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LINEAR MONOTONE METHOD FOR NONLINEAR BOUNDARY VALUE PROBLEMS IN BANACH SPACES

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

1. Introduction. One of the most useful techniques in proving the existence of multiple solutions of nonlinear boundary value problems (BVP for short) is the monotone iterative method, which yields monotone sequences that converge to extremal solutions of the problem. Recently, because of applications, this technique has attracted much attention, see **[1, 2, 4-7, 11, 12, 14, 15]**. To explain this method, let us consider the scalar BVP

$$
\begin{aligned}
&u'' = f(t, u, u'),\ 0 < t < 1, \\
&B^i u = \alpha_i u(i) + (-1)^{i+1}\beta_i u'(i) = b_i,\ i = 0, 1,
\end{aligned}
\tag{1.1}
$$

where α_i, $\beta_i \geqq 0$, $\alpha_i^2 + \beta_i^2 \neq 0$ and $f \in C[I \times \mathbf{R} \times \mathbf{R}, \mathbf{R}]$, I being the interval $[0,1]$. Suppose that $v_0, w_0 \in C^2[I, \mathbf{R}]$ with $v_0(t) \leqq w_0(t)$ on I and

$$
\begin{aligned}
&v_0'' \geqq f(t, v_0, v_0'),\ B^i v_0 \leqq b_i, \\
&w_0'' \leqq f(t, w_0, w_0'),\ B^i w_0 \geqq b_i.
\end{aligned}
\tag{1.2}
$$

Then v_0, w_0 are called lower and upper solutions of (1.1). Suppose also that f_u, $f_{u'}$ exist and f satisfies a Nagumo condition. In order to obtain monotone iterations, one considers the auxiliary BVP

$$
u'' = F(t, u, u'),\ B^i u = b_i \tag{1.3}
$$

where $F(t, u, u') = f(t, \eta(t), u') + M_1(c)(u - \eta(t))$, $v_0(t) \leqq \eta(t) \leqq w_0(t)$, $|f_u(t, u, u')| \leqq M(c)$ for $t \in I$, $v_0(t) \leqq u \leqq w_0(t)$ and $|u'| \leqq c$ for some suitable $c > 0$ which is related to the Nagumo constant. To proceed further with the monotone method it becomes necessary to show that there exists a unique solution for the BVP (1.3). For this purpose, one proves that (i) v_0, w_0 are also lower and upper solutions of (1.3) and (ii) F satisfies a Nagumo condition. Then it follows from known results **[3, 8, 9, 10, 12,**

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13] that the auxiliary BVP (1.3) possesses a solution. The uniqueness of solutions follows by the maximum principle in view of the fact that F is linear in u and $M_1(c) > 0$. We recall that these known results crucially depend on the modification of f, namely $\tilde{f}$, where

$$(1.4)\qquad \begin{aligned} \tilde{f}(t, u, u') &\equiv f(t, p(t, u), q(u')) + r(t, u), \\ p(t, u) &= \max[v_0(t), \min(u, w_0(t))], \\ q(u') &= \max[-c, \min(u', c)], \end{aligned}$$

and

$$r(t, u) = \begin{cases} (u - w_0(t))/(1 + u^2) & \text{if } u > w_0(t), \\ 0 & \text{if } v_0(t) \leqq u \leqq w_0(t), \\ (u - v_0(t))/(1 + u^2) & \text{if } u < v_0(t), \end{cases}$$

$c > 0$ being a number such that $c > |v_0'(t)|, |w_0'(t)|$ on I. These results have been extended to finite and countably infinite systems of BVP's where the inequalities between vectors are understood as componentwise, see [**3**, 7, **10**, **13**].

If one desires to extend this attractive monotone method to BVP's in an arbitrary Banach space E, one needs to induce a partial ordering by means of a cone K in E. Then it is easy to define lower and upper solutions and a Nagumo condition, as before. Corresponding to this setting, we need an existence and uniqueness result for the BVP (1.3) where F is now a nonlinear function in a Banach Space. The modified function approach followed earlier makes sense only when $E = \mathbf{R}^n$ and $K = \mathbf{R}^n_+$, the standard cone. Consequently, the problem of proving existence of solutions of the BVP (1.3) becomes difficult. One is faced with the problem of proving such an existence result directly by other means (say by Lyapunov-like methods [3, **8**]) which need extra assumptions. Unfortunately, the results obtained in [**4**] for the general case, are correct only in special Banach spaces since the modified function approach which is not valid in general, is followed there.

