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A CLASS OF MEROMORPHIC STARLIKE FUNCTIONS

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ABSTRACT. Let $\Lambda^*(p)$ be the class of functions $f(z)$ univalent and meromorphic in $\Delta = \{z/|z| < 1\}$ with simple pole at $z = p$, $0 < p < 1$, $f(0) = 1$ and which map Δ onto a domain whose complement is starlike with respect to the origin. We discuss the integral means $\int_{-\pi}^{\pi} |f(p + re^{i\theta})|^\lambda d\theta$, $-\infty < \lambda < \infty$, $0 < r < 1 - p$, for a function $f(z)$ in $\Lambda^*(p)$. The results for $\lambda > 0$ are the best possible. Estimates on $\int_{-\pi}^{\pi} | f^{(n)}(p + re^{i\theta})|^\lambda d\theta$, $n = 1, 2, \ldots$, $\lambda \geq 1$ are also obtained.

1. Introduction. Let $\Sigma(p)$ denote the class of functions $f(z)$ which are meromorphic and univalent in $\Delta = \{z/|z| < 1\}$ with a simple pole at $z = p$, $0 < p < 1$, and with $f(0) = 1$. If, further, there exists δ, $p < \delta < 1$, such that $\mathrm{Re}(zf'(z)/f(z)) < 0$ and

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}(zf'(z)/f(z))d\theta = -1$$

for $\delta < |z| < 1$ and $z = re^{i\theta}$, we say that $f(z)$ is in $\Lambda(p)$. We let $\Lambda^*(p)$ denote the class of those functions in $\Sigma(p)$ which map Δ onto a domain whose complement is starlike with respect to the origin. It is obvious that $\Lambda(p) \subset \Lambda^*(p)$. However, if $p \geqq \sqrt{3 - 2\sqrt{2}}$, the containment is proper [5] and if $0 < p < \sqrt{3 - 2\sqrt{2}}$, $\Lambda(p) = \Lambda^*(p)$ [8]. In [1] it was proven that $\Lambda^*(p)$ is equivalent to the class of functions $f(z)$ which have the representation

$$f(z) = -pzg(z)/(z - p)(1 - pz) \tag{1.1}$$

where $g(z)$ is in Σ^*, the class of normalized starlike functions with pole at the origin.

We note at this stage that from (1.1) it is easily seen that

$$F(z) = -p(1 + z)^2/(z - p)(1 - pz)$$

and

$$G(z) = -p(1 - z)^2/(z - p)(1 - pz)$$

are in $\Lambda^*(p)$. In the sequel, $F(z)$ and $G(z)$ will always designate these

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functions.

The purpose of this paper is to discuss the behavior of $f(z)$ in $\Lambda^*(p)$ when z is near the pole $z = p$. Some immediate results can be obtained from the representation (1.1) and known properties of functions in Σ^*. For example, making use of the inequalities $(1 - |z|)^2 \leqq |zg(z)| \leqq (1 + |z|)^2$ we obtain

$$(1.2)\qquad \begin{aligned} p(1 - p - r)^2/r(1 - p^2 + pr) &\leqq |f(p + re^{i\theta})| \\ &\leqq p(1 + p + r)^2/r(1 - p^2 - pr) \end{aligned}$$

for $r < 1 - p$ and $0 \leqq \theta \leqq 2\pi$. Equality is attained on the right side of (1.2) by $F(z)$ at the point $z = p + r$. The left side does not appear to be sharp. It is likely that $|f(p + re^{i\theta})| \geqq \min\{|G(p + re^{i\theta})| : 0 \leqq \theta \leqq 2\pi\}$.

If $f(z)$ is in $\Lambda^*(p)$ and $f(z) = \sum_{n=-1}^{\infty} b_n(z - p)^n$ for $|z - p| < 1 - p$, then making use of (1.2) and the integral formula for b_n it follows that $|b_n| = O((1 - p)^{-(n+2)})$ as $p \to 1$. However, this result is also true for the larger class of functions in $\Sigma(p)$ which are different from 0, as we now prove.

THEOREM 1. *If $f(z)$ is in $\Sigma(p)$ with $f(z) \neq 0$ and $f(z) = \sum_{n=-1}^{\infty} b_n(z - p)^n$ for $|z - p| < (1 - p)$, then $|b_n| = O((1 - p)^{-(n+2)})$ as $p \to 1$. The order estimate is best possible.*

