ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 13, Number 1, Winter 1983

REACTION-DIFFUSION EQUATIONS FOR COMPETING SPECIES SINGULARLY PERTURBED BY A SMALL DIFFUSION RATE

ANTHONY LEUNG

1. Introduction and preliminaries. This article considers the system of reaction-diffusion equations

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} &= \Delta u_1 + u_1[a - bu_1 - cu_2] \\
\frac{\partial u_2}{\partial t} &= \varepsilon \Delta u_2 + u_2[e - fu_1 - gu_2]
\end{aligned} \tag{1.1}
$$

with prescribed initial condition at $t = 0$, and boundary Dirichlet condition at the boundary of a fixed space domain for all $t \geqq 0$. Here, a, b, c, e, f, g and ε are positive constants and $\Delta = \sum_{i=1}^{n}(\partial^2/\partial x_i^2)$. The system is a model for two competing biological species with Volterra-Lotka type of reaction. We will discuss the situation when $\varepsilon > 0$ is small, describing the behavior when the diffusion rate of the second species u_2 is small compared with that of the first species u_1.

For large time, the usual formal singular perturbation procedure is to set $\varepsilon = 0$ and $\partial u_2/\partial t = 0$, solve u_2 in terms of u_1 in the equation

$$u_2[e - fu_1 - gu_2] = 0, \tag{1.2}$$

and substitute back into the first equation in (1.1). One then analyzes the resulting scaler equation for u_1 alone and finally uses (1.2) again to study the behavior of u_2. This procedure reduces the study of the full system (1.1) to that of a scalar equation, and is therefore of significant simplification for numerical as well as analytical investigation.

One difficulty for our problem is that (1.2) describes two natural solutions of u_2 in terms of u_1, namely:

$$u_2 = 0, \tag{1.3}$$

or

$$u_2 = g^{-1}(e - fu_1). \tag{1.4}$$

This research was partially supported by a grant from the National Science Foundation, MCS 80-01851.

Received by the editors on June 5, 1981.

There is, therefore, a choice between them when we express u_2 in terms of u_1 in the first equation in (1.1). It turns out that the appropriate procedure is to switch between the two choices as u_1 crosses the value ef^{-1}. More precisely, we will use the following solution of (1.1):

$$u_2 = h(u_1) = \begin{cases} g^{-1}(e - fu_1) & \text{if } u_1 \leqq ef^{-1} \\ 0 & \text{if } u_1 \geqq ef^{-1}. \end{cases} \tag{1.5}$$

Substituting into the first equation in (1.1), and setting $\partial u_1/\partial t = 0$, we obtain, (after replacing u_1 by u)

$$\Delta u + u[a - bu - ch(u)] = 0. \tag{1.6}$$

This equation will play an important role in the construction upper and lower bounds for $u_i(x, t)$. (Compare with equations (2.5) and (3.14), which are modifications of (1.6).) Substituting the upper and lower bounds for u_1 into the function h in (1.5), we will eventually obtain respectively lower and upper bounds for $u_2(x, t)$. Theorem 2.1 gives lower bound for u_1 and upper bound for u_2. It looks difficult to apply at first sight. However, applying the same technique, together with the assumption $ab^{-1} > ef^{-1}$, one obtains very convenient results in Theorem 3.1. It essentially gives simple sufficient conditions on the initial and boundary conditions, so that $u_2(x, t)$ becomes arbitrarily small for large t in the interior, except for "boundary layer" adjustments near the boundary. One can compare the results of Theorem 3.1 with the case of the system of ordinary differential equations:

$$\begin{aligned} \frac{du_1}{dt} &= u_1[a - bu_1 - cu_2] \\ \frac{du_2}{dt} &= u_2[e - fu_1 - gu_2]. \end{aligned} \tag{1.7}$$

If $ab^{-1} > ef^{-1}$, phase plane analysis easily shows that for $u_1(0) > 0$, $u_2(0) \geqq 0$, $(u_1(0), u_2(0))$ close to $(ab^{-1}, 0)$, one has $(u_1(t), u_2(t)) \to (ab^{-1}, 0)$ as $t \to \infty$.

Theorem 3.2 gives an upper bound for u_1 and a lower bound for u_2, under the assumption $ab^{-1} > ef^{-1}$. This assumption is the more interesting case because u_1 is likely to achieve large enough values so that $h(u_1)$ will be switched to become 0.

For a study of (1.1) with initial and Dirichlet conditions, and $ab^{-1} < ef^{-1}$ (even with ε not small), one is referred to [5, p. 212]. For a study of the equilibrium solutions (i.e., $\partial u_1/\partial t = \partial u_2/\partial t \equiv 0$), one is referred to [2]. (However, in many situations here, the assumptions in [2] are not satisfied.)

From the point of view of applications, Theorem 2.1 is useful in simplifying calculations to locate regions where $u_2(x, t)$ will eventually be close to extinction (see remark 3 below). Combining Theorem 2.1 and 3.2, one can obtain convenient estimates for the large-time behavior of $u_1(x, t)$ and $u_2(x, t)$ (see remark 10 below).

We will now clarify the notations and conventions. Let ℓ, $0 < \ell < 1$, be a fixed number. For an open set G in $\mathbf{R}^n$, let $H^{2+\ell}(\bar{G})$ denotes the Banach space of all real-valued functions u continuous in $\bar{G}$ with all first and second derivatives also continuous in $\bar{G}$, and with finite value for the norm $|u|^G_{2+\ell}$ (as described in [**14**, p. 159]). Equation (1.1) will be considered for $x = (x_1, \ldots, x_n) \in \mathscr{D}$, where $\mathscr{D}$ is a bounded open connected subset of $\mathbf{R}^n$, $n \geqq 1$, with boundary $\partial\mathscr{D}$. We assume that $\partial\mathscr{D} \in H^{2+\ell}$, (see [**14**]). For any $T > 0$, let $\mathscr{D}_T = \mathscr{D} \times (0, T)$. $H^{2+\ell,(2+\ell)/2}(\bar{\mathscr{D}}_T)$ denotes the Banach space of all real-valued functions u having all derivatives of the form $D^\alpha D_t^r u$, with $2r + |\alpha| \leqq 2$, continuous on $\bar{\mathscr{D}}_T$ and having finite norm $|u|^{(2+\ell)}_{\mathscr{D}_T}$(as described in [**14**]). Solutions of (1.1) will mean solutions in $H^{2+\ell,(2+\ell)/2}(\bar{\mathscr{D}}_T)$, $T > 0$. When $\partial u_i/\partial t$, $i = 1, 2$, on the left of (1.1) are replaced by 0, solutions will mean solutions in $H^{2+\ell}(\bar{\mathscr{D}})$.