However, if f does not depend on u', the situation becomes quite simple, that is, we will have

$$(1.5)\qquad u'' = F(t, u) = f(t, \eta(t)) + M_1(u - \eta(t)), \; B^i u = b_i.$$

In view of the fact, F is linear in u and $M_1 > 0$, there results

$$((u - v), F(t, u) - F(t, v))_- \geqq M_1|u - v|^2$$

where $(x, y)_-$ is the generalized inner product and hence existence and uniqueness of solutions of (1.5) follows from abstract results. This is precisely the method adapted in [**15**] for extending the monotone method to a Banach space for the case where f does not depend on u'.

In this paper, we wish to develop the monotone method for those f that are linear in u' by means of a linear procedure; that is, we consider the BVP

$$(1.6) \qquad u'' = f(t, \eta, \eta') + M_1(c)(u - \eta) + M_2(u' - \eta'), \; B^iu = b_i,$$

for $\eta \in [v_0, w_0]$ and $\|\eta'\| \leqq c$ on I, instead of (1.3). Here M_1 is as before. Unfortunately, M_1 depends on c which is the bound on $|u'|$ which causes considerable problems since we must make sure that every solution of (1.6) for every η, satisfies $|u'| \leqq c$ for some $c > 0$. However, it turns out that these difficulties can be surmounted and the entire procedure of the linear monotone method works well to our advantage. As a rich dividend, one is pleased to have the iterates as unique solutions of linear BVP's (1.6) with constant coefficients instead of the solutions of nonlinear BVP (1.3) considered so far. Thus, our results are more constructive than before as a result of computable iterates one gets in the process. We have also obtained an extension of the results in [6] where only the scalar equation was considered. Our method would be applicable to other situations such as partial differential equations and we plan to attempt this elsewhere. Finally, we mention that attempts are being made to remove the linearity assumption of f on u'. This leads to two different approaches: one continues in the same spirit of this paper; the other follows the nonlinear procedure alluded to earlier in the introduction.

2. Preliminaries. Let E be a real Banach space with $\|\cdot\|$ and let E^* denote the dual of E. Let $K \subset E$ be a cone, that is, a closed convex subset such that $\lambda K \subset K$, for every $\lambda \geqq 0$ and $\{K\} \cap \{-K\} = \{0\}$. By means of K a partial order $\leqq$ is defined as $v \leqq u$ if $u - v \in K$. We let $K^* = \{\phi \in E^*: \phi(u) \geqq 0 \text{ for all } u \in K\}$. A cone K is said to be normal if there exists a real number $N > 0$ such that $0 \leqq v \leqq u$ implies $\|v\| \leqq N \|u\|$ where N is independent of u, v. We shall always assume in this paper that K is a normal cone.

Let α denote the Kuratowski measure of noncompactness, the properties of which may be found in [8].

For any $v_0, w_0 \in C[I, E]$ such that $v_0(t) \leqq w_0(t)$ on I where $I = [0, 1]$, we define the conical segment

$$[v_0, w_0] = \{u \in C[I, E]: v_0 \leqq u \leqq w_0\}.$$

We consider the boundary value problem (BVP for short)

$$(2.1) \qquad \begin{aligned} u'' &= f(t, u, u'), \\ B^iu &= \alpha_i u(i) + (-1)^{i+1}\beta_i u'(i) = b_i, \; i = 0, 1, \end{aligned}$$

where $f \in C[I \times E \times E, E]$, $b_i \in E$, α_i, $\beta_i \geqq 0$ and $\alpha_i^2 + \beta_i^2 > 0$.

