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MULTIPARAMETER BIFURCATION PROBLEMS FOR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday

1. Introduction. Consider the operator equation

$$u = F(\lambda, u), \tag{1.1}$$

where F: $\Lambda \times E \to E$ is a completely continuous map, Λ, E are real Banach spaces, and Λ is viewed as a space of parameters. Assuming $F(\lambda, 0) = 0$ for all $\lambda \in \Lambda$, several interesting questions arise. Among these are for what values of λ, if any, are there solutions (λ^*, u^*) of (1.1), with $\|(\lambda^*, u^*) - (\lambda, 0)\|_{\Lambda \times E}$ arbitrarily small, and given that such parametric values exist, are there connected branches of solutions emanating from $(\lambda, 0)$ and existing globally. Points such as $(\lambda, 0)$ are known as bifurcation points, and the questions above are included in the branch of mathematics known as bifurcation theory.

One class of problems for which (1.1) is an appropriate setting in which to study branching behavior is the class of nonlinear Sturm-Liouville boundary value problems in ordinary differential equations. Investigations into this set of problems using the topological degree of Leray and Schauder have been made by Crandall and Rabinowitz **[5]**, Rabinowitz **[10]**, Turner **[12]**, and others.

The purpose of this paper is to examine via the topological degree of Leray and Schauder the bifurcation phenomena associated with a multiparameter generalization of Sturm-Liouville type problems which has been investigated by Browne and Sleeman (**[1]**, **[2]**). We give a precise statement of the problem in §2. In §3, a number of theorems necessary to the analysis of the problem are stated. These theorems for the most part are established by means of the Leray-Schauder degree. Finally, in §4, the analysis of the problem is presented.

2. Statement of the problem. We let $[d_i, b_i]$ be a closed bounded interval

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on the real line $\mathbf{R}$ and let y_i be a function defined on $[d_i, b_i]$ for $i = 1, 2, \ldots, k$. We consider the following system of equations:

$$(2.1)\qquad (p_i(t_i)y_i'(t_i))' + q_i(t_i)y_i(t_i) + \left[\sum_{j=1}^{k} \lambda_j a_{ij}(t_i)\right]y_i(t_i) + r_i(t_i, \lambda, y_i(t_i), y_i'(t_i)) = 0,$$

$i = 1, 2, \ldots, k$, subject to the boundary conditions

$$(2.2\text{-i})\qquad \alpha_i y_i(d_i) + \alpha_i' y_i'(d_i) = 0,$$

$$(2.2\text{-ii})\qquad \beta_i y_i(b_i) + \beta_i' y_i'(b_i) = 0,$$

i = 1, 2, ..., k. The following conditions are assumed to hold:

(2.3-i) $(|\alpha_i| + |\alpha_i'|)(|\beta_i| + |\beta_i'|) > 0$ for $i = 1, 2, \ldots, k$.

(2.3-ii) p_i: $[d_i, b_i] \to \mathbf{R}$ is positive and continuously differentiable for $i = 1, 2, \ldots, k$.

(2.3-iii) q_i: $[d_i, b_i] \to \mathbf{R}$ is continuous for $i = 1, 2, \ldots, k$.

(2.3-iv) a_{ij}: $[d_i, b_i] \to \mathbf{R}$ is positive and continuous for $i, j = 1, 2, \ldots, k$.

(2.3-v) h: $\prod_{i=1}^{k} [d_i, b_i] \to \mathbf{R}$ is positive, where if $(t_1, \ldots, t_k) \in \prod_{i=1}^{k} [d_i, b_i]$, $h(t_1, \ldots, t_k) = \det[a_{ij}(t_i)]^{k}{}_{i,j=1}$.

(2.3-vi) $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbf{R}^k$.

(2.3-vii) $r_i(t, \lambda, u, v) = o(|u| + |v|)$ uniformly for $t \in [d_i, b_i]$ and λ contained in compact subsets of $\mathbf{R}^k$.

The problem is to examine the structure of the solution set to (2.1)–(2.3). There are two tasks involved in such an examination. The first of these is to determine which points λ in $\mathbf{R}^k$ are bifurcation points for (2.1)–(2.3) and to give a geometric description of this set. The second task is to determine the existence of global branches of solutions to (2.1)–(2.3), relating, if possible, the structure of a point on a solution branch to the structure of an eigenfunction of an associated linear problem.