Proof. Let Σ be the class of functions $g(z)$ analytic and univalent for $0 < |z| < 1$ with a simple pole of residue one at the origin. If $f(z)$ is in $\Sigma(p)$ and $f(z) \neq 0$, it is easily seen that $f(z) = (b_{-1}/1 - p^2)g((z - p)/(1 - pz))$ where $g(z)$ is in Σ and $g(z) \neq 0$. Let $w = (z - p)/(1 - pz)$ and $z = p + re^{i\theta}$, then for $r < 1 - p$, $|w| \geqq r/3(1 - p)$. Since $g(z)$ is in Σ and $g(z) \neq 0$, $|g(w)| \leqq (1 + |w|)^2/|w| \leqq 3(1 - p)/r + 3$. Therefore

$$|f(p + re^{i\theta})| \leqq \frac{|b_{-1}|}{1 - p^2}\left[\frac{3(1 - p)}{r} + 3\right] \leqq \frac{2}{(1 - p)^2}\left[\frac{3(1 - p)}{r} + 3\right].$$

where we have used the fact that $|b_{-1}| \leqq p(1 + p)/(1 - p) \leqq 2/(1 - p)$. This can be seen by noting that $(f(z) - 1)/f'(0)$ is in $S(p)$, a class discussed by Kirwan and Schober [4]. They proved that the residue of a function in $S(p)$ is bounded in absolute value by $p^2/(1 - p^2)$. Combining this with the fact that $|f'(0)| \leqq (1 + p)^2/p$ [5] gives the bound on $|b_{-1}|$. Since $b_n = 1/2\pi \int_{-\pi}^{\pi} f(p + re^{i\theta})/r^n e^{in\theta} d\theta$ we obtain $|b_n| \leqq 2/(1 - p)^2\,[3(1 - p)/r + 3]\,1/r^n$. Letting $r \to (1 - p)$ we obtain $|b_n| \leqq 12/(1 - p)^{n+2}$.

To see that the order is best possible, we note that if $F(z) = \sum_{n=-1}^{\infty} b_n(z - p)^n$, $|z - p| < (1 - p)$, then $b_n = -p^n/(1 - p)^{n+2}(1 + p)^n$ for $n \geqq 1$.

2. Integral means of f(z). The integral means

$$\int_{-\pi}^{\pi} |f^{(n)}(re^{i\theta})|^{\lambda} d\theta$$

have been discussed in [1] and [6].

In this section we consider the integral means $\int_{-\pi}^{\pi}|f(p + re^{i\theta})|^{\lambda}d\theta$ for $f(z)$ in $\Lambda^*(p)$ and $-\infty < \lambda < \infty$. We will prove that $F(z)$ maximizes the integral means when $\lambda > 0$ and conjecture that $G(z)$ maximizes the integral means when $\lambda < 0$. We will make use of the following result [3]. If $f(x)$ is nonnegative and measurable on $[-a, a]$, let $f^*(x)$ denote its symmetrically decreasing rearrangement as defined in [3, p. 278].

LEMMA 1. *If $f(x)$ and $g(x)$ are nonnegative integrable functions on the interval $[-a, a]$, then $\int_{-a}^{a} f(x)g(x)dx \leqq \int_{-a}^{a} f^*(x)g^*(x)dx$.*

THEOREM 2. *If $f(z)$ is in $\Lambda^*(p)$, then*

$$\int_{-\pi}^{\pi} |f(p + re^{i\theta})|^{\lambda} d\theta \leqq \int_{-\pi}^{\pi} |F(p + re^{i\theta})|^{\lambda} d\theta \tag{2.1}$$

for $0 < r < 1 - p$ and $\lambda > 0$.

PROOF. Since $f(z)$ is in $\Lambda^*(p)$ it has the representation (1.1). Since $g(z)$ is in Σ^* there exists $m(t)$ increasing on $[-\pi, \pi]$ with $\int_{-\pi}^{\pi} dm(t) = 1$ such that

$$g(z) = \frac{1}{z} \exp \int_{-\pi}^{\pi} \log(1 - e^{-it}z)^2 dm(t) .$$

Thus

$$f(z) = \frac{-p}{(z - p)(1 - pz)} \exp \int_{-\pi}^{\pi} \log(1 - e^{-it}z)^2 dm(t) .$$

Making use of the continuous form of the arithmetic geometric mean inequality [11], we have

$$\begin{aligned} &|f(p + re^{i\theta})|^{\lambda} \\ &= \frac{p^{\lambda}}{r^{\lambda}|1 - p^2 - pre^{i\theta}|^{\lambda}} \exp \int_{-\pi}^{\pi} \log|1 - pe^{-it} - re^{i(\theta - t)}|^{2\lambda} dm(t) \qquad (2.2)\\ &\leqq \frac{p^{\lambda}}{r^{\lambda}|1 - p^2 - pre^{i\theta}|^{\lambda}} \int_{-\pi}^{\pi} |1 - pe^{-it} - re^{i(\theta - t)}|^{2\lambda} dm(t). \end{aligned}$$