Let us list the following assumptions for convenience:

(A_1) $v_0, w_0 \in C^2[I, E]$ with $v_0(t) \leqq w_0(t)$ on I such that

$$v_0'' \geqq f(t, v_0, v_0'), B^i v_0 \leqq b_i,$$
$$w_0'' \leqq f(t, w_0, w_0'), B^i w_0 \geqq b_i, i = 0, 1.$$

(A_2) $f(t, x, y) - f(t, \bar{x}, \bar{y}) \leqq M_1(c)(x - \bar{x}) + M_2(y - \bar{y})$, whenever $v_0(t) \leqq \bar{x} \leqq x \leqq w_0(t)$, $t \in I$ and $\|y\|, \|\bar{y}\| \leqq c$, c being any positive constant (recall (A_2) is equivalent to f being linear in y).

(A_3) $\|f(t, x, y)\| \leqq j(\|y\|)$ whenever $t \in I$, $v_0(t) \leqq x \leqq w_0(t)$ where j is is continuous on $[0, \infty)$.

(A_4) j, M_1 and M_2 are increasing and positive functions satisfying

$$h(s) \equiv j(s) + RM_1(s) + 2M_2 s = o(s^2) \text{ as } s \to \infty$$

(this is equivalent to $j(s) = o(s^2)$ since the other terms are linear in s) where $R = \max[\max\|v_0(t) - w_0(t)\|, \max\|v_0(t)\|, \max\|w_0(t)\|, t \in [0, 1])$.

(A_5) $\alpha(f(I \times A \times B)) \leqq L \max(\alpha(A), \alpha(B))$, for any bounded sets A, B in E.

REMARKS. (i) The condition (A_2) implies that f is quasimonotone nonincreasing relative to K. that is, if $x \leqq y$, $\phi(x - y) = 0$ and $\phi(x' - y') = 0$ for some $\phi \in K^*$ then $\phi(f(t, x, x') - f(t, y, y')) \geqq 0$.

(ii) The assumption (A_5) implies that f maps bounded sets into bounded sets.

Let us consider the linear BVP

$$u'' = F(t, u, u'), B^i u = b_i, i = 0, 1, \tag{2.2}$$

where $F(t, u, u') = f(t, \eta(t), \eta'(t)) + M_1(c)(u - \eta(t)) + M_2(u' - \eta'(t))$, and $\eta \in C^1[I, E]$ with $\eta \in [v_0, w_0]$ and $\|\eta'(t)\| \leqq c$.

We want to show that F satisfies a Nagumo condition in order to guarantee a bound on u' for any solution u of (2.2). To do this we need condition (A_4). If $M_1(c)$ is independent of c, then $j(s) = o(s^2)$ as $s \to \infty$ is sufficient. In the case of scalar BVP (1.1) we only need $\lim_{s\to\infty} j(s)/s^2$ is finite, or more specifically, a quadratic growth on j. In this case, if we further suppose that $M_1(c)$ in (1.3) is independent of c, then clearly F satisfies a Nagumo condition. This is in essence what is assumed in **[1]** where the use of a truncation argument seems to indicate that one can avoid requiring $M_1(c)$ to be independent of c. When this assumption is not imposed implicitly, it becomes necessary as in **[2]** to demand $M_1(c) = o(c^2)$ as $c \to \infty$.

Let us begin with the following lemma whose proof is adapted from **[13]**.

LEMMA 2.1. *Let (A_3) and (A_4) hold and let $u(t)$ be any solution of* (2.2) *such that $u \in [v_0, w_0]$. Then there exists a $c^* > 0$ depending only on v_0, w_0 and h such that $\|u'(t)\| \leqq c^*$ on I.*

PROOF. Let $c = \|u'(t_0)\| = \max\|u'(t)\|$. If μ is a real number such that $|\mu| \leqq 1/2$ and $t_0 + \mu \in I$, then it follows from

$$u(t_0 + \mu) = u(t_0) + \mu u'(t_0) + \int_0^1 u''(t_0 + s\mu)(\mu^2/2)ds$$

that $|\mu|c \leqq 2R + (\mu^2/2)h(c)$, where R is the number defined in (A_4). If μ is such that $|\mu|c = 3R$ and this choice is possible for $c \geqq 6R$, then $R \leqq (1/2)|\mu^2|h(c) = (9/2)R^2(h(c)/c^2)$ which leads to a contradiction for large c. Thus, it is enough to choose $c = c^*$ such that $c^* \geqq 6R$ and $h(c^*)/(c^*)^2 \leqq 1/6R$. It then follows that $\|u'(t)\| \leqq c^*$ on I proving the lemma.

