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THE CONTROL MEASURE PROBLEM AND THE UNIVERSAL MEASURE TOPOLOGY

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In this paper we characterize submeasures with control measures and pathological submeasures in terms of the universal measure topology of Graves. The results give new interpretations of several problems equivalent to the control measure problem (Maharam submeasure problem). Our main tool is the complete Boolean algebra of projections in the universal measure space; the setting for our results is the complete lattice of Fréchet-Nikodým topologies on an algebra of sets.

1. Preliminaries. Let $\mathscr{B}$ be a Boolean algebra. A *submeasure* on $\mathscr{B}$ is a map $\lambda: \mathscr{B} \to [0, \infty)$ such that

(1) $\lambda(0) = 0$,

(2) $\lambda(A) \leqq \lambda(B)$ whenever $A \leqq B$ in $\mathscr{B}$,

(3) $\lambda(A \vee B) \leqq \lambda(A) + \lambda(B)$ for all A and B in $\mathscr{B}$.

A submeasure λ is *exhaustive* if $\lambda(A_n) \to 0$ whenever (A_n) is a disjoint sequence in $\mathscr{B}$ and *continuous* if $\lambda(A_n) \to 0$ whenever (A_n) is a decreasing sequence in $\mathscr{B}$ and $\bigwedge A_n = 0$. Call λ *strictly positive* when $\lambda(A) = 0$ if and only if $A = 0$.

Let $\mathscr{C}$ be the algebra of clopen subsets of the Stone space X of $\mathscr{B}$. Then submeasures on $\mathscr{B}$ are in $1 - 1$ correspondence with submeasures on $\mathscr{C}$.

For submeasures λ and μ on $\mathscr{C}$, say λ is *μ-continuous* if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\lambda(A) < \varepsilon$ whenever $\mu(A) < \delta$. Call λ and μ *mutually continuous* if λ is μ-continuous and μ is λ-continuous.

A *Fréchet-Nikodým topology* on $\mathscr{C}$ is a topology making the map $(A, B) \to A \Delta B$ from $\mathscr{C} \times \mathscr{C}$ (with the product topology) to $\mathscr{C}$ continuous and making the map $A \to A \cap B$ continuous at $\emptyset$ uniformly for B in $\mathscr{C}$. A Fréchet-Nikodým topology on $\mathscr{C}$ makes $\mathscr{C}$ into a topological group in which intersection is uniformly continuous. Fréchet-Nikodým topologies were introduced and studied by Drewnowski [5], [6].

If λ is a submeasure on $\mathscr{C}$, then we may define a semimetric d_λ on $\mathscr{C}$ by $d_\lambda(A, B) = \lambda(A \Delta B)$, Of course, d_λ is a metric if and only if λ is strictly positive. The semimetric topology G_λ is a Fréchet-Nikodým topology

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on $\mathscr{C}$. Note that λ is μ-continuous if and only if $G_\lambda \subseteqq G_\mu$. We wish to compare G_λ with the universal measure topology τ on $\mathscr{C}$.

For background, we give a short description of the universal measure space $L(\mathscr{C})$. For details, see Graves **[8]** and Brook **[1]**, **[2]**. Let $S(\mathscr{C})$ be the vector space of all complex-valued $\mathscr{C}$-simple functions on X. Define the *universal measure* $\chi\colon \mathscr{C} \to S(\mathscr{C})$ by $\chi(A) = \chi_A$, the characteristic function of A.

A finitely additive map ϕ from $\mathscr{C}$ to a locally convex space W is *strongly bounded* or *exhaustive* if $\phi(A_n) \to 0$ whenever (A_n) is a disjoint sequence in $\mathscr{C}$. Say ϕ is *strongly countably additive* if it is strongly bounded and countably additive. Call ϕ a *measure* if it is at least finitely additive. Since $\mathscr{C}$ is operationally finite (nontrivial infinite unions and intersections do not exist), each strongly bounded measure is automatically strongly countably additive.

Now each finitely additive ϕ from $\mathscr{C}$ to W induces a linear map $\tilde{\phi}$: $S(\mathscr{C}) \to W$ defined by $\tilde{\phi}(f) = \int f d\phi$. Let τ be the weakest topology on $S(\mathscr{C})$ making $\tilde{\phi}$ continuous for every strongly countably additive ϕ from $\mathscr{C}$ to W for every locally convex space W. Then τ is a locally convex topology on $S(\mathscr{C})$, called the *universal measure topology*. The *universal measure space* $L(\mathscr{C})$ is the τ-completion of $S(\mathscr{C})$. With respect to τ, the measure $\chi\colon \mathscr{C} \to L(\mathscr{C})$ is strongly countably additive.