Integrating (2.2) over $[-\pi, \pi]$ with respect to θ and changing the order of integration we obtain

$$\int_{-\pi}^{\pi} |f(p + re^{i\theta})|^{\lambda} d\theta \leqq \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{p^{\lambda}|1 - pe^{-it} - re^{i(\theta - t)}|^{2\lambda}}{r^{\lambda}|1 - p^2 - pre^{i\theta}|^{\lambda}} d\theta dm(t) . \tag{2.3}$$

We let

$$I(t) = \int_{-\pi}^{\pi} |1 - pe^{-it} - re^{i(\theta - t)}|^{2\lambda} / |1 - p^2 - pre^{i\theta}|^{\lambda} d\theta$$

and note that the theorem will be proven if we can prove that

$$(2.4) \qquad I(t) \leqq \int_{-\pi}^{\pi} |1 + p + re^{i\theta}|^{2\lambda}/|1 - p^2 - pre^{i\theta}|^{\lambda} d\theta$$

for $-\pi \leqq t \leqq \pi$. We make use of Lemma 1 to prove (2.4). The rearrangement of $|1 - p^2 - pre^{i\theta}|^{-\lambda}$ is itself and the rearrangement of $|1 - pe^{-it} - re^{i(\theta-t)}|^{\lambda}$ is $\big||1 - pe^{-it}| + re^{i\theta}\big|^{\lambda}$ for any fixed t, $-\pi \leqq t \leqq \pi$. Thus by Lemma 1

$$(2.5) \qquad I(t) \leqq \int_{-\pi}^{\pi} \big||1 - pe^{-it}| + re^{i\theta}|^{2\lambda}/|1 - p^2 - pre^{i\theta}|^{\lambda} d\theta\,.$$

It is easily seen that $\big||1 - pe^{-it}| + re^{i\theta}\big| \leqq |1 + p + re^{i\theta}|$ for any fixed t, $-\pi \leqq t \leqq \pi$, and thus (2.4) follows from (2.5).

The methods of Theorem 1 will not quite give the best possible estimates on $\int_{-\pi}^{\pi}|f(p + re^{i\theta})|^{\lambda} d\theta$ when $\lambda < 0$. One suspects that $\int_{-\pi}^{\pi}|f(p + re^{i\theta})|^{\lambda} d\theta \leqq \int_{-\pi}^{\pi}|G(p + re^{i\theta})|^{\lambda} d\theta$ where $\lambda < 0$ and $G(z) = -p(1 - z)^2/(z - p)(1 - pz)$. The next theorem comes very close to giving this.

THEOREM 3. *If $f(z)$ is in $\Lambda^*(p)$, then*

$$\int_{-\pi}^{\pi} |f(p + re^{i\theta})|^{\lambda} d\theta \leqq \frac{p^{\lambda}}{r^{\lambda}} \int_{-\pi}^{\pi} \frac{|1 - p - re^{i\theta}|^{2\lambda}}{|1 - p^2 + pre^{i\theta}|^{\lambda}} d\theta$$

for $\lambda < 0$ and $0 < r < 1 - p$.

REMARK. The only difference between the inequality given in Theorem 3 and the conjectured inequality is that the term $|1 - p^2 + pre^{i\theta}|^{\lambda}$ would be replaced by $|1 - p^2 - pre^{i\theta}|^{\lambda}$.

PROOF. As in Theorem 2 we have the existence of $m(t)$ increasing on $[-\pi, \pi]$ with $\int_{-\pi}^{\pi} dm(t) = 1$ such that

$$f(z) = \frac{-p}{(z - p)(1 - pz)} \exp \int_{-\pi}^{\pi} \log(1 - e^{-it}z)^2 dm(t)\,.$$

Letting $\mu > 0$ and again making use of the continuous form of the arithmetic geometric mean inequality we have

$$|f(p + re^{i\theta})|^{-\mu} \leqq \frac{r^{\mu}|1 - p^2 - pre^{i\theta}|^{\mu}}{p^{\mu}} \cdot \int_{-\pi}^{\pi} |1 - pe^{-it} - r^{i(\theta-t)}|^{-2\mu} dm(t)\,.$$

Thus

$$\int_{-\pi}^{\pi} |f(p + re^{i\theta}|^{-\mu} d\theta \leqq \frac{r^{\mu}}{p^{\mu}} \int_{-\pi}^{\pi} I(t) dm(t)$$

where

$$I(t) = \int_{-\pi}^{\pi} |1 - p^2 - pre^{i\theta}|^{\mu}/|1 - pe^{-it} - re^{i(\theta - t)}|^{2\mu} d\theta$$