Let us fix $c = c^*$ and assume that (A_2) holds for this c^*. We next show that the BVP (2.2) possesses a unique solution on I.

LEMMA 2.2. *Assume that (A_2), (A_3) and (A_4) hold. Then there exists a unique solution $u \in C^2[I, E]$ for the* BVP (2.2) *such that if $u \in [v_0, w_0]$ then $\|u'(t)\| \leqq c^*$ on I.*

PROOF. Let $\eta \in C^1[I, E]$ be such that $\eta \in [v_0, w_0]$ and $\|\eta'(t)\| \leqq c^*$ on I. Write (2.2) in the form

$$u'' - M_2u' - M_1(c^*)u = \sigma(t), \; B^iu = b_i,$$

where $\sigma(t) = f(t, \eta(t), \eta'(t)) - M_1(c^*)\eta(t) - M_2\eta'(t)$. Let $G(t, s)$ be the Green's function associated with the scalar BVP

$$u'' - M_2u' - M_1u = \sigma(t), \; B^iu = 0$$

and let ψ be the unique function satisfying $\psi'' - M_2\psi' - M_1\psi = 0$ and $B^i\psi = b_i$, $i = 0, 1$. The existence of $G(t, s)$ is a consequence of the fact $M_1 > 0$ and the maximum principle. Then is easy to see that the unique solution of the BVP (2.2) is given by

$$u(t) = \int_0^1 G(t, s)\sigma(s)ds + \psi(t). \tag{2.3}$$

As a result, if $u \in [v_0, w_0]$, it follows by Lemma 2.1 that $\|u'(t)\| \leqq c^*$ on I.

We now prove that $u \in [v_0, w_0]$.

LEMMA 2.3. *Assume that (A_1), (A_2), (A_3) and (A_4) hold. Then $u \in [v_0, w_0]$.*

PROOF. For any $\phi \in K^*$, we set $p(t) = \phi(u(t) - v_0(t))$. Then using (A_1), (A_2) we have

$$\begin{aligned} p'' &= \phi(u'' - v_0'') \leqq \phi[f(t, \eta, \eta') + M_1(u - \eta) + M_2(u' - \eta') - f(t, v_0, v_0'] \\ &\leqq \phi[M_1(\eta - v_0) + M_2(\eta' - v_0') + M_1(u - \eta) + M_2(u' - \eta')] \\ &= M_1p + M_2p'. \end{aligned}$$

Also $B^ip \geqq 0$, $i = 0, 1$. Hence, by the maximum principle, we get $p(t) \geqq 0$ on I. Since $\phi \in K^*$ is arbitrary, it follows that $v_0(t) \leqq u(t)$ on I. Using a similar argument, we can show that $u(t) \leqq w_0(t)$ on I. Hence $u \in [v_0, w_0]$ and the proof is complete.

For each $\eta \in C^1[I, E]$ such that $\eta \in [v_0, w_0]$ and $\|\eta'(t)\| \leqq c^*$ on I, we define a mapping A by $A\eta = u$ where u is the unique solution of the BVP (2.2) corresponding to η. The following result concerning the map A holds.

LEMMA 2.4. *Suppose that the assumptions* (A_1), (A_2), (A_3) *and* (A_4) *hold. Then*
(i) $v_0 \leqq Av_0$; $Aw_0 \leqq w_0$; *and*
(ii) A *is monotone on* $[v_0, w_0]$.

PROOF. Suppose that $Av_0 = v_1$. Set $p(t) = \phi[v_1(t) - v_0(t)]$ for any $\phi \in K^*$. Then

$$\begin{aligned} p'' &\leqq \phi[f(t, v_0, v_0') + M_1(v_1 - v_0) + M_2(v_1' - v_0') - f(t, v_0, v_0')] \\ &= M_1 p + M_2 p', \end{aligned}$$

in view of (A_1). Furthermore, the respective boundary conditions yield $B^i p \geqq 0$. Hence, we have $p(t) \geqq 0$ on I which implies $v_0 \leqq v_1 = Av_0$. Similarly, we can show that $Aw_0 \leqq w_0$ proving (i).