3. Preliminary results. Let E_i be a real Banach space, for $i = 1, 2, \ldots, k$. Then consider the equation

$$(3.1)\qquad u = F(\lambda, u),$$

where $u = (u_1, \ldots, u_k) \in E_1 \times \cdots \times E_k$ and $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbf{R}^k$. F: $\mathbf{R}^k \times E_1 \times \cdots \times E_k \to E_1 \times \cdots \times E_k$ has the following form:

$$F(\lambda, u) = (F_1(\lambda, u_1), \ldots, F_k(\lambda, u_k)), \tag{3.2}$$

where $F_i: \mathbf{R}^k \times E_i \to E_i$ is given by

$$F_i(\lambda, u_i) = \sum_{j=1}^{k} \lambda_j A_{ij} u_i + r_i(\lambda, u_i). \tag{3.3}$$

$A_{ij}: E_i \to E_i$ is a compact linear operator and $r_i: \mathbf{R}^k \times E_i \to E_i$ is completely continuous and satisfies $\lim_{\|u_i\|_i \to 0} r_i(\lambda, u_i)/\|u_i\|_i = 0$ uniformly for λ contained in compact subsets of $\mathbf{R}^k$, $\| \quad \|_i$ the norm in E_i.

Note that $F(\lambda, u)$ can now be expressed in the form

$$F(\lambda, u) = A(\lambda)u + r(\lambda, u), \tag{3.4}$$

with

$$A(\lambda)(u_1, \ldots, u_k) = \sum_{j=1}^{k} \lambda_j (A_{1j}u_1, \ldots, A_{ij}u_i, \ldots, A_{kj}u_k) \tag{3.5}$$

and

$$r(\lambda, u) = (r_1(\lambda, u_1), \ldots, r_i(\lambda, u_i), \ldots, r_k(\lambda, u_k)). \tag{3.6}$$

Since $(\lambda, 0)$ solves (3.1) for any $\lambda \in \mathbf{R}^k$, a bifurcation analysis of (3.1) is quite natural. Thus we let B denote the set $\{\lambda \in \mathbf{R}^k: (\lambda, 0) \text{ is a bifurcation point for (3.1)}\}$. The following results then obtain and are established by means of the Leray-Schauder topological degree in [3] and [4] (for definition and properties, see [8] and [9]).

DEFINITION 3.1. $\lambda = (\lambda_1, \ldots, \lambda_k)$ is called a generalized characteristic value of $A(\lambda)$ (3.5) provided the null space $N(I - A(\lambda))$ of the linear operator $I - A(\lambda)$ is nontrivial. For such a λ, the algebraic multiplicity of λ, denoted mult λ, is the dimension of the subspace $\bigcup_{j=1}^{\infty} N\{I - A(\lambda))^j\}$ of $E = E_1 \times \cdots \times E_k$.

PROPOSITION 3.2. *B is closed in $\mathbf{R}^k$.*

PROPOSITION 3.3. *Σ_A is closed in $\mathbf{R}^k$.*

THEOREM 3.4. $B \subseteq \Sigma_A$.

THEOREM 3.5. *Let $\lambda_0 \in \mathbf{R}^k$ have odd algebraic multiplicity and suppose that there exist λ_1 and λ_2, elements of $\mathbf{R}^k \backslash \Sigma_A$ with the following properties:*
(i) *$0, \lambda_0, \lambda_1, \lambda_2$ are collinear.*
(ii) *There is a $t \in (0, 1)$ such that $\lambda_0 = t\lambda_1 + (1 - t)\lambda_2$.*
(iii) *If $h(t) = t\lambda_1 + (1 - t)\lambda_2$, $t \in [0, 1]$, then $h([0, 1]) \cap \Sigma_A = \{\lambda_0\}$.*
Then $(\lambda_0, 0)$ is a bifurcation point for (3.1)–(3.3).

COROLLARY 3.6. *If λ_0, λ_1, and λ_2 are as in the statement of Theorem* 3.5, *then $\mathbf{R}^k \backslash \Sigma_A$ is not connected.*

DEFINITION 3.7. Let $\lambda \in \mathbf{R}^k$ be a generalized characteristic value of $A(\lambda)$, where $A(\lambda)$ is as in (3.5). Let $h: \mathbf{R} \to \mathbf{R}^k$ be a line such that $h(0) = \lambda$ and $\mathbf{u}_h$ is a unit vector in the direction of h. We say that $\mathbf{u}_h$ is a direction of changing degree at λ if the following condition holds: there is a number $\varepsilon_h > 0$ such that

(i) $\deg_{LS}(I - A(h(t)), B(0, 1), 0)$ is defined, for all t such that $|t| < \varepsilon_h$ and $t \neq 0$;

(ii) $\deg_{LS}(I - A(h(\tau)), B(0, 1), 0) = \operatorname{sgn}(\tau\beta) \cdot \deg_{LS}(I - A(h(\beta)), B(0, 1), 0)$ for all $\tau, \beta \in (-\varepsilon_h\, \varepsilon_h)$, $\tau \neq 0$, $\beta \neq 0$.