Let sca($\mathscr{C}$) denote the Banach space of complex-valued strongly countably additive measures on $\mathscr{C}$ with the total variation norm. Since a scalar measure on an algebra of sets is bounded if and only if it is strongly bounded, and since strongly bounded measures on $\mathscr{C}$ must be strongly countably additive, sca($\mathscr{C}$) = ba($\mathscr{C}$), the space of complex-valued bounded additive measures on $\mathscr{C}$. Let sca$(\mathscr{C})^+$ and ba$(\mathscr{C})^+$ denote the spaces of nonnegative measures in sca($\mathscr{C}$) and ba($\mathscr{C}$).

Now sca($\mathscr{C}$) is the τ-dual of $S(\mathscr{C})$, and $L(\mathscr{C})$ may be identified with the norm-dual of sca($\mathscr{C}$) **[8]**. Since $S(\mathscr{C})$ with pointwise multiplication and complex conjugation is an algebra with involution, we may define an Arens multiplication and natural involution on $L(\mathscr{C})$. With its dual norm, $L(\mathscr{C})$ becomes a commutative C^*-algebra with unit in which multiplication is jointly τ-continuous. The set $\mathscr{P}$ of projections (self-adjoint idempotents) in $L(\mathscr{C})$ is a complete Boolean algebra which contains $\chi(\mathscr{C}) \cong \mathscr{C}$. Every increasing (resp. decreasing) net in $\mathscr{P}$ τ-converges to its supremum (resp. infimum), and $\mathscr{P}$ is the τ-closure of $\chi(\mathscr{C})$ in $L(\mathscr{C})$ **[1]**.

Finally, the collection of all polars (taken in $L(\mathscr{C})$) of τ-equicontinuous subsets of sca$(\mathscr{C})^+$ is a neighborhood base for τ on $L(\mathscr{C})$. Each such neighborhood $U = D^\circ$ has the property that P is in U whenever $P \leqq Q$, P and Q are in $\mathscr{P}$, and Q is in U.

2. Submeasures with control measures. Say a submeasure λ on $\mathscr{C}$ has a *control measure* if there is a measure μ in $\mathrm{ba}(\mathscr{C})^+$ such that λ and μ are mutually continuous. In this section we show that a submeasure λ has a control measure if and only if the Fréchet-Nikodým topology G_λ is weaker than the universal measure topology τ on $\mathscr{C}$.

For P and Q in $\mathscr{P}$, define the *symmetric difference* $P\Delta Q$ by $P\Delta Q = P(I-Q) + Q(I-P)$. It is easy to see that the map $(P, Q) \to P\Delta Q$ from $\mathscr{P} \times \mathscr{P}$ to $\mathscr{P}$ is τ-continuous and the map $P \to PQ$ from $\mathscr{P}$ to $\mathscr{P}$ is τ-continuous at 0 uniformly for Q in $\mathscr{P}$. In other words, τ is a Fréchet-Nikodým topology on $\mathscr{P}$, and hence on $\mathscr{C}$.

Combining this result with those of §1, we have the following theorem.

THEOREM 2.1. *With the operation of symmetric difference and the topology* τ, *the set* $\mathscr{P}$ *of projections in* $L(\mathscr{C})$ *is a complete topological group containing* $\mathscr{C}$ *as a dense subgroup.*

Now let λ be a submeasure on $\mathscr{C}$. From the inequality $|\lambda(A) - \lambda(B)| \leqq \lambda(A\Delta B)$ it follows that λ is uniformly τ-continuous on $\mathscr{C}$ if and only if λ is τ-continuous at $\varnothing$.

Say that λ is *χ-continuous*, where χ is the universal measure on $\mathscr{C}$, if for every $\varepsilon > 0$ there is a τ-neighborhood U such that $\lambda(A) < \varepsilon$ whenever χ_A is in U.

LEMMA 2.2. *For a submeasure* λ *on* $\mathscr{C}$, *the following are equivalent*:
(1) λ *is* χ*-continuous*,
(2) λ *is* τ*-continuous on* $\mathscr{C}$,
(3) λ *is uniformly* τ*-continuous on* $\mathscr{C}$, *and*
(4) *the topology* G_λ *is weaker than* τ *on* $\mathscr{C}$.