2. Boundary conditions and lower-upper bounds. We will study the solutions $(u_1(x, t), u_2(x, t))$ for system (1.1) satisfied on $(x, t) \in \mathscr{D} \times [0, \infty)$, with initial-boundary conditions:

$$
\begin{aligned}
&u_i(x, 0) = \phi_i(x), \ x \in \bar{\mathscr{D}}, \ i = 1, 2 \\
&u_i(x, t) = \theta_i(x), \ (x, t) \in \partial\mathscr{D} \times [0, \infty)
\end{aligned} \tag{2.1}
$$

$i = 1, 2$. Here, $\phi_i(x) = \theta_i(x)$ for $x \in \partial\mathscr{D}$, $i = 1, 2$ and $\phi_i(x)$, $\theta_i(x)$ satisfy the compatibility conditions of order 1 on $\partial\mathscr{D}$ at $t = 0$. Further, for $i = 1, 2$, ϕ_i are nonnegative functions in $H^{2+\ell}(\bar{\mathscr{D}})$, and θ_i are nonnegative functions satisfying conditions of [**4**, Theorem 5.2, p. 320]. By [**14**, Remark 1], the existence and uniqueness of the solution $(u_1(x, t), u_2(x, t))$ in $H^{2+\ell,(2+\ell)/2}(\bar{\mathscr{D}}_T)$, each $T > 0$, have been proved for the initial-boundary value problem (1.1), (2.1).

The following lemma will be used extensively in the proof of our theorems.

LEMMA 2.1. *Let $v_i(x, t)$, $w_i(x, t) \in \bar{\mathscr{D}} \times [0, \infty)$, $i = 1, 2$ be functions in $H^{2+\ell,1+\ell/2}(\bar{\mathscr{D}}_T)$, each $T > 0$, satisfying the inequalities*:

$$
\begin{aligned}
&0 \leqq v_i \leqq w_i, \qquad i = 1, 2, \\
&\Delta v_1 + v_1[a - bv_1 - cw_2] - \frac{\partial v_1}{\partial t} \geqq 0, \\
&\Delta w_1 + w_1[a - bw_1 - cv_2] - \frac{\partial w_1}{\partial t} \leqq 0, \\
&\varepsilon\Delta v_2 + v_2[e - fw_1 - gv_2] - \frac{\partial v_2}{\partial t} \geqq 0, \\
&\varepsilon\Delta w_2 + w_2[e - fv_1 - gw_2] - \frac{\partial w_2}{\partial t} \leqq 0.
\end{aligned} \tag{2.2}
$$

Let $(u_1(x, t), u_2(x, t))$, *with* $u_i \in H^{2+\ell, 1+\ell/2}(\bar{\mathscr{D}}_T)$, *each* $T > 0$, $i = 1, 2$, *be a solution of the reaction-diffusion equations* (1.1) *with initial-boundary conditions such that*

$$(2.3) \qquad v_i(x, 0) \leqq u_i(x, 0) \leqq w_i(x, 0), \; x \in \mathscr{D}, \; i = 1, 2,$$

and

$$(2.4) \quad v_i(x, t) \leqq u_i(x, t) \leqq w_i(x, t), \; (x, t) \in \partial\mathscr{D} \times [0, \infty), \; i = 1, 2.$$

Then $(u_1(x, t), u_2(x, t))$ *will satisfy*

$$v_i(x, t) \leqq u_i(x, t) \leqq w_i(x, t), \; (x, t) \in \bar{\mathscr{D}} \times [0, \infty).$$

This lemma is a special case of Lemma 2.1 in [5].

We are now ready to use the following reduced problem to construct a lower bound for $u_1(x, t)$ and an upper bound for $u_2(x, t)$:

$$\Delta y + y[a - by - c\, h(y) - c(2\delta + I(x) + L(x))] = 0 \text{ in } \mathscr{D}$$

$$(2.5) \qquad y(x) = \theta_1(x) \text{ on } \partial\mathscr{D}$$

Here, $\delta > 0$ is a small constant, and $I(x)$, $L(x)$ will be respectively chosen to adapt to $\phi_2(x)$ and $\theta_2(x)$. Note that the first equation in (2.5) is a slight modification of (1.6).

THEOREM 2.1. *Let* $\delta > 0$ *be an arbitrary small number,* $I(x)$ *and* $L(x)$ *be respectively nonnegative and positive functions in* $H^{2+\alpha}(\bar{\mathscr{D}})$, *and* $y(x)$ *be a nonnegative solution of the boundary value problem* (2.5) *above. Suppose that the nonnegative function* $h(y(x))$ *has a "smooth trunction"* $M^y(x)$ *in the following sense:*

(i) $M^y(x) \in H^{2+\alpha}(\bar{\mathscr{D}})$,

(ii) $M^y(x) = h(y(x))$ *if* $h(y(x)) > \delta$ *(i.e., if* $y(x) < f^{-1}(e - g\delta)$*)*,

and

(iii) $0 \leqq M^y(x) \leqq \delta$ *if* $0 \leqq h(y(x)) \leqq \delta$ *(i.e., if* $y(x) \geqq f^{-1}(e - g\delta)$*)*.