for $-\pi \leqq t \leqq \pi$. We note at this stage that if one could prove that $I(t) \leqq I(0)$ for $-\pi \leqq t \leqq \pi$, then the conjectured inequality would follow. It can be proven that $I(0)$ is a local maximum of $I(t)$ but the author was unable to prove that it is an absolute maximum. However an application of lemma 1 gives us the inequality of Theorem 3. We note that the rearrangement of $|1 - p^2 - pre^{i\theta}|^{\mu}$ is $|1 - p^2 + pre^{i\theta}|^{\mu}$ and the rearrangement of $|1 - pe^{-it} - re^{i(\theta-t)}|^{-2\mu}$ is $\big||1 - pe^{-it}| - re^{i\theta}\big|^{-2\mu}$ for each fixed t, $-\pi \leqq t \leqq \pi$. Thus by Lemma 1

$$I(t) \leqq \int_{-\pi}^{\pi} |1 - p^2 + pre^{i\theta}|^{\mu}/\big||1 - pe^{-it}| - re^{i\theta}\big|^{2\mu} d\theta\,.$$

It is easily proven that $\big||1 - pe^{-it}| - re^{i\theta}\big| \geqq |1 - p - re^{i\theta}|$ for $-\pi \leqq t \leqq \pi$. Therefore

$$I(t) \leqq \int_{-\pi}^{\pi} |1 - p^2 + pre^{i\theta}|^{\mu}/|1 - p - re^{i\theta}|^{2\mu} d\theta\,,$$

giving us

$$\int_{-\pi}^{\pi} |f(p + re^{i\theta})|^{-\mu}\, d\theta \leqq \frac{r^{\mu}}{p^{\mu}} \int_{-\pi}^{\pi} \frac{|1 - p^2 + pre^{i\theta}|^{\mu}}{|1 - p - re^{i\theta}|^{2\mu}} d\theta$$

for $\mu > 0$. This is equivalent to the inequality given in the theorem.

3. Integral means of the derivative and arc length. In this section we consider estimates on the integral means $\int_{-\pi}^{\pi} |f'(p + re^{i\theta})|^{\lambda} d\theta$, $0 < r < 1 - p$, $\lambda \geqq 1$. For the case $\lambda > 1$ the order estimates obtained are the best possible. The case $\lambda = 1$ gives us some information on the arc length of the image of the circle $|z - p| = r$, $0 < r < 1 - p$, for a function $f(z)$ in $\Lambda^*(p)$. Sharp results concerning the length of the image of $|z| = r$, $0 < r < 1$, were obtained in [1].

We will make use of the following result of Pommerenke [9]. For $0 < r < 1$,

$$\text{(3.1)} \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{|1 - re^{i\theta}|^{\mu}} \sim \begin{cases} \dfrac{2^{-\mu+1}\Gamma(\mu - 1)}{[\Gamma(\mu/2)]^2(1 - r)^{\mu-1}}, & \mu > 1 \\ (1/\pi) \log(1/(1 - r)), & \mu = 1\,. \end{cases}$$

This implies the existence of positive constants C_{μ} and C so that

$$\text{(3.2)} \qquad \int_{-\pi}^{\pi} \frac{d\theta}{|1 - re^{i\theta}|^{\mu}} \leqq \begin{cases} C_{\mu}(1 - r)^{-(\mu-1)}, & \mu > 1 \\ C_1 \log(1/(1 - r)) + C, & \mu = 1\,. \end{cases}$$

In what follows K and C represent constants which are independent of $f(z)$ and r, though they may change their values from line to line.

THEOREM 4. *If $f(z)$ is in $\Lambda^*(p)$, then for* $0 < r < 1-p$

$$(3.3)\quad \int_{-\pi}^{\pi} |f'(p + re^{i\theta})|^\lambda d\theta \leqq \begin{cases} \dfrac{C_\lambda}{r^{2\lambda}(1-p-r)^{\lambda-1}}, & \lambda > 1 \\ (1/r^2)[C_1 \log(1/(1-p-r)) + C], & \lambda = 1, \end{cases}$$

where C_λ and C are constants dependent on p but independent of $f(z)$ and r. Moreover $|C_\lambda| \leqq D_\lambda(1-p)^{-3\lambda}$ where D_λ is a constant depending only on λ.

PROOF. As observed in [1] the function $P(z) = -(z-p)(1-pz)f'(z)/f(z)$ has positive real part in Δ with $P(0) = pf'(0)$. Hence

$$(3.4)\quad f'(z) = -f(z)P(z)/(z-p)(1-pz) = pzg(z)P(z)/(z-p)^2(1-pz)^2$$

where $g(z)$ is in Σ^*. Using the fact that $|g(z)| \leqq (1 + |z|)^2/|z|$ we obtain for $0 < r < 1-p$,

$$\begin{aligned} |f'(p + re^{i\theta})| &\leqq \frac{p(1 + |p + re^{i\theta}|^2)|P(p + re^{i\theta})|}{r^2|1 - p^2 - pre^{i\theta}|^2} \\ &\leqq \frac{p(1 + p + r)^2}{r^2(1 - p^2 - pr)^2} |P(p + re^{i\theta})| . \\ &\leqq \frac{4}{r^2(1 - p)^2} |P(p + re^{i\theta})| . \end{aligned}$$