THEOREM 2.3. *Let* λ *be a submeasure on* $\mathscr{C}$. *If* λ *is* τ*-continuous, then* λ *has a unique* τ*-continuous extension to* $\mathscr{P}$, *and this extension* (*still called* λ) *is a submeasure on* $\mathscr{P}$.

PROOF. By 2.1 and 2.2, the map λ has a unique extension to a uniformly τ-continuous map $\lambda\colon \mathscr{P} \to [0, \infty)$.

Clearly $\lambda(0) = \lambda(\varnothing) = 0$. Now $\mathscr{C} \times \mathscr{C}$ is dense in $\mathscr{P} \times \mathscr{P}$. Since the maps $(P, Q) \to P\Delta Q$ and $(P, Q) \to PQ$ from $\mathscr{P} \times \mathscr{P}$ to $\mathscr{P}$ are τ-continuous and since $P \vee Q = (P\Delta Q)\, \Delta\, PQ$, the map $(P, Q) \to P \vee Q$ from $\mathscr{P} \times \mathscr{P}$ to $\mathscr{P}$ is τ-continuous. It follows that the maps $(P, Q) \to \lambda(P \vee Q) - \lambda(P)$ and $(P, Q) \to \lambda(P) + \lambda(Q) - \lambda(P \vee Q)$ from $\mathscr{P} \times \mathscr{P}$ to $\mathbf{R}$ are τ-continuous. Since λ is a submeasure on $\mathscr{C}$, both maps are nonnegative on $\mathscr{C} \times \mathscr{C}$, hence on $\mathscr{P} \times \mathscr{P}$. If $P \leqq Q$, then $\lambda(P) \leqq \lambda(P \vee Q) = \lambda(Q)$; for all P and Q in $\mathscr{P}$ we have $\lambda(P \vee Q) \leqq \lambda(P) + \lambda(Q)$. Therefore λ is a submeasure on $\mathscr{P}$.

If λ is a submeasure which is τ-continuous on $\mathscr{C}$, then for each P in $\mathscr{P}$ we may define a map $\lambda \circ P$ on $\mathscr{P}$ by $(\lambda \circ P)(Q) = \lambda(PQ)$. It is easy to check that $\lambda \circ P$ is a submeasure on $\mathscr{P}$ and $\lambda \circ P \leqq \lambda$. Since both λ and multiplication by P are τ-continuous, $\lambda \circ P$ is τ-continuous on $\mathscr{P}$.

THEOREM 2.4. *Let λ be a submeasure on $\mathscr{C}$. If λ is τ-continuous, then there is a smallest projection P_λ in $\mathscr{P}$ such that $\lambda \circ P_\lambda = \lambda$ on $\mathscr{P}$.*

PROOF. Put $\mathscr{J} = \{P \in \mathscr{P} \mid \lambda \circ P = \lambda\}$. Then $\mathscr{J}$ is nonempty (since it contains I) and closed under multiplication. So $\mathscr{J}$ is directed downwards by the order on $\mathscr{P}$. Regarded as a net indexed by itself, $\mathscr{J}$ is a decreasing net in $\mathscr{P}$. Put $P_\lambda = \wedge\{P \in \mathscr{P} \mid P \in \mathscr{J}\}$. Then P_λ is the τ-limit of the net $\mathscr{J}$. Fix Q in $\mathscr{P}$. Since λ and multiplication by Q are τ-continuous, we have

$$\begin{aligned}(\lambda \circ P_\lambda)(Q) &= \lambda(P_\lambda Q) = \lim\{\lambda(PQ) | P \in \mathscr{J}\} \\ &= \lim\{(\lambda \circ P)(Q) \mid P \in \mathscr{J}\} = \lambda(Q)\,.\end{aligned}$$

Therefore $\lambda \circ P_\lambda = \lambda$ on $\mathscr{P}$. Clearly P_λ is the smallest such projection.

LEMMA 2.5. *Let λ be a submeasure which is τ-continuous on $\mathscr{C}$. For Q in $\mathscr{P}$, the following are equivalent*:

(1) $P_\lambda Q = 0$,

(2) $\lambda(Q) = 0$, *and*

(3) $\lambda \circ (I - Q) = \lambda$.

PROOF. (1) implies (2) since $\lambda = \lambda \circ P_\lambda$.

