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LINEAR TRANSFORMATIONS PRESERVING SETS OF RANKS

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ABSTRACT. Let T be a linear transformation on $M_{m,n}(F)$, the set of all $m \times n$ matrices over the algebraically closed field F, and let R_j denote the subset of all matrices of rank j. Further let $R_E = \bigcup_{j \in E} R_j$ where E is a subset of $\{0, 1, \ldots, \min(m, n)\}$. We explore the structure of T when $T(R_E) \subseteqq R_E$.

1. Introduction. Let $M_{m,n}(F)$ denote the set of all $m \times n$ matrices over the algebraically closed field F and let $\rho(A)$ denote the rank of the matrix A. Let R_j denote the set of all matrices $A \in M_{m,n}(F)$ such that $\rho(A) = j$. If E is a subset of $\{0, 1, \ldots, \min(m, n)\}$, let $R_E = \bigcup_{j \in E} R_j$. In this notation consider the following problem: if $T: M_{m,n}(F) \rightarrow M_{m,n}(F)$ is a linear transformation, $E \subseteqq \{0, 1, \ldots, \min(m, n)\}$, and $T(R_E) \subseteqq R_E$, then what is the structure of T? There are two trivial cases: $E = \{0, 1, \ldots, \min(m, n)\}$ and $E = \{1, \ldots, \min(m, n)\}$. In the first case T need only be linear and in the second case T need only be linear and nonsingular.

Throughout the remainder of the paper we will assume that T is a linear transformation on $M_{m,n}(F)$ and that $m = \min(m, n)$.

Some research has been done for the case $E = \{k\}$ **[1, 6, 7]** and, in fact, in each known case when E is a proper subset of $\{1, \ldots, m\}$ the structure of T is the same **[1, 2, 3, 6, 7]**. We demonstrate that structure in the following theorem of Marcus, Moyls and Westwick **[6, 7]**.

THEOREM 1. *If $T(R_1) \subseteqq R_1$, then there exist $m \times m$ and $n \times n$ nonsingular matrices U and V respectively such that either*

i) *$T: A \rightarrow UAV$ for all $A \in M_{m,n}(F)$*

or

ii) *$m = n$ and $T: A \rightarrow UA^tV$ for all $A \in M_{m,n}(F)$ where A^t denotes the transpose of A.*

For easy reference we define a transformation T satisfying (i) or (ii) in Theorem 1 as a rank-1-preserver. We note that as a consequence of **[2, Thm. 4]** we have the following theorem.

THEOREM 2. *If E is a subset of $\{0, 1, \ldots, m\}$, $E \neq \{1, 2, \ldots, m\}$, and if T is nonsingular, then T is a rank-1-preserver.*

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We use the notation $A[\alpha_1, \ldots, \alpha_s|\beta_1, \ldots, \beta_t]$ to denote the submatrix of A on rows $\alpha_1, \ldots, e_s$ and columns $\beta_1, \ldots, \beta_t$.

2. An extension of a result of Botta. In [3] Botta proves the interesting result that, if $m = n$ and $E = \{0, 1, \ldots, m - 1\}$, and if $T(R_E) \subseteqq R_E$, then either T is nonsingular (hence a rank-1-preserver) or $T(M_{m,n}(F)) \subseteqq R_E$.

THEOREM 3. *If* $E = \{0, \ldots, k\}$, $1 \leqq k \leqq m$, *and* $T(R_E) \subseteqq R_E$, *then* either $T(M_{m,n}(F)) \subseteqq R_E$ or $\dim \ker T \leqq mn - (k + 1)^2$.

The proof of this theorem and a later one rely heavily on the following lemma.

LEMMA 1. *If* $T(R_E) \subseteqq R_E$, *where* $E = \{0, 1, \ldots, k\}$, $1 \leqq k \leqq m$, and $T(M_{m,n}(F)) \nsubseteq R_E$, *then there exist nonsingular matrices* R, $U \in M_m(F)$, S, $V \in M_n(F)$, *and a positive integer s such that*

$$UT\left(R^{-1}\begin{bmatrix} I_{k+s} & 0 \\ 0 & 0 \end{bmatrix} S^{-1}\right)V = \begin{bmatrix} I_{k+t} & 0 \\ 0 & 0 \end{bmatrix}$$

for some $t > 0$ *and*

$$T'(A) = UT\left(R^{-1}\begin{bmatrix} A & 0 & 0 \\ 0 & a_{11}I_{s-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} S^{-1}\right)V[1, \ldots, k + 1|1, \ldots, k + 1]$$

is a nonsingular linear transformation of $M_{k+1}(F)$ *to* $M_{k+1}(F)$ *mapping* I_{k+1} *to* I_{k+1}.