Thus for $0 < r < 1-p$,

$$(3.5)\quad \int_{-\pi}^{\pi} |f'(p + re^{i\theta})|^\lambda d\theta \leqq \frac{4^\lambda}{r^{2\lambda}(1 - p)^{2\lambda}} \int_{-\pi}^{\pi} |P(p + re^{i\theta})|^\lambda d\theta .$$

Since Re $P(z) > 0$ for z in Δ with $P(0) = pf'(0)$, it follows from the Herglotz representation for normalized functions of positive real part [10] that there exists an increasing function $m(t)$ on $[-\pi, \pi]$ with $\int_{-\pi}^{\pi} dm(t) = 1$ such that $P(z) = \int_{-\pi}^{\pi}(A + \bar{A}e^{-it}z)/(1-e^{-it}z)dm(t)$ where $A = pf'(0)$. Using Hölder's inequality we have for $\lambda \geqq 1$

$$(3.6)\quad \begin{aligned} |P(p + re^{i\theta})|^\lambda &\leqq \int_{-\pi}^{\pi} \frac{|A + \bar{A}pe^{-it} + \bar{A}re^{i(\theta-t)}|^\lambda}{|1 - pe^{-it} - re^{i(\theta-t)}|^\lambda} dm(t) \\ &\leqq 2^\lambda |A|^\lambda \int_{-\pi}^{\pi} \frac{1}{|1 - pe^{-it} - re^{i(\theta-t)}|^\lambda} dm(t) . \end{aligned}$$

Integrating (3.6) with respect to θ and interchanging the order of integration we obtain

$$(3.7)\quad \begin{aligned} &\int_{-\pi}^{\pi} |P(p + re^{i\theta})|^\lambda d\theta \\ &\leqq 2^\lambda |A|^\lambda \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{1}{|1 - pe^{-it} - re^{i(\theta-t)}|^\lambda} d\theta dm(t) . \end{aligned}$$

We will make use of Lemma 1 to estimate the integral on the right side of (3.7). For each fixed t the rearrangement of $|1 - pe^{-it} - re^{i(\theta-t)}|^{-\lambda}$ is $\big||1 - pe^{-it}| - re^{i\theta}\big|^{-\lambda}$. Thus by Lemma 1

$$(3.8)\quad \int_{-\pi}^{\pi} \frac{1}{|1 - pe^{-it} - re^{i(\theta-t)}|^{\lambda}}\, d\theta \leqq \int_{-\pi}^{\pi} \frac{1}{\big||1 - pe^{-it}| - re^{i\theta}\big|^{\lambda}}\, d\theta .$$

However, for each fixed t, $-\pi \leqq t \leqq \pi$, $\big||1 - pe^{-it}| - re^{i\theta}\big| \geqq |1 - p - re^{i\theta}|$ for $0 < r < 1 - p$. Thus from (3.8), (3.7) and (3.2) we obtain

$$(3.9)\quad \begin{aligned} \int_{-\pi}^{\pi} |P(p + re^{i\theta})|^{\lambda}\, d\theta &\leqq 2^{\lambda}|A|^{\lambda} \int_{-\pi}^{\pi} \frac{1}{|1 - p - re^{i\theta}|^{\lambda}}\, d\theta \\ &= \frac{2^{\lambda}|A|^{\lambda}}{(1 - p)^{\lambda}} \int_{-\pi}^{\pi} \frac{1}{|1 - (r/(1-p))e^{i\theta}|^{\lambda}}\, d\theta \\ &\leqq \begin{cases} C_{\lambda}/(1 - p - r)^{\lambda-1}, & \lambda > 1 \\ C_1 \log(1/(1 - p - r)) + C, & \lambda = 1 \end{cases} \end{aligned}$$

where we have used the fact that $|A| \leqq (1 + p)^2$ [5]. Combining (3.5) and (3.9) gives (3.3.)