(2) implies (3): Suppose $\lambda(Q) = 0$. For all P in $\mathscr{P}$,

$$\begin{aligned}\lambda(P - PQ) &\leqq \lambda(P) = \lambda((P - PQ) \vee PQ) \\ &\leqq \lambda(P - PQ) + \lambda(PQ) \\ &\leqq \lambda(P - PQ) + \lambda(Q) = \lambda(P - PQ)\,.\end{aligned}$$

Then $(\lambda \circ (I - Q))(P) = \lambda(P - PQ) = \lambda(P)$. So $\lambda \circ (I - Q) = \lambda$.

(3) implies (1): Suppose $\lambda \circ (I - Q) = \lambda$. Then $P_\lambda \leqq I - Q$, or $P_\lambda Q \to 0$.

Let λ be a submeasure on $\mathscr{C}$ and P a projection in $\mathscr{P}$. The map $A \to P\chi_A$ from $\mathscr{C}$ to $L(\mathscr{C})$ is strongly countably additive, so P defines a measure on $\mathscr{C}$. Say P is *λ-continuous* on $\mathscr{C}$ if for every τ-neighborhood U there is $\delta > 0$ such that $P\chi_A$ is in U whenever $\lambda(A) < \delta$. Similarly, say λ is *P-continuous* on $\mathscr{C}$ if for every $\varepsilon > 0$ there is a τ-neighborhood V such that $\lambda(A) < \varepsilon$ whenever $P\chi_A$ is in V.

THEOREM 2.6. *Let λ be a submeasure which is τ-continuous on $\mathscr{C}$. Then P_λ is λ-continuous and λ is P_λ-continuous on $\mathscr{C}$.*

PROOF. First we show that P_λ is λ-continuous on $\mathscr{P}$, that is, for every τ-neighborhood U there is $\delta > 0$ such that $P_\lambda P$ is in U whenever $\lambda(P) < \delta$. Suppose not. Then there is an absolutely convex τ-neighborhood U with the property that Q is in U whenever $Q \leqq P$ and P is in U such that for every $n \geqq 1$ there is P_n in $\mathscr{P}$ such that $\lambda(P_n) < (1/2)^n$ but $P_\lambda P_n$ is not in U.

For each $n \geqq 1$, put $Q_n = \bigvee_{i \geqq n} P_i$. Then (Q_n) is a decreasing sequence and $P_n \leqq Q_n$ for all n. Fix $n \geqq 1$. For each $k \geqq 1$, we have

$$\begin{aligned}\lambda(P_n \vee \cdots \vee P_{n+k}) &\leqq \lambda(P_n) + \cdots + \lambda(P_{n+k}) \\ &< 1/2^n + \cdots + 1/2^{n+k} < 1/2^{n-1}.\end{aligned}$$

Since $P_n \vee \cdots \vee P_{n+k} \to Q_n$ with respect to τ, we have $\lambda(P_n \vee \cdots \vee P_{n+k}) \to \lambda(Q_n)$. Then $\lambda(Q_n) \leqq 1/2^{n-1}$. So $\lambda(Q_n) \to 0$.

Put $Q = \bigwedge Q_n$. Since $Q_n \to Q$ with respect to τ, we have $\lambda(Q_n) \to \lambda(Q)$. Then $\lambda(Q) = 0$. By 2.5, $P_\lambda Q = 0$. But $P_\lambda Q_n \to P_\lambda Q$ with respect to τ, so $P_\lambda Q_n$ is eventually in U. Since $P_\lambda P_n \leqq P_\lambda Q_n$ for each n, $P_\lambda P_n$ is eventually in U. This contradiction shows that P_λ is λ-continuous on $\mathscr{P}$, hence on $\mathscr{C}$.

To see that λ is P_λ-continuous on $\mathscr{C}$, let $\varepsilon > 0$. Since λ is τ-continuous on $\mathscr{P}$, there is a τ-neighborhood V such that $\lambda(P) < \varepsilon$ whenever P is in U. So if $P_\lambda \chi_A$ is in U, then $\lambda(A) = (\lambda \circ P_\lambda)(A) = \lambda(P_\lambda \chi_A) < \varepsilon$. This completes the proof.

In [2] it is shown that for every strongly countably additive map ϕ from $\mathscr{C}$ to a complete locally convex space W, there is a projection P_ϕ in $\mathscr{P}$ such that ϕ and P_ϕ are mutually continuous as measures. In particular, for each μ in $\mathrm{sca}(\mathscr{C})^+$ there is a projection P_μ in $\mathscr{P}$ such that μ and P_μ are mutually continuous on $\mathscr{C}$. Moreover, the set $\mathscr{L} = \{P_\mu \in \mathscr{P} | \mu \in \mathrm{sca}(\mathscr{C})^+\}$ is a dense σ-ideal in the Boolean algebra $\mathscr{P}$.