PROOF. Since $T(M_{m,n}(F)) \nsubseteq R_E$, there is $G \in M_{m,n}(F)$ such that $\rho(T(G)) > k$. Since $T(R_E) \subseteqq R_E$, $\rho(G) > k$. Choose $B \in M_{m,n}(F)$ of smallest rank such that $\rho(T(B)) > k$. Say $\rho(B) = k + s$ and $\rho(T(B)) = k + t$. Let R, $U \in M_m(F)$ and S, $V \in M_n(F)$ be nonsingular matrices such that

$$RBS = \begin{bmatrix} I_{k+s} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad UT(B)V = \begin{bmatrix} I_{k+t} & 0 \\ 0 & 0 \end{bmatrix}.$$

Define T_1: $M_{m,n}(F) \to M_{m,n}(F)$ by $T_1(X) = UT(R^{-1}XS^{-1})V$ for all $X \in M_{m,n}(F)$ so that,

$$T_1\begin{bmatrix} I_{k+s} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I_{k+t} & 0 \\ 0 & 0 \end{bmatrix}.$$

Further, since R, S, U and V are nonsingular and $T(R_E) \subseteqq R_E$, we have that $T_1(R_E) \subseteqq R_E$.

Now define T': $M_{k+1}(F) \to M_{k+1}(F)$ by

$$T'(A) = T_1\begin{bmatrix} A & 0 & 0 \\ 0 & a_{11}I_{s-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}[1, 2, \ldots, k + 1|1, \ldots, k + 1]$$

for all $A \in M_{k+1}(F)$. It is easily checked that T' is linear. Also, if $C \in M_{k+1}(F)$ and $\rho(C) \leqq k$, then

$$\rho\begin{bmatrix} C & 0 & 0 \\ 0 & c_{11}I_{s-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \leqq k + s - 1$$

so that

$$\rho\left(T_1\begin{bmatrix} C & 0 & 0 \\ 0 & c_{11}I_{s-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \leqq k$$

since the matrix of smallest rank whose image under T (and hence under T_1) has rank greater than k has rank $k + s$. Thus, $\rho(T'(C)) \leqq k$. That is, the image under T' of any singular matrix is singular. By [3, Thm. 1], either T' is nonsingular or $\rho(T'(Z)) \leqq k$ for all $Z \in M_{k+1}(F)$. Since

$$T'(I_{k+1}) = T_1\left(\begin{bmatrix} I_{k+1} & 0 & 0 \\ 0 & I_{s-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)[1, \ldots, k+1 | 1, \ldots, k+1]$$

$$= \begin{bmatrix} I_{k+t} & 0 \\ 0 & 0 \end{bmatrix}[1, \ldots, k+1 | 1, \ldots, k+1] = I_{k+1},$$

we must conclude that T' is nonsingular, and the lemma is proven.

PROOF OF THEOREM 3. Suppose $T(M_{m,n}(F)) \nsubseteq R_E$. By Lemma 1, there exist nonsingular R, $U \in M_m(F)$, S, $V \in M_n(F)$, and a positive integer s such that

$$T'(A) = UT\left(R^{-1}\begin{bmatrix} A & 0 & 0 \\ 0 & a_{11}I_{s-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}S^{-1}\right)V[1, \ldots, k+1 | 1, \ldots, k+1]$$

is nonsingular. That is, dim im $T' = (k+1)^2$. Now, dim im $T \geqq$ dim im T' so that dim im $T \geqq (k+1)^2$. Thus dim ker $T \leqq mn - (k+1)^2$.

3. Main results. One of our main results is contained in the following theorem.

THEOREM 4. *If* $0 \notin E$, $\max\{j : j \in E\} = k < m$ *and* $T(R_E) \subseteq R_E$, *then either* $T(M_{m,n}(F)) \subseteq R_G$, *where* $G = \{0, 1, \ldots, k\}$ *or* T *is nonsingular* (*and hence a rank*-1-*preserver*).