We now consider the sharpness for the case $\lambda > 1$. Since $f'(z)$ has a pole of order two at $z = p$ it is easily seen that the term $1/r^{2\lambda}$ is necessary. We will now prove that the exponent $(\lambda - 1)$ on $(1-p-r)$ cannot be replaced by a smaller exponent. For this purpose we use a function which was used in [1]. It is easily seen that the function $g(z) = (1 - z)^s(1 - pz)/z$, $0 \leqq s \leqq 1$ is a member of Σ^*. Thus the function $f(z) = -p(1 - z)^s/(z - p)$, $0 \leqq s \leqq 1$ is a member of $\Lambda^*(p)$, and

$$f'(p + re^{i\theta}) = pr^{-2}(1 - p - re^{i\theta})^{s-1}(1 - p - (1 - s)re^{i\theta}) .$$

Let δ, $0 < \delta < \lambda - 1$, be given and chose s so that $0 < s < (\lambda - 1 - \delta)/\lambda$. With s fixed and $(1-p)/2 < r < (1-p)$ we have

$$\begin{aligned} \int_{-\pi}^{\pi} |f'(p + re^{i\theta})|^{\lambda} d\theta &= (p^{\lambda}/r^{\lambda}) \int_{-\pi}^{\pi} \frac{|1 - p - (1 - s)re^{i\theta}|^{\lambda}}{|1 - p - re^{i\theta}|^{\lambda-\lambda s}}\, d\theta \\ &\geqq C \int_{-\pi}^{\pi} \frac{1}{|1 - p - re^{i\theta}|^{\lambda-\lambda s}}\, d\theta . \end{aligned}$$

By the choice of s, $\lambda - \lambda s > 1$, and thus by (3.1)

$$\int_{-\pi}^{\pi} |f'(p + re^{i\theta})|^{\lambda} d\theta \geqq C/(1 - p - r)^{\lambda-\lambda s-1} .$$

Therefore, by the choice of s,

$$\lim_{r\to(1-p)} (1 - p - r)^{\delta} \int_{-\pi}^{\pi} |f'(p + re^{i\theta})|^{\lambda} d\theta = \infty .$$

Let $L(r)$ be the length of the image of the circle $|z - p| = r$ for a function $f(z)$ in $\Lambda^*(p)$. The case $\lambda = 1$ of Theorem 4 gives us the following corollary.

COROLLARY. *If $f(z)$ is in $\Lambda^*(p)$, then*

$$L(r) = \int_{-\pi}^{\pi} r|f'(p + re^{i\theta})|d\theta = O\left(\frac{1}{r}\log(1/(1-p-r)\right).$$

4. Integral means of higher order derivatives. We will make use of a method employed by Feng and MacGregor [2] and also utilized in [1], to discuss the integral means $\int_{-\pi}^{\pi}|f^{(n)}(p + re^{i\theta})|^{\lambda}d\theta$, $\lambda \geqq 1$, $0 < r < (1 - p)$. We first need two lemmas similar to those proven in [1]. Again in this section letters signifying constants do not necessarily have the same value each time they appear.

LEMMA 2. *Let $0 < p < 1$ and $h(z)$ be analytic for $0 < |z - p| < 1-p$. If there exist positive constants A, α and β such that*

$$|h(p + re^{i\theta})| \leqq A/r^{\alpha}(1 - p - r)^{\beta} \tag{4.1}$$

for $0 < r < 1-p$ and $-\pi \leqq \theta \leqq \pi$, then there exists a positive constant B so that

$$|h'(p + re^{i\theta})| \leqq B/r^{\alpha+1}(1 - p - r)^{\beta+1} \tag{4.2}$$

for $0 < r < 1 - p$ and $-\pi \leqq \theta \leqq \pi$.

PROOF. Let $f(z) = h(p + (1 - p)z)$, then $f(z)$ is analytic for $0 < |z| < 1$ and (4.1) implies that $|f(z)| \leqq C|z|^{-\alpha}(1 - |z|)^{-\beta}$. Therefore the analytic function $g(z) = z^{\alpha}f(z)$ satisfies $|g(z)| \leqq C(1 - |z|)^{-\beta}$. Thus $|g'(z)| \leqq C(1 - |z|)^{-(\beta+1)}$ [7]. The last inequality implies that $|f'(z)| \leqq C|z|^{-(\alpha+1)}(1 - |z|)^{-(\beta+1)}$ which implies (4.2).

LEMMA 3. *Let $0 < p < 1$ and let $h(z)$ be analytic for $0 < |z - p| < 1-p$. If there exists a constant A_1 such that*

$$|h'(p + re^{i\theta})/h(p + re^{i\theta})| \leqq A_1/r(1 - p - r) \tag{4.3}$$

for $0 < r < 1-p$ and $-\pi \leqq \theta \leqq \pi$, then for each $n = 1, 2, \ldots$ there exists a constant A_n such that

$$|h^{(n)}(p + re^{i\theta})/h(p + re^{i\theta})| \leqq A_n/r^n(1 - p - r)^n \tag{4.4}$$

for $0 < r < 1-p$ and $-\pi \leqq \theta \leqq \pi$.