NOTATION 3.8. Let $\mathscr{S}$ denote the closure in $\mathbf{R}^k \times E_1 \times \cdots \times E_k$ of the set $\{(\lambda, u) \in \mathbf{R}^k \times E_1 \times \cdots \times E_k: (\lambda, u)$ solves (3.1)–(3.6) and $u \neq 0\}$.

THEOREM 3.9. *Consider* (3.1)–(3.6). *Let* $\lambda \in \Sigma_A$. *Suppose that* $\mathbf{u}_h$ *is a direction of changing degree at* λ. *Then there exists a subcontinuum* $\mathscr{C}$ *of* $\mathscr{S}$ *meeting* $(\lambda, 0)$ *such that either*

(i) $\mathscr{C}$ *is unbounded, or*

(ii) $\mathscr{C} \cap ([B \times \{0\}] \setminus \{(\lambda, 0)\}) \neq \varnothing$.

COROLLARY 3.10. *Let* $\mathscr{C}$ *denote the maximal continuum of* $\mathscr{S}$ *meeting* $(\lambda, 0)$, *where* λ *is as in Theorem* 3.9. *Then for each direction* $\mathbf{u}_h$ *of degree change at* λ, *there is a subcontinuum* $\mathscr{C}_h$ *of* $\mathscr{C}$ *satisfying the following conditions*:

(i) $\mathscr{C}_h$ *satisfies the alternatives of Theorem* 3.9,

(ii) *the projection of* $\mathscr{C}_h$ *into parameter space is contained in the line* h.

In addition to the above bifurcation results, a result from several variable perturbation theory is needed for the analysis in §4.

THEOREM 3. 11. *Suppose that* $A(\varepsilon_1, \ldots, \varepsilon_k)$ *is a bounded linear operator defined on a Hilbert space* $\mathscr{H}$ *which is a power series in* $(\varepsilon_1, \ldots, \varepsilon_k)$,

$$A(\varepsilon_1, \cdots, \varepsilon_k) = A_0 + \varepsilon_1 A_{11} + \varepsilon_2 A_{12} + \cdots + \varepsilon_k A_{1k} + \cdots \tag{3.7}$$

convergent in a neighborhood of $(0, \ldots, 0)$. *Suppose that for* $(\varepsilon_1, \ldots, \varepsilon_k) \in \mathbf{R}^k$, $A(\varepsilon_1, \ldots, \varepsilon_k)$ *is self-adjoint. Suppose furthermore that* λ *is a simple eigenvalue of the operator* $A(0, \ldots, 0) = A_0 = A$ *and that there are positive numbers* d_1 *and* d_2 *such that the spectrum of* A *in the interval* $(\lambda - d_1, \lambda + d_2)$ *consists of exactly the point* λ. *Then there is an ordinary power series* $\lambda(\varepsilon_1, \ldots, \varepsilon_k)$ *and a power series* $\phi(\varepsilon_1, \ldots, \varepsilon_k)$ *in* $\mathscr{H}$ *convergent in a neghborhood of* $(0, \ldots, 0)$ *such that*

(i) $\phi(\varepsilon_1, \ldots, \varepsilon_k)$ *is an eigenfunction of* $A(\varepsilon_1, \ldots, \varepsilon_k)$ *belonging to* $\lambda(\varepsilon_1, \ldots, \varepsilon_k)$, i.e., $A(\varepsilon_1, \ldots, \varepsilon_k)\phi(\varepsilon_1, \ldots, \varepsilon_k) = \lambda(\varepsilon_1, \ldots, \varepsilon_k)$ $\phi(\varepsilon_1, \ldots, \varepsilon_k)$, *and*

(ii) *for each pair of positive numbers* d_1', d_2', $d_1' < d_1$, $d_2' < d_2$, *there is* *a positive number ρ such that the spectrum of $A(\varepsilon_1, \ldots, \varepsilon_k)$ in $(\lambda - d_1', \lambda + d_2')$ consists of $\lambda(\varepsilon_1, \ldots, \varepsilon_k)$ if $(\varepsilon_1, \ldots, \varepsilon_k) \in \mathbf{R}^k$ and $|(\varepsilon_1, \ldots, \varepsilon_k)| < \rho$.*