In developing the arguments for this theorem we use the following lemmas, which appear to have some importance in themselves.

LEMMA 2. *If* $\rho(A) = s$ and $\rho(T(A)) = t$ *for any linear transformation*

T, then there exist $A_i \in M_{m,n}(F)$, $i = 1, \ldots, m - s$, *such that* $\rho(A_i) = s + i$ *and* $\rho(T(A_i)) \geqq t$.

PROOF. Let U and V be nonsingular matrices such that

$$UAV = \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}.$$

Define

$$B_i = U^{-1}\begin{bmatrix} 0_s & 0 & 0 \\ 0 & I_i & 0 \\ 0 & 0 & 0 \end{bmatrix}V^{-1},$$

$i = 1, \ldots, m - s$. Clearly, $\rho(xA + B_i) = s + i$ whenever $x \neq 0$. Let R and S be nonsingular matrices such that

$$RT(A)S = \begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix}.$$

Further, define T_1: $M_{m,n}(F) \to M_{m,n}(F)$ by $T_1(X) = RT(X)S$. Obviously, $\rho(T_1(X)) = \rho(T(X))$. Now det $T_1(xA + B_i)[1, \ldots, t|1, \ldots, t] = x^t + f(x)$ where the degree of $f(x)$ is less than t, since

$$T_1(xA + B_i) = x\begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix} + T_1(B_i).$$

Thus there is some nonzero x, say x_i, for which det $T_1(x_iA + B_i)[1, \ldots, t|1, \ldots, t]$ is nonzero. That is $\rho(T_1(x_iA + B_i) \geqq t$. Let $A_i = x_iA + B_i$. Here $\rho(A_i) = s + i$ and $\rho(T(A_i)) = \rho(T_1(A_i)) = \rho(T_1(x_iA + B_i)) \geqq t$.

LEMMA 3. *If* $E \subseteqq \{0, \ldots, m\}$, $\max_{j\in E} j = k$ *and* $T(R_E) \subseteqq R_E$, *then* $T(R_G) \subseteqq R_G$, *where* $G = \{0, \ldots, k\}$.

PROOF. If $k = m$, the lemma is trivial. Suppose $k < m$ and $T(R_G) \nsubseteq R_G$. In this case there is $A \in M_{m,n}(F)$ with $\rho(A) = s < k$ and $\rho(T(A)) = t > k$ Since $s < k$, there is some i such that $s + i = k$. By Lemma 2 there exists $B \in M_{m,n}(F)$ such that $\rho(B) = s + i$ and $\rho(T(B)) \geqq t > k$. That is, $B \in R_E$ and $T(B) \notin R_E$, a contradiction.

PROOF OF THEOREM 4. Suppose $T(M_{m,n}(F)) \nsubseteq R_G$. By Lemma 3, $T(R_G) \subseteqq R_G$ and hence by Lemma 1, there exist nonsingular matrices $R, U \in M_m(F)$, $S, V \in M_n(F)$, and a positive integer s such that

$$UT\left(R^{-1}\begin{bmatrix} I_{k+s} & 0 \\ 0 & 0 \end{bmatrix}S^{-1}\right)V = \begin{bmatrix} I_{k+t} & 0 \\ 0 & 0 \end{bmatrix}$$

for some $t > 0$ and

$$T'(A) = UT\left(R^{-1}\begin{bmatrix} A & 0 & 0 \\ 0 & a_{11}I_{s-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}S^{-1}\right)V[1, \ldots, k+1|1, \ldots, k+1]$$

is a nonsingular linear transformation of $M_{k+1}(F)$. Let T_1: $M_{m,n}(F) \to M_{m,n}(F)$ be defined by $T_1(X) = UT(R^{-1}XS^{-1})V$ for all $X \in M_{m,n}(F)$. Here $T'(A) = T_1(A)[1, \ldots, k+1|1, \ldots, k+1]$ for all $A \in M_{k+1}(F)$. Now let $k^{(1)} = \max\{\rho(T(A)): A \in R_E\}$ and let $j^{(1)} = \min\{\rho(A): \rho(T(A)) = k^{(1)}\}$. If $j^{(1)} \geqq k^{(1)}$, then let $k^{(2)} = \max\{\rho(T(A)): A \in R_E$ and $\rho(T(A)) < k^{(1)}\}$ and let $j^{(2)} = \min\{\rho(A): \rho(T(A)) = k^{(2)}\}$. If $j^{(2)} \geqq k^{(2)}$, continue the process until either $k^{(\ell)} = q$ where $q = \min_{j \in E} j$ or $j^{(\ell)} < k^{(\ell)}$.