PROOF. Assume that (4.4) holds for some n. Let $g(z) = h^{(n)}(z)/h(z)$, then by lemma 2

$$|g'(p + re^{i\theta})| \leqq B_n/r^{n+1}(1 - p - r)^{n+1} \tag{4.5}$$

for some constant B_n. Since

$$h^{(n+1)}(z)/h(z) = g'(z) + h^{(n)}(z)h'(z)/h(z)^2,$$

we have, making use of (4.3), (4.4) and (4.5)

$$\left|\frac{h^{(n+1)}(p+re^{i\theta})}{h(p+re^{i\theta})}\right|$$
$$\leqq \frac{B_n}{r^{n+1}(1-p-r)^{n+1}} + \frac{A_n}{r^n(1-p-r)^n}\frac{A_1}{r(1-p-r)}$$
$$\leqq \frac{A_{n+1}}{r^{n+1}(1-p-r)^{n+1}}$$

for some constant A_{n+1}. This completes the proof of Lemma 3 by induction.

The following lemma is similar to one proven in [1].

LEMMA 4. *If $f(a)$ is in $\Sigma(p)$ and $f(z) \neq 0$, there exists a positive constant A such that*

$$|f''(p+re^{i\theta})/f'(p+re^{i\theta})| \leqq A/r(1-p-r)$$

for $0 < r < 1-p$ and $-\pi \leqq \theta \leqq \pi$.

PROOF. If $F(z)$ is in Σ and $F(z) \neq 0$, then $|zF''(z)/F'(z)| \leqq 10/(1-|z|)$. This inequality can be obtained by noting that $g(z) = 1/F(z)$ is in S, the class of functions analytic and univalent for $|z| < 1$ with $g(0) = 0$ and $g'(0) = 1$. Since $zF''(z)/F'(z) = zg''(z)/g'(z) - 2zg'(z)/g(z)$, applying well known bounds for functions in S gives the desired inequality on $|zF''(z)/F'(z)|$. Applying this inequality to the function $F(z) = f(p + (1-p)z)$ gives Lemma 4.

THEROEM 5. *If $f(z)$ is in $\Lambda^*(p)$, then for $\lambda \geqq 1, n = 1, 2, \ldots,$ and $0 < r < 1-p$*

$$\int_{-\pi}^{\pi} |f^{(n)}(p+re^{i\theta})|^\lambda d\theta$$
$$\leqq \begin{cases} \dfrac{C_\lambda}{r^{(n+1)\lambda}(1-p-r)^{n\lambda-1}}, & \lambda > 1 \\[2ex] \dfrac{A_1}{r^{n-1}(1-p-r)^{n-1}}\left[\dfrac{B_1}{r^2}\log\left(\dfrac{1-p}{1-p-r}\right) + C_1\right], & \lambda = 1 \end{cases}$$

where C_λ, A_1 and B_1 are constants independent of $f(z)$ and r.

PROOF. Let $h(z) = f'(z)$, then by Lemma 4

$$|h'(p+re^{i\theta})/h(p+re^{i\theta})| \leqq A/r(1-p-r)$$

for $0 < r < 1-p$. Thus by Lemma 3

$$|h^{(n-1)}(p+re^{i\theta})/h(p+re^{i\theta})| \leqq A_{n-1}/r^{n-1}(1-p-r)^{n-1}$$

or

$$|f^{(n)}(p + re^{i\theta})/f'(p + re^{i\theta})| \leqq A_{n-1}/r^{n-1}(1 - p - r)^{n-1}$$

for some constant A_{n-1} and $0 < r < 1-p$.
Thus

$$\int_{-\pi}^{\pi} |f^{(n)}(p + re^{i\theta})|^{\lambda} d\theta \leqq \frac{A^{\lambda}_{n-1}}{r^{(n-1)\lambda}(1 - p - r)^{(n-1)\lambda}} \int_{-\pi}^{\pi} |f'(p + re^{i\theta})|^{\lambda} d\theta .$$

An application of Theorem 4 then gives Theorem 5.

Since $f(z)$ has a simple pole at $z = p$, it is easily seen that the exponent $(n + 1)\lambda$ on r cannot be reduced. We now prove that for $\lambda > 1$ the exponent $n\lambda - 1$ on $(1-p-r)$ cannot be reduced. For this purpose we use once again the function $f(z) = (1 - z)^s/(z - p)$, $0 \leqq s \leqq 1$.

LEMMA 5. *Let* $f(z) = (1 - z)^s/(z - p)$ *and* $g(z) = (z - p)f(z) = (1 - z)^s$, $0 < s < 1$. *Then for s sufficiently close to* 0, *there exist* $r(s) < 1 - p$ *and a constant k so that*

$$\int_{-\pi}^{\pi} |f^{(n)}(p + re^{i\theta})|^{\lambda} d\theta \geqq k \int_{-\pi}^{\pi} |g^{(n)}(p + re^{i\theta})|^{\lambda} d\theta$$

for $r(s) < r < 1 - p$.