4. Main results. For $i = 1, 2, \ldots, k$, define

$$L_i x = (p_i x')' + q_i x, \tag{4.1}$$

where the independent variable t_i has been suppressed, and consider the problem

$$-L_i x = \mu x \tag{4.2}$$

subject to boundary conditions (2.2), where $\mu \in \mathbf{C}$. Recall that a real (complex) number μ for which (4.2)–(2.2) has a nontrivial solution x on $[d_i, b_i]$ is called an eigenvalue of the problem. Furthermore, such a nontrivial solution x is called an eigenfunction or eigensolution. Using (4.1), (2.1) may be written as

$$-L_i y_i = \left[\sum_{j=1}^{k} \lambda_j a_{ij}\right] y_i + r_i(t_i, \lambda, y_i, y_i'), \tag{4.3}$$

$i = 1, 2, \ldots, k$. If we assume that 0 is not an eigenvalue of (4.2)–(2.2) and let $-G_i(\cdot, \cdot) : [d_i, b_i] \times [d_i, b_i] \to \mathbf{R}$ denote the Green's function associated with (4.2)–(2.2) for $i = 1, 2, \ldots, k$, then (4.3)–(2.2) is equivalent to the integral equations problem

$$\begin{aligned} y_i(t_i) = &\sum_{j=1}^{k} \lambda_j \int_{d_i}^{b_i} G_i(t_i, s_i) a_{ij}(s_i) y_i(s_i) ds_i \\ &+ \int_{d_i}^{b_i} G_i(t_i, s_i) r_i(s_i, \lambda, y_i(s_i), y_i'(s_i)) ds_i, \end{aligned} \tag{4.4}$$

$i = 1, 2, \ldots, k$.

We now make some observations concerning the righthand side of (4.4). Let E_i denote the real Banach space of continuously differentiable functions on $[d_i, b_i]$ satisfying boundary conditions (2.2) on $[d_i, b_i]$ with

$$\|u\|_i = \max_{t \in [d_i b_i]} |u(t)| + \max_{t \in [d_i b_i]} |u'(t)| \tag{4.5}$$

as norm on E_i.

LEMMA 4.1. *If for $u \in E_i$ we define $A_{ij} u$ by*

$$A_{ij} u(t) = \int_{d_i}^{b_i} G_i(t, s) a_{ij}(s) u(s) ds, \tag{4.6}$$

then A_{ij} maps E_i into E_i and is a compact linear operator.

Let $L^2_{a_{ij}}[d_i, b_i]$ denote the set of all measurable functions f on (d_i, b_i) such that

$$\int_{d_i}^{b_i} |f(t)|^2\, a_{ij}(t)dt < \infty.$$

It is well-known that $L^2_{a_{ij}}[d_i, b_i]$ is a Hilbert space with inner product

$$(f, g) = \int_{d_i}^{b_i} a_{ij}(t)f(t)g(t)dt,$$

LEMMA 4.2. *If A_{ij} is defined on $L^2_{a_{ij}}[d_i, b_i]$ by* (4.6), *then A_{ij}: $L^2_{a_{ij}}[d_i, b_i] \to L^2_{a_{ij}}[d_i, b_i]$ is a compact self-adjoint linear operator.*

LEMMA 4.3. *If $u \in E_i$ we define $R_i(\lambda, u)$ by*

$$R_i(\lambda, u)(t) = \int_{d_i}^{b_i} G_i(t, s)r_i(s, \lambda, u(s), u'(s))ds.$$

Then R_i: $\mathbf{R}^k \times E_i \to E_i$ is completely continuous and satisfies

$$\lim_{\|u\|_i \to 0} R_i(\lambda, u)/\|u\|_i = 0$$

uniformly for λ contained in compact subsets of $\mathbf{R}^k$.

Observe that (4.4) may be written as $y_i = \sum_{j=1}^k \lambda_j A_{ij} y_i + R_i(\lambda, y_i)$. Then Lemma 4.1 and Lemma 4.3 imply the following result.

THEOREM 4.4. *For $\lambda \in \mathbf{R}^k$ and $y_i \in E_i$, define $F_i(\lambda, y_i)$ by*

$$F_i(\lambda, y_i) = \sum_{j=1}^k \lambda_j A_{ij} y_i + R_i(\lambda, y_i). \tag{4.7}$$

Then setting

$$F(\lambda, y) = (F_1(\lambda, y_1), \ldots, F_k(\lambda, y_k)) \tag{4.8}$$

for $y = (y_1, \ldots, y_k) \in E_1 \times \cdots \times E_k$ and $\lambda \in \mathbf{R}^k$ satisfies conditions (3.1)–(3.3). *Thus Theorem* 3.5 *is applicable to the situation.*