CASE 1. $j^{(\ell)} \geqq k^{(\ell)} = q$. If $q \neq 1$, then $T(R_q) \subseteqq R_q$. By [1, Thm. 2.1] either T is nonsingular or there is $B \in M_{m,n}(F)$ such that $\rho(B) < q$ and $\rho(T(B)) = q$, contradicting that $j^{(\ell)} \geqq q$. Thus T is nonsingular. If $q = 1$, then by Theorem 1, T is nonsingular.

CASE 2. $j^{(\ell)} < k^{(\ell)}$. In this case we can assume without loss of generality that $\ell = 1$ (i.e., $j^{(l)} < k^{(l)}$). Choose $H \in M_{m,n}(F)$ such that $\rho(H) = j^{(l)}$ and $\rho(T(H)) = k^{(l)}$. Let

$$T(H) = K = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}$$

where K_1 is $(k+1) \times k \times 1)$. Let $U_1, V_1 \in M_{k+1}(F)$, $U_2 \in M_{m-k-1}(F)$ and $V_2 \in M_{n-k-1}(F)$ be nonsingular matrices chosen so that:

$$U_1K_1V_1 = \begin{bmatrix} 0 & I_u \\ 0 & 0 \end{bmatrix},$$

$$U_1K_2V_2 = \begin{bmatrix} 0 & K_1^{(2)} \\ I_v & 0 \\ 0 & 0 \end{bmatrix}$$

where $K_1^{(2)}$ is $u \times (n-k-1-v)$

$$U_2K_3V_1 = \begin{bmatrix} 0 & I_w & 0 \\ 0 & 0 & K_1^{(3)} \end{bmatrix}$$

where $K_1^{(3)}$ is $(m-k-1-w) \times u$ and

$$U_2K_4V_2 = \begin{bmatrix} K_1^{(4)} & K_2^{(4)} \\ K_3^{(4)} & K_4^{(4)} \end{bmatrix}$$

where $K_1^{(4)}$ is $w \times v$.

To see the existence of U_1, U_2, V_1 and V_2, let

$$U_1 = \begin{bmatrix} I_u & -K_2^{(2)} \\ 0 & I_{k+1-u} \end{bmatrix} \begin{bmatrix} I_u & 0 \\ 0 & P_2 \end{bmatrix} P_1,$$

and

$$V_1 = Q_1 \begin{bmatrix} Q_2 & 0 \\ 0 & I_u \end{bmatrix} \begin{bmatrix} I_{k+1-u-w} & 0 & 0 \\ 0 & I_w & -K_2^{(3)} \\ 0 & 0 & I_u \end{bmatrix}$$

where P_1 and Q_1 are chosen so that

$$P_1 K_1 Q_1 = \begin{bmatrix} 0 & I_u \\ 0 & 0 \end{bmatrix},$$

then P_2 and V_2 are chosen so that

$$\begin{bmatrix} I_u & 0 \\ 0 & P_2 \end{bmatrix} P_1 K_2 V_2 = \begin{bmatrix} K_2^{(2)} & K_1^{(2)} \\ I_v & 0 \\ 0 & 0 \end{bmatrix}.$$

Choose Q_2 and U_2 so that

$$U_2 K_3 Q_1 \begin{bmatrix} Q_2 & 0 \\ 0 & I_u \end{bmatrix} = \begin{bmatrix} 0 & I_w & K_2^{(3)} \\ 0 & 0 & K_1^{(3)} \end{bmatrix}.$$