Then the solution of the initial boundary value problem (1.1), (2.1) *will satisfy*

$$(2.6) \qquad u_1(x, t) \geqq y(x)$$

and

$$(2.7) \qquad 0 \leqq u_2(x, t) \leqq M^y(x) + \delta + I(x)\, e^{-rt} + L(x)$$

for $(x, t) \in \bar{\mathscr{D}} \times [0, \infty)$ *provided that* $\varepsilon > 0$ *is small enough, and*

$$(2.8) \qquad \phi_1(x) \geqq y(x) \text{ for } x \in \bar{\mathscr{D}},$$

$$(2.9a) \qquad 0 \leqq \phi_2(x) \leqq M^y(x) + \delta + I(x) + L(x) \text{ for } x \in \bar{\mathscr{D}}$$

and

(2.9b) $$0 \leqq \theta_2(x) \leqq M^y(x) + \delta + L(x) \text{ for } x \in \partial\mathscr{D}.$$

Here r is any constant with $0 < r < \delta g$.

REMARK 1. Since $\theta_1(x) \geqq 0$, the zero function is a lower solution for the scalar problem (2.5). An upper solution for (2.5) is a constant function with a sufficiently large positive constant. Consequently, there exists a nonnegative solution $y(x)$ to (2.5), as stated in Theorem 2.1. Moreover, the smoothness of the nonlinear expression in (2.5) implies that $y \in H^{2+\alpha}(\bar{\mathscr{D}})$ (see, e.g., [10]).

REMARK 2. When one restricts to the case $x \in \bar{\mathscr{D}} \subset \mathbf{R}^1$, i.e., $n = 1$, one can readily prove that $M^y(x)$, satisfying conditions (i) to (iii) as stated in Theorem 2.1, does exist.

REMARK 3. In order to apply Theorem 2.1 effectively, one may choose $L(x)$ to be arbitrarily small for x outside a small neighborhood of $\partial\mathscr{D}$, and to grow quickly to slightly larger than $\theta_2(x) - \delta$ at $x \in \partial\mathscr{D}$. ($L(x)$ therefore plays the role of a "boundary layer" correction for u_2). Then one may choose $I(x) \geqq \phi_2(x) - \delta - L(x)$ for $x \in \bar{\mathscr{D}}$, so (2.9a, b) are always satisfied. Inequality (2.7) will then imply that $u_2(x, t)$ is nearly dominated by $M^y(x)$, or $h(y(x))$, for x outside a small neighborhood of $\partial\mathscr{D}$, when t is sufficiently large. One can therefore use the reduced problem (2.5) to approximate the asymptotic behavior of the full problem (1.1), (2.1), as $t \to +\infty$, provided ε is small enough and (2.8), (2.9a, b) are satisfied. Consider those x outside a neighborhood of $\partial\mathscr{D}$, so that $L(x)$ is defined arbitrarily small. If $y(x) \geqq e/f$, then $h(y(x))$ and $M^y(x)$ will be small, and $u_2(x, t)$ will tend to small values as $t \to +\infty$ (by means of (2.7)). In other words, those will be locations where u_2 becomes extinct in the long run. Theorem 3.1 below describes a variant of this situation when $h(y(x)) \equiv 0$, which is a simple, but important, case.

REMARK 4. Let $F(u_1, u_2) \equiv u_2(e - fu_1 - gu_2)$. We have $F(ef^{-1}, 0) = 0$ and $(\partial F/\partial u_2)(ef^{-1}, 0) = 0$. The relation $F(u_1, u_2) = 0$ defines u_2 as two smooth functions of u_1 (namely, $u_2 = 0$ and $u_2 = g^{-1}(e - fu_1)$). These two functions coalesce when $(u_1 u_2) = (ef^{-1}, 0)$. This is usually the difficult case when one studies the full problem (1.1), (2.1) by means of the reduced problem through setting $\varepsilon = 0$. Theorem 2.1 essentially treats this situation when $u_2 = h(u_1)$ switches from one smooth choice of the implicit function defined by $F(u_1, u_2) = 0$ to another (cf. Fife [2]).

PROOF OF THEOREM 2.1. We will use Lemma 2.1 above by constructing appropriate lower and upper solutions v_i, w_i, $i = 1, 2$. Let $v_1(x, t) = y(x)$, $w_2(x, t) = M^y(x) + \delta + I(x)e^{-rt} + L(x)$, for $(x, t) \in \bar{\mathscr{D}} \times [0, \infty)$. We have

$$\begin{aligned}
&\Delta v_1 + v_1[a - bv_1 - cw_2] - \frac{\partial v_1}{\partial t} \\
&\quad = \Delta y + y[a - by - cM^y(x) - c\delta - cI(x)e^{-rt} - cL(x)] \\
&\quad \geqq \Delta y + y[a - by - c(h(y(x) + \delta) - c(\delta + I(x)e^{-rt} + L(x))] \\
&\quad \geqq \Delta y + y[a - by - ch(y) - c(2\delta + I(x) + L(x))] = 0
\end{aligned}$$

for $(x, t) \in \bar{\mathscr{D}} \times [0, \infty)$. On the other hand,

$$\begin{aligned}
&\varepsilon\Delta w_2 + w_2[e - fv_1 - gw_2] - \frac{\partial w_2}{\partial t} \\
&\quad = \varepsilon\Delta w_2 + w_2[e - fy - gM^y(x) - g\delta - gI(x)e^{-rt} - gL(x)] + rI(x)e^{-rt} \\
&\quad = \varepsilon\Delta w_2 + w_2[e - fy - gM^y(x)] - w_2[g\delta + gL(x)] + I(x)e^{-rt}(r - w_2 g),
\end{aligned}$$

which is less than zero for ε sufficiently small (because $e - fy(x) - gM^y(x) = e - fy(x) - gh(y(x)) = 0$, if $y(x) < (e - g\delta)f^{-1}$, and $e - fy(x) - gM^y(x) \leqq e - fy(x) \leqq g\delta$ if $y(x) \geqq (e - g\delta)f^{-1}$), for $(x, t) \in \bar{\mathscr{D}} \times [0, \infty)$.