In light of Theorem 3.4, we next determine Σ_A. Observe that

$$A(\lambda)\begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_k \end{bmatrix} = \sum_{j=1}^k \lambda_j \begin{bmatrix} A_{1j} & 0 & \cdots & 0 \\ 0 & A_{2j} & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & \cdot & \cdots & A_{kj} \end{bmatrix} \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_k \end{bmatrix} \tag{4.9}$$

where the operators in the *i-th* column map E_i into E_i. Denote by $\tilde{A}_j$ the operator on $E = E_1 \times \cdots \times E_k$ given by

$$\tilde{A}_j = \begin{bmatrix} A_{1j} & 0 & \cdots & 0 \\ 0 & A_{2j} & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & \cdot & \cdots & A_{kj} \end{bmatrix} \tag{4.10}$$

Then

$$\Sigma_A = \left\{\lambda \in \mathbf{R}^k : N\left[I - \sum_{j=1}^{k} \lambda_j \tilde{A}_j\right] \neq \{0\}\right\}.$$

It is easy to see that this set is the same as the set of $\lambda \in \mathbf{R}^k$ for which at least one equation in the system of equations

$$(p_i u_i')' + q_i u_i + \sum_{j=1}^{k} \lambda_j a_{ij} u_i = 0, \tag{4.11}$$

$i = 1, 2, \ldots, k$, has a nontrivial solution satisfying (2.2).

We now give a geometric description of Σ_A and determine which points of Σ_A are bifurcation points. We proceed as follows.

First fix an $i \in \{1, 2, \ldots, k\}$. Consider (4.11)–(2.2) for this particular i, and write (4.11)–(2.2) as

$$(p_i u_i')' + \left(q_i + \sum_{j=1}^{k-1} \lambda_j a_{ij}\right) u_i + \lambda_k a_{ik} u_i = 0. \tag{4.12}$$

u_i subject to (2.2). Now fix $\lambda_j = \mu_j, j < k$. By condition (2.3-iv), $a_{ik} > 0$. Then by classical Sturm-Liouville theory for boundary value problems, there is a countable increasing sequence of real numbers $\{\lambda_k^n\}_{n=1}^{\infty}$ converging to $+\infty$ such that (4.12)–(2.2) has a nontrivial soultion. Then if $\tilde{\lambda}_n \in \mathbf{R}^k$ is given by $\tilde{\lambda}_{nj} = \mu_j$, $j \neq k$, $\tilde{\lambda}_{nk} = \lambda_k^n$, $\tilde{\lambda}_n$ is a generalized characteristic value for the operator $\sum_{j=1}^{k} \lambda_j \tilde{A}_j^i$ associated with (4.12)–(2.2), where $\tilde{A}_j^i$ is given by

$$\tilde{A}_j^i = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & A_{ij} & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \tag{4.13}$$

It follows from classical Sturm-Liouville theory that mult $\tilde{\lambda}_n = 1$. Thus Corollary 3.6 implies there is a separation of $\mathbf{R}^k$ through $\tilde{\lambda}_n$ consisting of generalized characteristic values of $\sum_{j=1}^{k} \lambda_j \tilde{A}_j^i$. Furthermore, the nontrivial solutions to (4.12)–(2.2) at $\tilde{\lambda}_n$ have $n - 1$ simple zeros in (d_i, b_i). We have thus established the following result.

THEOREM 4.5. *The elements of Σ_A which are also generalized characteristic values of $\sum_{j=1}^{k} \lambda_j \tilde{A}_j^i$ all lie on separations of $\mathbf{R}^k$ consisting of generalized characteristic values of $\sum_{j=1}^{k} \lambda_j \tilde{A}_j^i$.*

We now examine these sets more closely. Let $L_i(\lambda_1, \ldots, \lambda_{k-1})$ denote the operator given by

$$L_i(\lambda_1, \ldots, \lambda_{k-1})u = (p_i u')' + \left(q_i + \sum_{j=1}^{k-1} \lambda_j a_{ij}\right)u. \tag{4.14}$$

The formal expression $L_i(\lambda_1, \ldots, \lambda_{k-1})$ is analytic in $\lambda_1, \ldots, \lambda_{k-1}$. Then by uniqueness of initial value problems, solutions to initial value problems for the equation $L_i(\lambda_1, \ldots, \lambda_{k-1})u = \mu u$ depend analytically on $\lambda_1, \ldots, \lambda_{k-1}$. Let $(\lambda_1^0, \ldots, \lambda_{k-1}^0, \lambda_k^0)$ be a generalized characteristic value for $\sum_{j=1}^k \lambda_j \tilde{A}_j^i$ and assume for the moment that $\lambda_k^0 \neq 0$ and that 0 is not an eigenvalue for $L_i(\lambda_1^0, \ldots, \lambda_{k-1}^0)$. Then by the construction of the Green's function, 0 is not an eigenvalue of $L_i(\lambda_1, \ldots, \lambda_{k-1})$ and $G_i(t, s, \lambda_1, \ldots, \lambda_{k-1})$ depends analytically on $\lambda_1 \ldots, \lambda_{k-1}$ in a neighborhood of $(\lambda_1^0, \ldots, \lambda_{k-1}^0) \in \mathbf{C}^{k-1}$. Thus the map