THEOREM 2.7. *Let λ be a submeasure on $\mathscr{C}$. If λ is τ-continuous on $\mathscr{C}$, then λ has a control measure.*

PROOF. We show that $P_\lambda = P_\mu$ for some μ in $\mathrm{ba}(\mathscr{C})^+$. First notice that λ is exhaustive on $\mathscr{P}$. For if (P_i) is a disjoint sequence in $\mathscr{P}$, then $P_i \to 0$ with respect to τ, so $\lambda(P_i) \to 0$. Now a standard argument shows that λ satisfies the countable chain condition on $\mathscr{P}$, that is, if $(P_\alpha)_{\alpha \in I}$ is a disjoint family in $\mathscr{P}$ and $\lambda(P_\alpha) \neq 0$ for all α in I, then I is countable.

Suppose that $(P_\alpha)_{\alpha \in I}$ is a disjoint family of nonzero subprojections of P_λ. For each α in I, $P_\lambda P_\alpha = P_\alpha \neq 0$, so $\lambda(P_\alpha) \neq 0$ by 2.5. Then I is countable. Thus every disjoint family of nonzero subprojections of P_λ is countable.

Since $\mathscr{L}$ is dense in $\mathscr{P}$, every nonzero projection has a nonzero subprojection in $\mathscr{L}$. Using Zorn's lemma, we may find a maximal disjoint family of subprojections of P_λ which are in $\mathscr{L}$. This family must be countable, so we may write it (Q_n). Since $\mathscr{L}$ is dense in $\mathscr{P}$, $\bigvee Q_n = P_\lambda$. Since $\mathscr{L}$ is a σ-ideal, $\bigvee Q_n$ is in $\mathscr{L}$. So P_λ is in $\mathscr{L}$. Then $P_\lambda = P_\mu$ for some μ in $\mathrm{sca}(\mathscr{C})^+ = \mathrm{ba}(\mathscr{C})^+$.

Since μ and P_μ are mutually continuous, by 2.6 λ and μ are mutually continuous on $\mathscr{C}$.

COROLLARY 2.8. *Let λ be a submeasure on $\mathscr{C}$. Then λ has a control measure if and only if the Fréchet-Nikodým topology G_λ is weaker than the universal measure topology τ on $\mathscr{C}$.*

PROOF. Suppose λ has a control measure μ. Then $G_\lambda = G_\mu$, where G_μ is the Fréchet-Nikodým topology μ induces on $\mathscr{C}$. By definition of τ, the linear map $\tilde{\mu}$: $S(\mathscr{C}) \to \mathbf{C}$ is τ-continuous. Then μ is τ-continuous on $\mathscr{C}$. By 2.2 G_μ is weaker than τ.

The converse follows from 2.2 and 2.7.

3. Pathological submeasures. A submeasure λ on $\mathscr{C}$ is *pathological* if λ is nonzero but λ dominates no nonzero measure in ba$(\mathscr{C})^+$. Christensen and Herer **[3]**, **[4]**, among others, have studied pathological submeasures and given examples. Here we apply a theorem of Christensen to show that a nonzero submeasure λ is pathological if and only if the Fréchet-Nikodým topology G_λ is orthogonal to the universal measure topology τ.

The set of all Fréchet-Nikodým topologies on $\mathscr{C}$ with the order relation $\subseteqq$ is a complete lattice **[6]**. The indiscrete and discrete topologies are the least and greatest Fréchet-Nikodým topologies on $\mathscr{C}$.

Let λ and μ be submeasures on $\mathscr{C}$. Say λ and μ are *topologically orthogonal* if for every $\varepsilon > 0$ there is A in $\mathscr{C}$ such that $\lambda(A) < \varepsilon$ and $\mu(X - A) < \varepsilon$. Similarly, say λ and the universal measure χ are topologically orthogonal if for every $\varepsilon > 0$ and every τ-neighborhood U there is A in $\mathscr{C}$ such that $\lambda(A) < \varepsilon$ and χ_{X-A} is in U.