Now

$$U_1 K_1 V_1 = \begin{bmatrix} I_u & -K_2^{(2)} \\ 0 & I_{k+1-u} \end{bmatrix} \begin{bmatrix} I_u & 0 \\ 0 & P_2 \end{bmatrix} P_1 K$$

$$\cdot Q_1 \begin{bmatrix} Q_2 & 0 \\ 0 & I_u \end{bmatrix} \begin{bmatrix} I_{k+1-u-w} & 0 & 0 \\ 0 & I_w & -K_2^{(3)} \\ 0 & 0 & I_u \end{bmatrix}$$

$$= \begin{bmatrix} I_u & -K_2^{(2)} \\ 0 & I_{k+1-u} \end{bmatrix} \begin{bmatrix} I_u & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} 0 & I_u \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_2 & 0 \\ 0 & I_u \end{bmatrix} \cdot$$

$$\begin{bmatrix} I_{k+1-u-w} & 0 & 0 \\ 0 & I_w & -K_2^{(3)} \\ 0 & 0 & I_u \end{bmatrix} = \begin{bmatrix} I_\sigma & -K_2^{(2)} \\ 0 & I_{k+1-u} \end{bmatrix} \begin{bmatrix} 0 & I_u \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{k+1-u-w} & 0 & 0 \\ 0 & I_w & -K_2^{(3)} \\ 0 & 0 & I_u \end{bmatrix}$$

$$= \begin{bmatrix} 0 & I_u \\ 0 & 0 \end{bmatrix}.$$

Also, $U_1 K_2 V_2$ and $U_2 K_3 V_1$ have the desired forms.

Now

$$\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} K \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & I_u & 0 & K_1^{(2)} \\ 0 & 0 & 0 & I_v & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I_w & 0 & K_1^{(4)} & K_2^{(4)} \\ 0 & 0 & K_1^{(3)} & K_3^{(4)} & K_4^{(4)} \end{bmatrix} = C_1 .$$

Let

$$U_3 = \begin{bmatrix} 0 & -K_1^{(4)} & 0 \\ -K_1^{(3)} & -K_3^{(4)} & 0 \end{bmatrix} \in M_{m-k-1,\ k+1}(F)$$

and

$$V_3 = \begin{bmatrix} 0 & 0 \\ 0 & -K_2^{(4)} \\ 0 & -K_1^{(2)} \end{bmatrix} \in M_{k+1,\ n-k-1}(F) .$$

Now

$$\begin{bmatrix} I_{k+1} & 0 \\ U_3 & I_{m-k-1} \end{bmatrix} C_1 \begin{bmatrix} I_{k+1} & V_3 \\ 0 & I_{n-k-1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & I_u & 0 & 0 \\ 0 & 0 & 0 & I_v & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I_w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L \end{bmatrix} = C_2 ,$$

where $\rho(L) = k^{(1)} - u - v - w = x$. Let $U_4 \in M_{m-k-1-w}(F)$ and $V_4 \in M_{n-k-1-v}(F)$ be nonsingular matrices such that

$$U_4 L V_4 = \begin{bmatrix} I_x & 0 \\ 0 & 0 \end{bmatrix} .$$

Now

$$\begin{bmatrix} I_{k+1+w} & 0 \\ 0 & U_4 \end{bmatrix} C_2 \begin{bmatrix} I_{k+1+v} & 0 \\ 0 & V_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & I_u & 0 & 0 & 0 \\ 0 & 0 & 0 & I_v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = C_3 ,$$

where $u + v + w + x = \rho(K) = k^{(1)}$. Define $T_2 \colon M_{m,n}(F) \to M_{m,n}(F)$ by

$$T_2(X) = \begin{bmatrix} I_{k+1+w} & 0 \\ 0 & U_4 \end{bmatrix} \begin{bmatrix} I_{k+1} & 0 \\ U_3 & I_{m-k-1} \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} T_1(X)$$
$$\cdot \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \begin{bmatrix} I_{k+1} & V_3 \\ 0 & I_{n-k-1} \end{bmatrix} \begin{bmatrix} I_{k+1+v} & 0 \\ 0 & V_4 \end{bmatrix}$$

for all $X \in M_{m,n}(F)$. Hence $T_2(H) = C_3$.