$$u \to \int_{d_i}^{b_i} G_i(t, s, \lambda_1, \ldots, \lambda_{k-1}) a_{ik}(s)\, u(s)\, ds$$

is a compact (Lemma 4.2) linear map on $L^2_{a_{ik}}[d_i, b_i]$ which is analytic in $\lambda_1, \ldots, \lambda_{k-1}$ in a neighborhood of $(\lambda_1^0, \ldots, \lambda_{k-1}^0)$ and self-adjoint for $(\lambda_1, \ldots, \lambda_{k-1}) \in \mathbf{R}^{k-1}$. Since $1/\lambda_1^0$ is a simple isolated eigenvalue of the operator for $(\lambda_1^0, \ldots, \lambda_{k-1}^0)$, Theorem 3.11 is applicable.

REMARK 4.6. The assumption that 0 is not an eigenvalue of $L_i(\lambda_1^0, \ldots, \lambda_{k-1}^0)$ can be removed by considering instead $\tilde{L}_i(\lambda_1^0, \ldots, \lambda_{k-1}^0) = L_i(\lambda_1^0, \ldots, \lambda_{k-1}^0) + \eta I$ for $\eta > 0$ and sufficiently small. Since $(0, \ldots, 0)$ is not a generalized characteristic value of $\sum_{j=1}^k \lambda_j \tilde{A}_j^i$, the assumption that $\lambda_k^0 \neq 0$ is also not an intrinsic problem.

Now apply Theorem 3.11 at $(\lambda_1^0, \ldots, \lambda_{k-1}^0, \lambda_k^0)$. Let N denote the neighborhood guaranteed by Theorem 3.11, and let $(\lambda_1^n, \ldots, \lambda_{k-1}^n)$ be any sequence in $N \cap \mathbf{R}^{k-1}$ converging to $(\lambda_1^0, \ldots, \lambda_{k-1}^0)$. Then $\lambda_k^n \to \lambda_k^0$ and if $\{\phi^n\}_{n=1}^\infty$, ϕ^0 denote the corresponding eigenfunctions, a simple argument shows that $\phi^n \to \phi^0$ in $C^1[d_i, b_i]$. Thus the eigenfunctions associated with $(\lambda_1, \ldots, \lambda_k)$ in a neighborhood of $(\lambda_1^0, \ldots, \lambda_k^0)$ have the same nodal structure as the eigenfunctions for $(\lambda_1^0, \ldots, \lambda_k^0)$. Hence the following theorem is established.

THEOREM 4.7. *Let $i \in \{1, 2, \ldots, k\}$. Let $\Gamma_i = \{\lambda \in \Sigma_A : \lambda$ is a generalized characteristic value for $\sum_{j=1}^k \lambda_j \tilde{A}_j^i\}$. Then $\Gamma_i = \bigcup_{\ell=1}^\infty \Gamma_i^\ell$, where Γ_i^ℓ is characterized as follows:*

(i) *$\Gamma_i^\ell = \{\lambda \in \Gamma_i :$ if ϕ is an eigenfunction associated with λ, then ϕ has $\ell - 1$ simple zeros in $(d_i, b_i)\}$.*
(ii) *Γ_i^ℓ is a $k-1$ analytic manifold which separates $\mathbf{R}^k$ for $\ell = 1, 2, 3, \ldots$.*
(iii) *$\Gamma_i^\ell \cap \Gamma_i^{\ell'} = \phi$ if $\ell \neq \ell'$.*
(iv) *Γ_i^ℓ is unbounded for $\ell = 1, 2, 3, \ldots$.*

(v) $\{\Gamma_i^{\ell}\}_{\ell=1}^{\infty}$ *may be ordered in the following sense: Let* $(\mu_1, \ldots, \mu_{k-1}) \in \mathbf{R}^{k-1}$. *Then there is a unique* $\lambda_k(\ell)$ *such that* $\bar{\lambda}^{\ell} \in \Gamma_i^{\ell}$, *where*

$$\lambda_j^{\ell} = \begin{cases} \mu_j & \text{if } j < k \\ \lambda_k(\ell) & \text{if } j = k. \end{cases}$$

Furthermore, $\lambda_k(1) < \lambda_k(2) < \cdots < \lambda_k(\ell) < \cdots \to +\infty$.