We next let $v_2(x, t) \equiv 0$, $w_1(x, t) \equiv C$ where C is a large positive constant, $C > \max\{a/b, \max\{\phi_1(x) : x \in \bar{\mathscr{D}}\}\}$. Clearly

$$\varepsilon\,\Delta v_2 + v_2[e - fw_1 - gv_2] - \frac{\partial v_2}{\partial t} \geqq 0$$

and

$$\Delta w_1 + w_1[a - bw_1 - cv_2] - \frac{\partial w_1}{\partial t} \leqq 0$$

for $(x, t) \in \bar{\mathscr{D}} \times [0, \infty)$.

Finally, conditions (2.8) and (2.9a, b) imply that $v_i(x, 0) \leqq u_i(x, 0) \leqq w_i(x, 0)$ for $x \in \mathscr{D}$, and $v_i(x, t) \leqq u_i(x, t) \leqq w_i(x, t)$ for $(x, t) \in \partial\mathscr{D} \times [0, \infty)$, $i = 1, 2$. Lemma 2.1 therefore asserts that $y(x) \leqq u_1(x, t) \leqq C$ and $0 \leqq u_2(x, t) \leqq M^y(x) + \delta + I(x)e^{-rt} + L(x)$, for all $(x, t) \in \bar{\mathscr{D}} \times [0, \infty)$.

3. Extinction of u_2 or upper-lower bounds. The following theorem is analogous to Theorem 2.1. It illustrates conditions, on the inital and boundary data, which will imply extinction of u_2 for all x except at the boundary, as $t \to +\infty$. To avoid excessive technicalities, we restrict to the case in $\mathbf{R}^1$, i.e., $n = 1$. In Theorem 3.1, we therefore assume $\mathscr{D} = (a_0, b_0)$, $a_0 < b_0$, $\partial\mathscr{D} = \{a_0, b_0\}$.

THEOREM 3.1. *Suppose that $a/b > e/f$. Let R be a number satisfying $0 < R < a/b - e/f$. Assume that the initial conditions satisfy*

$$\phi_1(x) \geqq \frac{a}{b} - R \qquad \textit{for all } x \in [a_0, b_0] \tag{3.1}$$

and

(3.2) $$\phi_2(x) < c^{-1}bR \qquad \textit{for all } x \in [a_0, b_0].$$

Let $\bar{a}$, $\bar{b}$ be arbitrary numbers satisfying $a_0 < \bar{a} < \bar{b} < b_0$. Then for any arbitrary small $\sigma > 0$, the solution $(u_1(x, t), u_2(x, t))$ of (1.1), (2.1) *will satisfy*

(3.3) $$0 \leqq u_2(x, t) < \sigma$$

for all $(x, t) \in [\bar{a}, \bar{b}] \times [K, \infty)$ for large enough $K > 0$, provided $\varepsilon > 0$ is sufficiently small.

REMARK 5. One can consider this as a generalized version of the ordinary differential equations (1.7), with $a/b > e/f$, where extinction of u_2 will occur if $(u_1(0), u_2(0))$ are in an appropriate region (see §1).

PROOF. We first proceed to construct $L(x)$, $I(x)$ for $x \varepsilon [a_0, b_0]$ by a procedure similar to that described in Remark 3. Let $\delta > 0$ be such that $\delta < \min\{c^{-1}bR, \sigma/2\}$ and $\max\{\phi_2(x)\colon x \in [a_0, b_0]\} + 4\delta < c^{-1}bR$. For $x = a_0$ or b_0, define $L(x)$ and $I(x)$ be arbitrary numbers satisfying

(3.4) $$\begin{gathered} \max\{\phi_2(a_0), \phi_2(b_0)\} + \delta < L(x) < c^{-1}bR - 3\delta, \\ 0 < I(x) < \delta \end{gathered}$$

We therefore have

(3.5) $$\max\{\phi_2(a_0), \phi_2(b_0)\} + 2\delta < I(x) + L(x) + \delta < c^{-1}bR - \delta$$

for $x = a_0$ or b_0. We will now define $I(x)$, $L(x)$ as functions in $H^{2+\alpha}(\bar{\mathscr{D}})$ by the following procedures so that

(3.6) $$\phi_2(x) + \delta < I(x) + L(x) + \delta < c^{-1}bR$$

for all $x \in [a_0, b_0]$. Let $\tilde{\sigma}$ be a number satisfying $0 < \tilde{\sigma} < \min\{\sigma - 2\delta, \delta\}$. Define $L(x)$ and $I(x)$ in $[\bar{a}, \bar{b}]$ as any functions in $H^{2+\alpha}(\bar{\Omega})$, $\bar{\Omega} = [\bar{a}, \bar{b}]$ satisfying $0 < L(x) < \tilde{\sigma}$, $\phi_2(x) + \delta < I(x) < c^{-1}b\,R - 2\delta - \tilde{\sigma}$, for $x \in [\bar{a}, \bar{b}]$. If we let $h(x) = I(x) + L(x)$ for $x \in \{a_0, b_0\} \cup [\bar{a}, \bar{b}]$, we clearly have for such x, the inequalities

(3.7) $$\phi_2(x) + 2\delta < h(x) + \delta < c^{-1}bR - \delta.$$

Extend $h(x)$ to be a function in $H^{2+\alpha}(\bar{\mathscr{D}})$, $\bar{\mathscr{D}} = [a_0, b_0]$, so that

(3.8) $$\phi_2(x) + \delta < h(x) + \delta < c^{-1}bR,$$

for all $x \in [a_0, b_0]$. We next extend the definition of $L(x)$ to $(a_0, \bar{a}) \cup (\bar{b}, b_0)$ so that $L(x)$ is in $H^{2+\alpha}(\bar{\mathscr{D}})$ and $0 < L(x) < h(x)$ on $(a_0, \bar{a}) \cup (\bar{b}, b_0)$. Finally, set $I(x) = h(x) - L(x)$ on $(a_0, \bar{a}) \cup (\bar{b}, b_0)$. We therefore have inequalities (3.6) valid for all $x \in [a_0, b_0]$.