To prove (ii), let $\eta_1, \eta_2 \in C^1[I, E]$ such that $\eta_1, \eta_2 \in [v_0, w_0]$, $\eta_1 \leqq \eta_2$ and $\|\eta_1'\|$, $\|\eta_2'\| \leqq c^*$. Suppose that $A\eta_1 = u_1$ and $A\eta_2 = u_2$ and set $p(t) = \phi[u_2(t) - u_1(t)]$ for $\phi \in K^*$. Using (A_2), we get

$$\begin{aligned} p'' &= \phi[f(t, \eta_2, \eta_2') + M_1(u_2 - \eta_2) + M_2(u_2' - \eta_2') - f(t, \eta_1, \eta_1') \\ &\qquad - M_1(u_1 - \eta_1) - M_2(u_1' - \eta_1')] \\ &\leqq \phi[M_1(u_2 - \eta_1) + M_2(u_2' - \eta_1') - M_1(u_1 - \eta_1) - M_2(u_1' - \eta_1')] \\ &= M_1 p + M_2 p'. \end{aligned}$$

Clearly, $B^i p = 0$ and hence $p(t) \geqq 0$ on I which yields $A\eta_1 \leqq A\eta_2$. This proves the Lemma.

In view of Lemma 2.4, we can define the sequences $\{v_n\}$, $\{w_n\}$ as follows: $v_n = Av_{n-1}$ and $w_n = Aw_{n-1}$. It is easy to see that $\{v_n\}$, $\{w_n\}$ are monotone sequences such that $v_n \leqq w_n$, v_n, $w_n \in [v_0, w_0]$ and $\|v_n'\|$, $\|w_n'\| \leqq c^*$ for all n. We shall now show that there exist subsequences of $\{v_n, v_n'\}$, $\{w_n, w_n'\}$ which converge uniformly on I.

LEMMA 2.5. *Let K be normal and let the assumptions* (A_1), (A_2), (A_3), (A_4) *and* (A_5) *hold. Suppose further that* $Q(L + M_1 + M_2) < 1$ *where* $Q = \max[\max_{I\times I} |G(t, s)|, \max_{I\times I}|G_t(t, s)|]$. *Then the sequences* $\{v_n, v_n'\}$, $\{w_n, w_n'\}$ *are uniformly bounded, equicontinuous and relatively compact on I.*

PROOF. Since the cone K is assumed to be normal, it follows from v_n,

$w_n \in [v_0, w_0]$ that $\{v_n\}$, $\{w_n\}$ are uniformly bounded on I. We also know that $\|v_n'\|$, $\|w_n'\| \leqq c^*$ on I. This implies the equicontinuity of the sequences $\{v_n, v_n'\}$, $\{w_n, w_n'\}$ because of the fact f maps bounded sets into bounded sets. Since v_n satisfies the relation

$$v_n(t) = \int_0^1 G(t, s)\sigma_{n-1}(s)ds + \phi(t),$$

where $\sigma_{n-1}(s) = f(s, v_{n-1}(s), v_{n-1}'(s)) - M_1 v_{n-1}(s) - M_2 v_{n-1}'(s)$, we see that

$$v_n'(t) = \int_0^t G_t(t, s)\sigma_{n-1}(s)ds + \phi'(t).$$

Letting $Q = \max[\max_{I\times I}|G(t, s)|, \max_{I\times I}|G_t(t, s)|]$ and using (A_5), we get, using arguments similar to those in [8], the estimates

$$\alpha(\{v_n(t)\}) \leqq Q(L + M_1 + M_2)\max(\alpha(\{v_n(t)\}), \alpha(\{v_n'(t)\})),$$
$$\alpha(\{v_n'(t)\}) \leqq Q(L + M_1 + M_2)\max(\alpha(\{v_n(t)\}), \alpha(\{v_n'(t)\})).$$

Since $Q(L + M_1 + M_2) < 1$, it follows from these estimates that $\alpha(\{v_n(t)\}) \equiv 0$ and $\alpha(\{v_n'(t)\}) \equiv 0$. Similar conclusions are true relative to the sequences $\{w_n\}$, $\{w_n'\}$. The proof of Lemma 2.5 is therefore complete.