REMARK 4.8. Theorem 4.7 states that Γ_i is the union of a collection of "hypersurfaces" which are "essentially parallel." In case the coefficient functions a_{ij} are constant, Γ_i^{ℓ} is a genuine hypersurface which is parallel to $\Gamma_i^{\ell'}$, $\ell \neq \ell'$.

COROLLARY 4.9. $$\Sigma_A = \bigcup_{i=1}^{k} \Gamma_i.$$

We now consider how the components of Γ_i meet those of Γ_j and determine the set B of bifurcation points for the problem. For this purpose, the following classical result is needed.

THEOREM 4.10 (KLEIN'S OSCILLATION THEOREM). *Consider the set of equations* (4.11)–(2.2) *for* $i = 1, 2, \ldots, k$, *and assume that* (2.3-i)–(2.3-v) *hold. Then the set of k-tuples* $(\lambda_1, \ldots, \lambda_k)$ *for which* (4.11)–(2.2) *has a nontrivial solution for all* $i = 1, 2, \ldots, k$ *is a countably infinite set in* $\mathbf{R}^k$ *accumulating only at* ∞. *Furthermore, given a k-tuple of nonnegative integers* $(p_1, \ldots, p_k)$, *there is a unique generalized characteristic value* $(\lambda_1, \ldots, \lambda_k)$ *for which the corresponding k-tuple* $(y_1, \ldots, y_k)$ *of nontrivial solutions to* (4.11)–(2.2), $(y_1, \ldots, y_k) \in E_k \times \cdots \times E_k$, *has the property that* y_i *has* p_i *simple zeros in* (d_i, b_i).

PROOF. See [6] or [7].

REMARK 4.11. If $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \Sigma_A$ is the point associated by Klein's Oscillation Theorem with the k-tuple of nonnegative integers $(p_1, p_2, \ldots, p_k)$, then

$$\{(\lambda_1, \lambda_2, \ldots, \lambda_k)\} = \Gamma_1^{p_1+1} \cap \Gamma_2^{p_2+1} \cap \cdots \cap \Gamma_k^{p_k+1}.$$

LEMMA 4.12. *Let g denote the radial ray in* $\mathbf{R}^k$ *emanating from the origin which passes through* $(\lambda_1, \ldots, \lambda_k)$, *where* $\lambda_1^2 + \cdots + \lambda_k^2 = 1$. *Then* $g \cap \Sigma_A$ *is an at most countable set. Furthermore, if* $\tilde{g}$ *is a compact subset of g, then* $\tilde{g} \cap \Sigma_A$ *is finite.*

PROOF. This result is an immediate consequence of the Riesz theory of compact operators [11].

Let us now consider the implications of Theorem 4.10 to the structure of Σ_A. By Theorem 4.7, Σ_A consists of k-countably infinite collections of "essentially parallel" "hypersurfaces." Theorem 4.10 adds that generically the intersection of two such "hypersurfaces" from different collections is a $k-2$ dimensional object, the intersection of three is a $k-3$ dimemsional object, and so on. Points in Σ_A lying in only one "hypersurface" have algebraic multiplicity 1 with respect to $A(\lambda)$. Thus they are bifurcation points by virtue of Lemma 4.12 and Theorem 3.5. Then in the generic situation one can see that all of Σ_A is contained in the set of bifurcation points B. This last statement holds since by Proposition 3.2, B is closed in $\mathbf{R}^k$. If the description of Σ_A above is not valid globally, we can still establish the equality of B and Σ_A by viewing $\lambda \in \Sigma_A$ as a generalized characteristic value for $\sum_{j=1}^k \lambda_j \tilde{A}_j^i$ for some $i \in \{1, 2, \ldots, k\}$ and noting that its multiplicity for this reduced problem is 1. This proves the following result.

THEOREM 4.13. $B = \Sigma_A$.

We now describe the nontrivial solutions which emanate from $\Sigma_A \times \{0\}$ in $\mathbf{R}^k \times E_1 \times \cdots \times E_k$. We begin with some notation.

NOTATION 4.14. Suppose $(\lambda_1, \ldots, \lambda_k) \in \Sigma_A$ is one of the points in $\mathbf{R}^k$ whose existence is guaranteed by Klein's Oscillation Theorem. Call $(\lambda_1, \ldots, \lambda_k)$ a Klein point and use Σ'_A to denote the set of all Klein points.

THEOREM 4.15. *Suppose* $\lambda^0 \in \Sigma_A \backslash \Sigma'_A$. *Suppose* $\{(\lambda^n, y_1^n, \ldots, y_k^n)\}_{n=1}^\infty$ *is a sequence of nontrivial solutions to* (2.1)–(2.2) *converging to* $(\lambda^0, 0, \ldots, 0)$ *in* $\mathbf{R}^k \times E_1 \times \cdots \times E_k$. *Then there is at least one* $i \in \{1, 2, \ldots, k\}$ *such that for all large* n, $y_i^n = 0$.