As in the proof of Theorem 2.1, we now construct appropriate lower and upper solutions v_i, w_i, $i = 1, 2$ and apply Lemma 2.1. Define $v_1(x, t)$

$\equiv a/b - R$, $w_2(x, t) = \delta + I(x)e^{-nt} + L(x)$, where $0 < n < \delta g$, for all $(x, t) \in [a_0, b_0] \times [0, \infty)$. We have

$$\Delta v_1 + v_1[a - bv_1 - cw_2] - \frac{\partial v_1}{\partial t} = v_1[bR - c(\delta + I(x)e^{-nt} + L(x))]$$
$$\geqq v_1[bR - c(\delta + I(x) + L(x)]$$
$$> v_1[bR - cc^{-1}bR] = 0,$$

for $(x, t) \in [a_0, b_0] \times [0, \infty)$, by (3.6). On the other hand,

$$\varepsilon\Delta w_2 + w_2[e - fv_1 - gw_2] - \frac{\partial w_2}{\partial t}$$
$$= \varepsilon\Delta w_2 + w_2\left[e - f\left(\frac{a}{b} - R\right) - g\delta - gI(x)e^{-nt} - gL(x)\right] + nI(x)e^{-nt}$$
$$< \varepsilon\Delta w_2 + w_2\left[e - f\left(\frac{a}{b} - R\right)\right] + I(x)e^{-nt}[-w_2 g + n]$$

which is less than zero provided ε is sufficiently small (because $e - f(a/b - R) < 0$, $-w_2 g + n < -\delta g + n < 0$), for all $(x, t) \in [a_0, b_0] \times [0, \infty)$. We next set $v_2(x, t) \equiv 0$, $w_1(x, t) \equiv C$, where C is a large positive constant, $C > \max\{a/b, \max\{\phi_1(x) : x \in [a_0, b_0]\}\}$. Clearly,

$$\varepsilon\Delta v_2 + v_2[e - fw_1 - gv_2] - \frac{\partial v_2}{\partial t} \geqq 0$$

and

$$\Delta w_1 + w_1[a - bw_1 - cv_2] - \frac{\partial w_1}{\partial t} \leqq 0$$

for $(x, t) \in [a_0, b_0] \times [0, \infty)$. Condition (3.1) and the choice of C imply that

$$v_i(x, 0) \leqq u_i(x, 0) \leqq w_i(x, 0), \quad x \in [a_0, b_0], \tag{3.9}$$

and

$$v_i(x, t) \leqq u_i(x, t) \leqq w_i(x, t), \; (x, t) \in \{a_0, b_0\} \times [0, \infty), \tag{3.10}$$

for $i = 1$. Inequality (3.8) implies that (3.9) is valid for $i = 2$. Inequality (3.4) implies that (3.10) is valid for $i = 2$. Consequently, by Lemma 2.1, we have $u_2(x, t) \leqq w_2(x, t) = \delta + I(x)e^{-nt} + L(x)$ for all $(x, t) \in [a_0, b_0] \times [0, \infty)$. Since by construction $L(x) < \bar{\sigma}$ for $x \in [\bar{a}, \bar{b}]$ and $2\delta + \bar{\sigma} < \sigma$, we have inequality (3.3) for those (x, t) as stated in the theorem.

As a duality to Theorem 2.1, we next use a reduced problem analogous to (1:6) to construct an upper bound for $u_1(x, t)$ and a lower bound for $u_2(x, t)$, where $(u_1(x, t), u_2(x, t))$ is a solution for (1.1), (2.1). We will only treat the case

(3.11) $$ab^{-1} > ef^{-1}$$

which is more interesting. One should be able to consider the other cases with appropriate modifications. However, they are too lengthy for our present purpose. To avoid excessive technicalities, we assume $x \in \mathbf{R}^1$, $\mathscr{D} = (a_0, b_0)$, $\bar{\mathscr{D}} = [a_0, b_0]$, $a_0 < b_0$, in Theorem 3.2. Let λ be an arbitrary number satisfying $0 < \lambda < e/3g$. Define $k(u)$ piecewise as follows. If $u \leqq f^{-1}(e - 3\lambda g)$, then

(3.12a) $$k(u) = a - bu - cg^{-1}(e - fu) + c_1\lambda$$

where $0 < c_1 < \min\{3bgf^{-1}, 3c\}$. If $f^{-1}(e - 3\lambda g) \leqq u \leqq f^{-1}(e - 2\lambda g)$, then

(3.12b) $$k(u) = (a - bef^{-1} + 3\lambda bgf^{-1}) + (3c - c_1)g^{-1}(fu - e + 2\lambda g).$$

If $f^{-1}(e - 2\lambda g) \leqq u \leqq f^{-1}e$, then

(3.12c) $$k(u) = a - bef^{-1} + 3\lambda bgf^{-1}.$$

If $f^{-1}e \leqq u \leqq f^{-1}(e + \lambda g)$, then

(3.12d) $$k(u) = (a - bef^{-1} + 3\lambda bgf^{-1}) - 4b(u - ef^{-1}).$$

If $f^{-1}(e + \lambda g) \leqq u$, then

(3.12e) $$k(u) = a - bu.$$

Note that

(3.13) $$a - bu - ch(u) \leqq k(u) \leqq a - bu - ch(u) + O(\lambda),$$

see Diagram 1. We will use the solution of

(3.14) $$\begin{gathered} \Delta z(x) + z(x)k(z(x)) = 0 \quad \text{for } x \in (a_0, b_0) \\ z(a_0) = \theta_1(a_0),\ z(b_0) = \theta_1(b_0), \end{gathered}$$

to construct the appropriate bounds for $u_1(x, t), u_2(x, t)$. Note that (3.13) implies that (3.14) is a slight perturbation of (1.6) for small λ.