Define T'': $M_{k+1}(F) \to M_{k+1}(F)$ by

$$T''(Y) = T_2 \begin{bmatrix} Y & 0 & 0 \\ 0 & y_{11} I_{s-1} & 0 \\ 0 & 0 & 0 \end{bmatrix} [1, \dots, k+1|1, \dots, k+1].$$

One sees by the structure of T_2 and the definition of T' that $T''(Y) = U_1 T'(Y) V_1$ for all $Y \in M_{k+1}(F)$. Since T' is nonsingular, T'' is and thus, for every pair (i, j), $1 \leqq i, j \leqq k + 1$, there is a matrix in $M_{k+1}(F)$ whose image has a nonzero (i, j) entry. Therefore, for any pair (i, j), $1 \leqq i, j \leqq k + 1$, there is a matrix in $M_{m,n}(F)$ whose image under T_2 has a nonzero (i, j) entry. Since every matrix in the image of T_2 is the sum of images of rank 1 matrices, there is a rank 1 matrix $P \in M_{m,n}(F)$ such that $T_2(P) = R$ has a nonzero $(k + 1, 1)$ entry. That is, $\rho(zH + P) \leqq j^{(1)} + 1$ for all $z \in F$ and $T_2(zH + P) \geqq k^{(1)} + 1$ for some $z \in F$ since $\det T_2(zH + P) \cdot [1, \dots, u + v, k + 1, k + 2, \dots, k + 1 + w + x|1, k + 1 - w - u, \dots, k + 1 + v + x] = \det(zC_3 + R)[1, \dots, u + v, k + 1, k + 2, \dots, k + 1 + w + x|1, k + 1 - w - u, \dots, k + 1 + v + x] = z^{k^{(1)}} \cdot r_{k+1,1} + f(z)$ and $\deg(f(z)) < k^{(1)}$, and thus for some choice of z, the above determinant is nonzero. That is $\rho(T_2(zH + P)) \geqq k^{(1)} + 1$. However since $T(R_{E'}) \subseteqq R_{E'}$ (and hence $T_2(R_{E'}) \subseteqq R_{E'}$) where $E' = \{0, 1, \dots, k^{(1)}\}$ and $j^{(1)} < k^{(1)}$, we have a contradiction. Thus T must be nonsingular.

The following corollary is a special case of Theorem 4.

COROLLARY 1. *If* $T(R_k) \subseteqq R_k$, $k > 0$, *then either* T *is nonsingular or* $\rho(T(A)) \leqq k$ *for all* $A \in M_{m,n}(F)$.

COROLLARY 2. *If* $T(R_k) \subseteqq R_k$, $k > 0$, *and if* $\dim \ker T \leqq (m - k)n$, *then* T *is nonsingular.*

PROOF. Suppose T is singular. From Corollary 1, $\rho(T(A)) \leqq k$ for all $A \in M_{m,n}(F)$. Thus, by [4, Theorem 1] $\dim \operatorname{im} T \leqq nk$, and therefore $\dim \ker T = mn - \dim \operatorname{im} T \geqq mn - nk = (m - k)n$, a contradiction.

THEOREM 5. *If* D *and* E *are nonempty disjoint subsets of* $\{0, 1, \dots, m\}$, $D \neq \{0\}$, $E \neq \{0\}$, $T(R_D) \subseteqq R_D$ *and* $T(R_E) \subseteqq R_E$, *then* T *is nonsingular.*

PROOF. Let $k = \max\{\rho(T(A)): A \in R_D\}$ and let $\ell = \max\{\rho(T(A)): A \in R_E\}$. Now $k \neq \ell$ since D and E are disjoint. Assume $k > \ell$. By

Theorem 4, T is nonsingular since there exists a matrix $A = M_{m,n}(F)$ such that $\rho(T(A)) = \ell > k$.

COROLLARY 3. *If* $T(R_k) \subseteqq R_k$ *and* $T(R_j) \subseteqq R_j$ *where* $k \neq j$ *and* $k, j > 0$, *then* T *is nonsingular.*

If E is a subset of $\{0, 1, \ldots, m\}$, we define E^c to be the complement of E in $\{0, 1, \ldots, m\}$.

THEOREM 6. *If* $E \subset \{0, \ldots, m\}$, $E \neq \emptyset$, $T(R_E) \subseteqq R_E$ *and* $T(R_{E^c}) \subseteqq R_{E^c}$, *then* T *is nonsingular.*

PROOF. By Theorem 5 either T is nonsingular or $E = \{0\}$ or $E^c = \{0\}$ Since $T(R_E) \subseteqq R_E$ and $T(R_{E^c}) \subseteqq R_{E^c}$, $T(A) \nsubseteq R_{\{0\}}$ if $A \neq 0$; that is, T is nonsingular.

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