We now apply Ascoli's theorem to the sequences $\{v_n, v_n'\}$, $\{w_n, w_n'\}$ to obtain subsequences $\{v_{n_k}, v_{n_k}'\}$, $\{w_{n_k}, w_{n_k}'\}$ which converge uniformly on I. Since the sequences $\{v_n\}$, $\{w_n\}$ are monotone, it follows that the full sequences $\{v_n\}$, $\{w_n\}$ converge uniformly and monotonically to continuous functions, that is, $\lim_{n\to\infty} v_n(t) = \rho(t)$ and $\lim_{n\to\infty} w_n(t) = r(t)$ on I. It is now easy to conclude that the full sequences $\{v_n'\}$, $\{w_n'\}$ also converge uniformly to $\rho'(t)$, $r'(t)$ respectively on I. It then follows from (2.2) and (2.3) that $\rho(t)$, $r(t)$ are solutions of BVP (2.1).

Finally, let us show that $\rho(t)$, $r(t)$ are minimal and maximal solutions respectively of BVP (2.1) in $[v_0, w_0]$ on I. To this end, let $u(t)$ be any solution of BVP (2.1) on I such that $u \in [v_0, w_0]$ on I. Assume that for some $n \geqq 0$, $v_n \leqq u \leqq w_n$ on I. Set $p(t) = \phi(u(t) - v_{n+1}(t))$ for $\phi \in K^*$ so that we obtain $B^i p = 0$. Then by (A_2), we have

$$\begin{aligned} p'' &= \phi[f(t, u, u') - f(t, v_n, v_n') - M_1(v_{n+1} - v_n) - M_2(v_{n+1}' - v_n')] \\ &\leqq \phi[M_1(u - v_n) + M_2(u' - v_n') - M_1(v_{n+1} - v_n) - M_2(v_{n+1}' - v_n')] \\ &= M_1 p + M_2 p'. \end{aligned}$$

It therefore follows that $v_{n+1} \leqq u$ on I. A similar argument yields $u \leqq w_{n+1}$ on I. Since $u \in [v_0, w_0]$, we have by induction $v_n \leqq u \leqq w_n$ on I for all n. Thus, we obtain by taking the limit as $n \to \infty$, $\rho(t) \leqq u(t) \leqq r(t)$ on I, proving ρ, r are minimal and maximal solutions of BVP (2.1) on I. We have therefore proved the following main result.

THEOREM 2.1. *Let K be a normal cone and let the assumptions* (A_1), (A_2), (A_3), (A_4) *and* (A_5) *hold. Then, if* $Q(L + M_1 + M_2) < 1$, *there exist monotone sequences* $\{v_n\}$, $\{w_n\}$ *which converge uniformly and monotonically to the minimal and maximal solutions* $\rho(t)$, $r(t)$ *respectively of the* BVP (2.1) *on I, that is, if u is any solution of the* BVP (2.1) *on I such that* $u \in [v_0, w_0]$, *then*

$$v_0 \leqq v_1 \leqq \cdots \leqq v_n \leqq \rho \leqq u \leqq r \leqq w_n \leqq \cdots \leqq w_1 \leqq w_0 \text{ on } I.$$

COROLLARY 2.1. *If the solutions of the* BVP (2.1) *are unique, then the assumptions of Theorem* 2.1 *imply that* $\rho(t) = u(t) = r(t)$ *on I.*

The smallness assumption $Q(L + M_1 + M_2) < 1$ is satisfied in general by choosing c^* small which in turn restricts the size of R (the bound on v_0, w_0). This is a price one has to pay to treat the BVP in an arbitrary Banach space. If on the other hand, $E = \mathbf{R}^n$, the assumptions (A_5) and $Q(L + M_1 + M_2) < 1$ are superfluous. We then have the following result which is new in itself.

COROLLARY 2.2. *The conclusions of Theorem* 2.1 *are valid under the assumptions* (A_1), (A_2), (A_3) *and* (A_4) *if* $E = \mathbf{R}^n$ *and* $K = \mathbf{R}^n_+$.

We remark that the linear procedure followed here has its origin in our recent announcement [5] where the scalar BVP is treated.

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