THEOREM 3.2. *Assume that* (3.11) *holds. Let $z(x)$ be a nonnegative solution of the boundary value problem* (3.14). *Suppose that*

(3.15) $$z'(\hat{x}) \neq 0, \text{ for all } \hat{x} \text{ where } z(\hat{x}) = e/f.$$

Then there is a "smooth truncation" $N^z(x) \in H^{2+\alpha}(\bar{\mathscr{D}})$ for $h(z(x))$ in the sense that:

(i) $N^z(x) = 0$ *if* $h(z(x)) = 0$ (*i.e., if* $z(x) \geqq ef^{-1}$),

(ii) $0 < N^z(x) < 2\lambda$ *if* $0 < h(z(x)) < 2\lambda$ (*i.e., if* $f^{-1}(e - 2g\lambda) < z(x) < ef^{-1}$), *and*

(iii) $N^z(x) = h(z(x)) - \sigma$ *for some positive* $\sigma < \lambda$ *if* $h(z(x)) \geqq 2\lambda$ (*i.e., if* $z(x) \leqq f^{-1}(e - 2\lambda g)$).

The functions $z(x)$ and $N^z(x)$ form upper and lower bounds respectively for $u_1(x, t)$, $u_2(x, t)$. More precisely,

$$u_1(x, t) \leqq z(x) \textit{ for all } (x, t) \in [a_0, b_0] \times [0, \infty) \tag{3.16}$$

$$u_2(x, t) \geqq N^z(x) \textit{ for all } (x, t) \in [a_0, b_0] \times [0, \infty) \tag{3.17}$$

provided $\varepsilon > 0$ is sufficiently small, and

$$0 \leqq \phi_1(x) \leqq z(x) \textit{ for all } x \in [a_0, b_0] \tag{3.18}$$

$$\phi_2(x) \geqq N^z(x) \textit{ for all } x \in [a_0, b_0]. \tag{3.19}$$

Here $(u_1(x, t), u_2(x, t))$ is the solution of (1.1), (2.1).

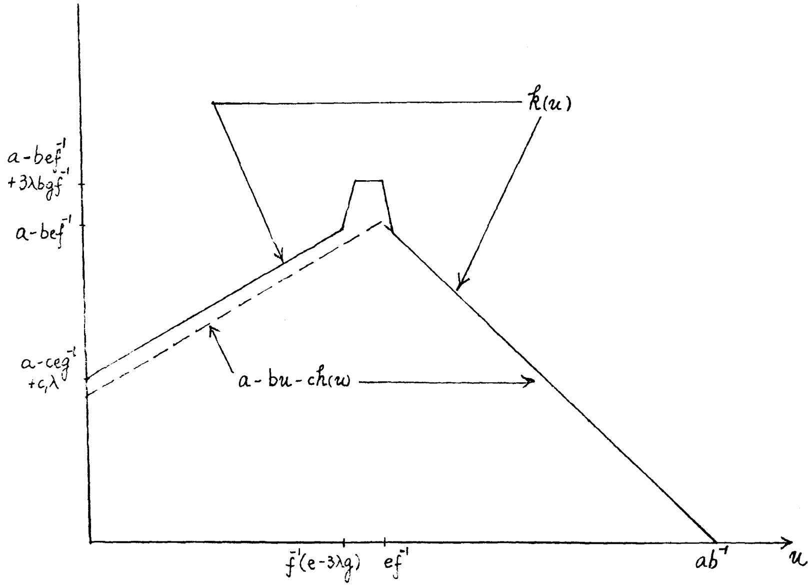

DIAGRAM 1

REMARK 6. As in Remark 1, we can readily prove the existence of a nonnegative solution $z(x) \in H^{2+\alpha}(\mathscr{D})$ of (3.14).

REMARK 7. Note that $N^z(x) = 0$ if $z(x) \geqq ef^{-1}$, and $N^z(x) \approx g^{-1}(e - fz(x))$ if $z(x) < ef^{-1}$. Therefore it is near two different choices of the root u_2 in the equation $u_2(e - fz(x) - gu_2) = 0$, depending on the size of $z(x)$.

REMARK 8. Hypothesis (3.15) seems analogous to that of (3.7) in **[2]**. However (3.15) is less restrictive.