PROOF. We have

$$f^{(n)}(z) = \frac{(-1)^n g^{(n)}(z) P(z)}{s(s - 1) \cdots (s - n + 1)(z - p)^{n+1}}$$

where

$$P(z) = (1 - z)^n + \sum_{k=1}^{n} (n!/k!) s(s - 1) \cdots (s - k + 1)(1 - z)^{n-k}(z - p)^k.$$

Thus

$$f^{(n)}(p + re^{i\theta}) = \frac{(-1)^n g^{(n)}(p + re^{i\theta}) P(p + re^{i\theta})}{s(s - 1) \cdots (s - n + 1) r^{n+1} e^{i(n+1)\theta}}$$

and hence there exists a positive constant C so that

$$(4.6) \qquad |f^{(n)}(p + re^{i\theta})| \geqq C|g^{(n)}(p + re^{i\theta})|\,|P(p + re^{i\theta})|$$

for $0 < s < 1$.
Now

$$P(p + re^{i\theta}) = (1 - p - re^{i\theta})^n$$
$$+ \sum_{k=1}^{n} (n!/k!)s(s - 1) \cdots (s - k + 1)(1 - p - re^{i\theta})^{n-k} r^k e^{ik\theta}$$

Since $P(1) \neq 0$, there exists α so that $P(p + (1 - p)e^{i\theta}) \neq 0$ for $|\theta| < \alpha$. If $|\theta| \geqq \alpha$, there exists γ so that $|1 - p - (1 - p)e^{i\theta}|^n \geqq \gamma$. Moreover

$$|P(p + (1 - p)e^{i\theta}) - (1 - p - (1 - p)e^{i\theta})^n|$$
$$\leqq \sum_{k=1}^{n} n!|s(s - 1) \cdots (s - k + 1)| 2^{n-k}(1 - p)^n .$$

Since the right side of the above inequality approaches zero as s approaches zero, it follows that there exists δ so that

$$|P(p + (1 - p)e^{i\theta} - (1 - p - (1 - p)e^{i\theta})^n| < \gamma/2$$

for $0 < s < \delta$ and all θ. Thus

$$|P(p + (1 - p)e^{i\theta})| \geqq |1 - p - (1 - p)e^{i\theta}|^n - \gamma/2 \geqq \gamma/2$$

if $|\theta| \geqq \alpha$ and $0 < s < \delta$. Therefore $P(p + (1 - p)e^{i\theta}) \neq 0$ for all θ if $0 < s < \delta$. Thus for each fixed s, $0 < s < \delta$. there exists $r(s) < 1 - p$ such that $P(p + re^{i\theta}) \neq 0$ for all θ and $r(s) \leqq r \leqq 1 - p$. Therefore there exists $D(s)$ so that $|P(p + re^{i\theta})| \geqq D(s) > 0$ for $r(s) \leqq r \leqq 1 - p$. Thus from (4.6) if follows that there exists a constant $K(s)$ so that

$$\int_{-\pi}^{\pi} |f^{(n)}(p + re^{i\theta})|^\lambda d\theta \geqq K(s) \int_{-\pi}^{\pi} |g^{(n)}(p + re^{i\theta})|^\lambda d\theta \tag{4.7}$$

for $0 < s < \delta$ and $r(s) < r < 1 - p$. This then proves the lemma.

Since sharpness of the exponent $n\lambda - 1$ in Theorem 5 was discussed earlier for $n = 1$ and $\lambda > 1$, we restrict our attention to $n \geqq 2$ and $\lambda \geqq 1$. Let $\lambda \geqq 1$ be given and let $\gamma < n\lambda - 1$. Choose s so that $0 < s < \min[1, n - (\gamma + 1)/\lambda]$ and also close enough to 0 so that (4.7) holds. Then with s fixed

$$\int_{-\pi}^{\pi} |g^{(n)}(p + re^{i\theta})|^\lambda d\theta$$
$$= (s|s - 1| \cdots |s - n + 1|)^\lambda \int_{-\pi}^{\pi} \frac{1}{|1 - p - re^{i\theta}|^{\lambda(n-s)}} d\theta$$
$$\geqq C/(1 - p - r)^{\lambda(n-s)-1}$$

for some constant C and r sufficiently close to $(1 - p)$. Here we have used the fact that $\lambda(n - s) > 1$ and (3.1). Since (4.7) holds we have

$$\int_{-\pi}^{\pi} |f^{(n)}(p + re^{i\theta})|^\lambda d\theta \geqq CK/(1 - p - r)^{\lambda(n-s)-1}$$

for r sufficiently close to $(1-p)$. Thus

$$\lim_{r\to(1-p)} (1-p-r)^{\gamma} \int_{-\pi}^{\pi} |f^{(n)}(p+re^{i\theta})|^{\lambda} d\theta = \infty$$

by the choice of s.

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