THEOREM 3.1 *For a nonzero submeasure λ on $\mathscr{C}$, the following are equivalent*:

(1) *λ is pathological,*

(2) *λ and μ are topologically orthogonal for every μ in* ba$(\mathscr{C})^+$,

(3) *λ and the universal measure χ are topologically orthogonal, and*

(4) *$G_\lambda \wedge \tau$ is the indiscrete topology,*

PROOF. (1) implies (2) is Theorem 2 **[3]**.

(2) implies (3): Let $\varepsilon > 0$ and let U be a τ-neighborhood. Find a τ-equicontinuous set D in sca$(\mathscr{C})^+ =$ ba$(\mathscr{C})^+$ such that $D^\circ \subseteqq U$. By Theorem 4.1 **[8]** there is ν in ba$(\mathscr{C})^+$ such that D is uniformly ν-continuous. Find $\delta > 0$ such that $\mu(B) \leqq 1$ for all μ in D whenever $\nu(B) < \delta$. Since λ and ν are topologically orthogonal, there is A in $\mathscr{C}$ such that $\lambda(A) < \varepsilon$ and $\nu(X - A) < \delta$. Then $\mu(X - A) \leqq 1$ for all μ in D, whence χ_{X-A} is in D°. Thus $\lambda(A) < \varepsilon$ and χ_{X-A} is in U.

(3) implies (4): Put $G = G_\lambda \wedge \tau$. Let 0 be a G-neighborhood of $\emptyset$. Find a G-neighborhood N of $\emptyset$ such that $\{C \cap B | \, C \in \mathscr{C}\} \subseteqq 0$ whenever

B is in N. Find a G-neighborhood M of $\emptyset$ such that $M \Delta M \subseteqq N$. Since M is a τ-neighborhood of $\emptyset$, there is a τ-neighborhood U in $L(\mathscr{C})$ such that $\{B \in \mathscr{C} | \chi_B \in U\} \subseteqq M$. Since M is a G_λ-neighborhood of $\emptyset$, there is $\varepsilon > 0$ such that $\{B \in \mathscr{C} | \lambda(B) < \varepsilon\} \subseteqq M$. Find A in $\mathscr{C}$ such that $\lambda(A) < \varepsilon$ and χ_{X-A} is in U. Then $X = A\Delta(X - A)$ is in $M\Delta M \subseteqq N$. So $0 = \mathscr{C}$. Therefore G is the indiscrete topology.

(4) implies (1): Suppose that λ is not pathological. Then there is nonzero μ in $\text{ba}(\mathscr{C})^+$ such that $\mu \leqq \lambda$. Then $G_\mu \subseteqq G_\lambda$. As in the proof of 2.8, $G_\mu \subseteqq \tau$. So $G_\mu \subseteqq G_\lambda \wedge \tau$. Then $G_\lambda \wedge \tau$ is not the indiscrete topology.

4. Conclusions. Finally we discuss the connection between these results and the Maharam submeasure problem. Consider the following statements.

(1) Let $\mathscr{B}$ be a σ-complete Boolean algebra and λ a strictly positive continuous submeasure on $\mathscr{B}$. Then there is a strictly positive σ-additive measure on $\mathscr{B}$.

(2) Let $\mathscr{B}$ be a Boolean algebra and λ an exhaustive submeasure on $\mathscr{B}$. Then there is a nonnegative bounded additive measure μ on $\mathscr{B}$ such that λ and μ are mutually continuous.

(3) Let $\mathscr{B}$ be a Boolean algebra and λ an exhaustive submeasure on $\mathscr{B}$. Then λ is not pathological.

In [9] Maharam asked if (1) were true. This problem, still unsolved, is called the control measure problem or Maharam submeasure problem. In an unpublished note [7] Fremlin gave ten statements, including (2) and (3), which are equivalent to (1).

Our results give a way to think about (2) and (3) in terms of lattices. Let λ be a nonzero submeasure on a Boolean algebra $\mathscr{B}$ and let τ be the universal measure topology on the algebra $\mathscr{C}$ of clopen subsets of the Stone space of $\mathscr{B}$. By 2.8 λ has a control measure if and only if $G_\lambda \subseteqq \tau$; by 3.1 λ is pathological if and only if G_λ is orthogonal to τ in the lattice of Fréchet-Nikodým topologies on $\mathscr{C}$.

In particular we have another version of the control measure problem. By 2.8, (2) is equivalent to the following statement.

(4) Let $\mathscr{B}$ be a Boolean algebra and λ an exhaustive submeasure on $\mathscr{B}$. Then λ is τ-continuous on $\mathscr{C}$.

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