PROOF OF THEOREM 3.2. Hypothesis (3.15) implies that there are only finite number of points $\hat{x}$ where $z(\hat{x}) = e/f$, and that the set $S_1 = \{x \in [a_0, b_0] \mid g^{-1}(e - fz(x)) < 0\}$ must be of the form $S_1 = \bigcup_{j=1}^{k}(\bar{a}_j, \bar{b}_j)$, when $[\bar{a}_j, \bar{b}_j]$, $j = 1, \ldots, k$, are mutually disjoint intervals. By choosing σ small enough, $0 < \sigma < \min\{c^{-1}c_1\lambda, \lambda\}$, the set $\{x \in [a_0, b_0] \mid g^{-1}(e - fz(x)) < \sigma\}$ will be of the form $\bigcup_{j=1}^{k}(a_j, b_j)$, where $a_j < \bar{a}_j < \bar{b}_j < b_j$, $[a_j, b_j]$ mutually disjoint, $j = 1, \ldots, k$. Let $s(t)$ be a three times continuously differentiable function for $t \in (-\infty, \infty)$, with $s(t) = 0$ when $t \geqq e/f$, $s(t) = \sigma$ when $t \leqq (e - 2\lambda g)f^{-1}$, $s'(t) < 0$ for $t \in (ef^{-1} - 2\lambda gf^{-1}, ef^{-1})$, and finally satisfying condition $s(t) < g^{-1}(e - ft)$ for $t < e/f$. We will modify and truncate the function $h(z(x))$ by first defining two functions $\omega^z(x)$ and $\chi^z(x)$ as follows: $\omega^z(x) = g^{-1}(e - fz(x)) - s(z(x))$, which is therefore in $H^{2+\alpha}(\bar{\mathscr{D}})$; $\chi^z(x)$ is a three times differentiable function on $[a_0, b_0]$, with $\chi^z(x) = 0$ when $x \in S_1$, $\chi^z(x) = 1$ when x is in the complement of $\bigcup_{j=1}^{k}(a_j, b_j)$, $(\chi^z)'(x) < 0$ for $x \in (a_j, \bar{a}_j)$, $(\chi^z)'(x) > 0$ for $x \in (\bar{b}_j, b_j)$, and $(\chi^z)'' > 0$ in intervals $(\bar{a}_j - \varepsilon_j, \bar{a}_j) \cup (\bar{b}_j, \bar{b}_j + \varepsilon_j)$ for some positive $\varepsilon_j < \min\{\bar{a}_j - a_j, b_j - \bar{b}_j\}$, for each $j = 1, \ldots, k$. We now define for each $x \in \bar{\mathscr{D}}$

$$N^z(x) = \omega^z(x)\chi^z(x),$$

which is in $H^{2+\alpha}(\bar{\mathscr{D}})$, and can be readily checked to satisfy conditions (i), (ii) and (iii) in the statement of the theorem. We now further find some important concavity properties of $N^z(x)$ for x immediately to the left and right of $\bar{a}_j$ and $\bar{b}_j$ respectively. Observe that

$$\begin{aligned}(\omega^z)''(\bar{a}_j) &= -g^{-1}fz''(\bar{a}_j) + s''(z(\bar{a}_j))[z'(\bar{a}_j)]^2 + s'(z(\bar{a}_j))\cdot z''(\bar{a}_j)\\ &= g^{-1}fz(\bar{a}_j)k(z(\bar{a}_j)) > 0\end{aligned}$$

(because $z(\bar{a}_j) = e/f$, $s''(e/f) = s'(e/f) = 0$, $k(e/f) > 0$);

$$(\omega^z)'(\bar{a}_j) = -g^{-1}fz'(\bar{a}_j) - s'(z(\bar{a}_j))z'(\bar{a}_j) = -g^{-1}fz'(\bar{a}_j) < 0$$

(by (3.15) and definition of $\bar{a}_j$). Consequently

$$(N^z)'' = (\omega^z)''\cdot\chi^z + 2(\omega^z)'(\chi^z)' + (\omega^z)(\chi^z)''$$

becomes greater than zero for x in an interval of the form $(\bar{a}_j - \bar{\varepsilon}_j, \bar{a}_j)$ for some positive $\bar{\varepsilon}_j < \varepsilon_j$, each $j = 1, \ldots, k$. Similarly, we can show $(\omega^z)'(\bar{b}_j) > 0$, $(\omega^z)''(\bar{b}_j) > 0$ and $(N^z)''(x) > 0$ for x in an interval of the form $(\bar{b}_j, \bar{b}_j + \bar{\bar{\varepsilon}}_j)$, where $\bar{\bar{\varepsilon}}_j$ is sufficiently small, $0 < \bar{\bar{\varepsilon}}_j < \varepsilon_j$. For the later convenience, we let

$$S_4 = \bigcup_{j=1}^{k}\{(\bar{a}_j - \bar{\varepsilon}_j, \bar{a}_j) \cup (\bar{b}_j, \bar{b}_j + \bar{\bar{\varepsilon}}_j)\}.$$

We are now ready to apply Lemma 2.1 by constructing appropriate upper and lower solutions. Let $v_2(x, t) = N^z(x)$, $w_1(x, t) = z(x)$, for $(x, t) \in [a_0, b_0] \times [0, \infty)$. Consider

$$\varepsilon \Delta v_2 + v_2[e - fw_1 - gv_2] - \frac{\partial v_2}{\partial t} = \varepsilon \Delta(N^z) + N^z[e - fz(x) - gN^z(x)] \tag{3.20}$$

in the three sets $\bar{S}_1 = \{x \in [a_0, b_0] \mid g^{-1}(e - fz(x)) \leqq 0\}$, $\bar{S}_2 = \{x \in [a_0, b_0] \mid g^{-1}(e - fz(x)) \geqq 2\lambda\}$ and $S_3 = \{x \in [a_0, b_0] \mid x \notin \bar{S}_1 \cup \bar{S}_2 \cup S_4\}$. In $\bar{S}_1$, $v_2 = 0$, so the expression in (3.20) is equal to zero. In $\bar{S}_2$

$$N^z[e - fz(x) - gN^z(x)] = \{g^{-1}(e - fz(x)) - \sigma\}[g\sigma] \geqq (2\lambda - \sigma)(g\sigma) > 0,$$

therefore the expression in (3.20) is greater than zero for ε sufficiently small. In S_3, x is bounded away from the zeroes of $N^z(x)$, so $N^z \geqq \ell$ for some $\ell > 0$. This implies that

$$g^{-1}(e - fz(x)) - s(z(x)) = \omega^z(x) \geqq \ell > 0,$$

$z(x) \leqq e/f - \ell_1$ for some $\ell_1 > 0$ and thus $s(z(x)) \geqq \ell_2 > 0$ for all $x \in S_3$. Consequently, for all $x \in S_3$,

$$[e - fz(x) - gN^z(x)] \geqq e - fz(x) - g\omega^z(x) = gs(z(x)) \geqq \ell_2 > 0.$$

This implies that the expression in (3.20) is greater than zero for all $x \in S_3$, provided that ε is sufficiently small. Finally, we consider (3.20) for $x \in S_4$. We have $\varepsilon \Delta N^z > 0$ and

$$[e - fz(x) - gN^z(x)] \geqq e - fz(x) - g\omega^z(x) = gs(z(x)) \geqq 0,$$

therefore the expression in (3.20) is greater than or equal to zero for all $x \in S_4$. Combining situations, we have

$$\varepsilon \Delta v_2 + v_2[e - fw_1 - gv_2] - \frac{\partial v_2}{\partial t} \geqq 0$$

for all $(x, t) \in [a_0, b_0] \times [0, \infty)$, provided that $\varepsilon > 0$ is sufficiently small.

