}(L) \cap \mathcal{R}$  or is non-positive there.) Since this holds for each  $h$  and since  $\tilde{\lambda}_i(h + V)$  depends continuously on  $h + V$ , it follows that  $\tilde{\lambda}_i(h + V)$  has fixed weak sign on all of  $\mathcal{D}(L) = \mathcal{N} \oplus (\mathcal{D}(L) \cap \mathcal{R})$ . (Note that, if  $\tilde{h} \in \mathcal{N}$  and if  $\tilde{\lambda}_i(\tilde{h} + V) = 0$  for all  $V \in \mathcal{D}(L) \cap \mathcal{R}$ , then the rank theorem would imply that (8) has a non-unique solution for  $h = \tilde{h}$  and for some  $f_0 \in \mathcal{R}$ .) By the same density argument as before, it follows that  $\tilde{\lambda}_i(h + V)$  has fixed weak sign on  $L^\infty(\Omega)$ . We show this is impossible. If  $u(x) = C = \text{constant}$  on  $\Omega$ ,  $Ly - Pg'(u)y = Ly - g'(C)y$  for  $y \in \mathcal{R}$ . Thus one easily sees that the distinct eigenvalues of (9) in this case are  $\{\tilde{\lambda}_j - g'(C)\}_{j \neq i}$  and  $\tilde{\lambda}_j - g'(C)$  has the same multiplicity as  $\tilde{\lambda}_j$ . Choose the smallest positive integer  $r$  such that  $\lambda_r = \tilde{\lambda}_i$  (where, as before, the  $\lambda_i$  are the eigenvalues of  $L$  counting multiplicity). Then, by our comments above,  $\tilde{\lambda}_r(C) = \tilde{\lambda}_{i+1} - g'(C)$ . Thus, since  $g'(y_1) > \tilde{\lambda}_{i+1}$ ,  $\tilde{\lambda}_r(y_1) < 0$ . On the other hand, by our assumptions on  $g$ , there exists  $y_2$  such that  $g'(y_2) < \tilde{\lambda}_{i+1}$ . Hence  $\tilde{\lambda}_r(y_2) > 0$ . This contradicts our earlier result that  $\tilde{\lambda}_r(u)$  has constant weak sign on  $L^\infty(\Omega)$ . Thus there exists  $h \in \mathcal{N}$  and  $f_0 \in \mathcal{R}$  such that the solution of (8) is not unique.

If  $g'(y)$  is bounded on  $\mathbf{R}$ , a more careful argument shows that for every  $h \in \mathcal{N}$ , there exists  $f_0 \in \mathcal{R}$  such that (8) has more than one solution.

Our methods could also be used if  $g$  depends on  $x$ . However, the result obtained contains terms which are harder to compute. Thus the result is more difficult to apply unless  $h_+(x)$  and  $h_-(x)$  are independent of  $x$  (where  $h_+$  and  $h_-$  were defined in the first application).

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