We next consider the expression

$$\Delta w_1 + w_1[a - bw_1 - cv_2] - \frac{\partial w_1}{\partial t} = \Delta z + z[a - bz - cN^z]. \tag{3.21}$$

For $x \in \bar{S}_1$, $z(x) \geqq e/f$ and (3.21) is equal to $\Delta z + z[a - bz] \leqq \Delta z + zk(z) = 0$. For $x \in \bar{S}_2$, $z(x) \leqq f^{-1}(e - 2\lambda g)$, (3.21) is equal to

$$\begin{aligned} \Delta z + z[a - bz - c\omega^z] &= \Delta z + z[a - bz - cg^{-1}(e - fz) + cs(z)] \\ &= \Delta z + z[a - bz - cg^{-1}(e - fz) + c\sigma] \\ &\leqq \Delta z + zk(z) = 0 \end{aligned}$$

(because $c\sigma < c_1\lambda \leqq 3\lambda bgf^{-1}$). For $x \notin \bar{S}_1 \cup \bar{S}_2$, $f^{-1}(e - 2\lambda g) < z(x) < f^{-1}e$,

$$\Delta z + z[a - bz - cN^z] \leqq \Delta z + z[a - bz] \leqq \Delta z + zk(z) = 0$$

(because for such x, $a - bz < a - bef^{-1} + 2\lambda bgf^{-1} < k(z)$). Combining all situations, we have

$$\Delta w_1 + w_1[a - bw_1 - cv_2] - \frac{\partial w_1}{\partial t} \leqq 0$$

for all $x \in [a_0, b_0]$.

Next, set $v_1(x, t) \equiv 0$, $w_2(x, t) \equiv K$, where $K \geqq \max\{e/g, \max\{\phi_2(x): x \in [a_0, b_0]\}\}$, for $(x, t) \in [a_0, b_0] \times [0, \infty)$. Clearly,

$$\Delta v_1 + v_1[a - bv_1 - cw_2] - \frac{\partial v_1}{\partial t} \geqq 0,$$

$$\varepsilon \Delta w_2 + w_2[e - fv_1 - gw_2] - \frac{\partial w_2}{\partial t} \leqq 0$$

for $(x, t) \in [a_0, b_0] \times [0, \infty)$. Finally, we consider the initial and boundary conditions. (3.18) implies that $v_1(x, 0) \leqq u_1(x, 0) = \phi_1(x) \leqq w_1(x, 0)$ for $x \in [a_0, b_0]$; and for $x = a_0$ or b_0, $v_1(x, t) \leqq u_1(x, t) = \theta_1(x) \leqq z(x) = w_1(x, t)$, for all $t \geqq 0$. (3.19) implies that for $x \in [a_0, b_0]$, $v_2(x, 0) = N^z(x) \leqq \phi_2(x) = u_2(x, 0) \leqq K = w_2(x, 0)$; and for $x = a_0$ or b_0, $v_2(x, t) \leqq \theta_2(x) = u_2(x, t) \leqq K \leqq w_2(x, t)$ for all $t \geqq 0$. Applying Lemma 2.1, the developments of the last two paragraphs lead to (3.16) and (3.17) as stated in the theorem.

REMARK 9. For $\mathbf{R}^n$, $n > 1$, we might still use a nonnegative solution of $\Delta z + zk(z) = 0$ in $\mathscr{D}$, $z = \theta_1$ on $\partial\mathscr{D}$ to construct upper and lower bounds for $u_1(x, t)$ and $u_2(x, t)$ respectively. This will be possible if $h(z(x))$ can be smoothly truncated into a nonnegative function $N^z(x)$ in $H^{2+\alpha}(\bar{\mathscr{D}})$ with properties (i) to (iii) as stated in Theorem 3.2, and the property that $\Delta N^z(x) > 0$ for those x where $N^z(x) > 0$ is small enough. The last property for N^z is contained in the proof of Theorem 3.2, when $n = 1$. This property is used to show that $v_2(x, t) = N^z(x)$ is a lower solution.

REMARK 10. When $n = 1$, and $ab^{-1} > ef^{-1}$, we can combine Theorems 2.1 and 3.2 to construct both upper and lower bounds for $u_1(x, t)$, $u_2(x, t)$, provided that appropriate conditions are satisfied at $t = 0$ and $x \in \partial\mathscr{D}$. More precisely, using (2.6), (2.7), (3.16) and (3.17), we have

$$y(x) \leqq u_1(x, t) \leqq z(x)$$

$$N^z(x) \leqq u_2(x, t) \leqq M^y(x) + \delta + I(x)e^{-rt} + L(x)$$

for $(x, t) \in \bar{\mathscr{D}} \times [0, \infty)$, provided that $\varepsilon > 0$ is small enough,

$$y(x) \leqq u_1(x, 0) = \phi_1(x) \leqq z(x)$$

$$N^z(x) \leqq u_2(x, 0) = \phi_2(x) \leqq M^y(x) + \delta + I(x) + L(x)$$

for $x \in \bar{\mathscr{D}}$,

$$N^z(x) \leqq u_2(x, t) = \theta_2(x) \leqq M^y(x) + \delta + L(x)$$

for $(x, t) \in \partial\mathscr{D} \times [0, \infty)$ and other minor conditions in Theorems 2.1, 3.2 are satisfied.

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Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221