PROOF. Suppose the result is false. Then for all $i \in \{1, 2, \ldots, k\}$, there is a subsequence of $\{y_i^n\}$, which we relabel if necessary, such that $y_i^n \neq 0$. Then $y_i^n = \sum_{j=1}^k \lambda_j^n A_{ij} y_i^n + R_i(\lambda^n, y_i^n)$. Then $y_i^n \neq 0$ implies

$$y_i^n/\|y_i^n\|_i = \sum_{j=1}^k \lambda_j^n A_{ij}(y_i^n/\|y_i^n\|_i) + (R_i(\lambda^n, y_i^n)/\|y_i^n\|_i).$$

$(R_i(\lambda^n, y_i^n)/\|y_i^n\|_i) \to 0$ as $n \to \infty$ by Lemma 4.3. That A_{ij} is compact for $j = 1, 2, \ldots, k$ guarantees the existence of another subsequence of $\{y_i^n\}$ (which we again relabel if necessary) such that $A_{ij}(y_i^n/\|y_i^n\|_i) \to y_{ij} \in E_i$. Hence $y_i^n/\|y_i^n\|_i \to v_i$, where $\|v_i\|_i = 1$ and $v_i = \sum_{j=1}^k \lambda_j^0 y_{ij}$. Now $y_i^n/\|y_i^n\|_i \to v_i$ and the continuity of A_{ij} together imply $y_{ij} = A_{ij}v_i$. Thus $v_i = \sum_{j=1}^k \lambda_j^0 A_{ij} v_i$ or $(p_i v_i')' + q_i v_i + \sum_{j=1}^k \lambda_j^0 a_{ij} v_i = 0$ with $v_i \in E_i$, $v_i \neq 0$. Thus $\lambda^0 \in \Sigma'_A$, a contradiction.

THEOREM 4.16. *Suppose* $\lambda^0 \in \Sigma_A$ *and* $\{(\lambda^n, y_1^n, \ldots, y_k^n)\}_{n=1}^\infty$ *is a sequence of nontrivial solutions to* (2.1)–(2.2) *converging to* $(\lambda^0, 0, \ldots, 0)$ *in* $\mathbf{R}^k \times E_1 \times \cdots \times E_k$. *Then for all* $i \in \{1, 2, \ldots, k\}$, *the sequence* $\{(\lambda^n, 0, \ldots, 0, y_i^n, 0, \ldots, 0)\}_{n=1}^\infty$ *converges to* $(\lambda^0, 0, \ldots, 0)$ *in* $\mathbf{R}^k \times E_1 \times \cdots \times E_k$.

Furthermore, there is at least one $i \in \{1, 2, \ldots, k\}$ *such that* $\{(\lambda^n, 0, \ldots, 0, y_i^n, 0, \ldots, 0)\}_{n=1}^{\infty}$ *has a subsequence, relabelled if need be, such that* $y_i^n \neq 0$ *for all n.*

In light of Theorem 4.16, our next step is to consider bifurcation in $\mathbf{R}^k \times \{0\} \times \cdots \times E_i \times \cdots \times \{0\}$ for $\lambda \in \Gamma_i$, where Γ_i is as in Theorem 4.7. We have the following result.

THEOREM 4.17. *Let* $\lambda^0 \in \Gamma_i$, *say* $\lambda_0 \in \Gamma_i^n$. *Then there is an unbounded "k-dimensional" continuum* $\mathscr{C}_{\lambda_0}$ *of nontrivial solutions to* (2.1)–(2.2) *in* $\mathbf{R}^k \times \{0\} \times \cdots \times E_i \times \cdots \times \{0\}$ *emanating from* $(\lambda_0, 0, \ldots, 0)$. *All solutions in* $\mathscr{C}_{\lambda_0}$ *have* $n-1$ *simple zeros in* (d_i, b_i).

PROOF. This result is an immediate consequence of Corollary 3.10 and nodal properties.

Using Theorem 4.17, a complete description of the nontrivial solutions emerging from $\mathbf{R}^k \times \{0\} \times \cdots \times \{0\}$ is possible. Let Π denote the projection operator from $\mathbf{R}^k \times E_1 \times \cdots \times E_k$ into $\mathbf{R}^k$, that is, if $(\lambda, y_1, \ldots, y_k) \in \mathbf{R}^k \times E_1 \times \cdots \times E_k$, $\Pi(\lambda, y_1, \ldots, y_k) = \lambda$.

THEOREM 4.18. *Let* $\lambda_0 \in \Sigma_A$. *Let* σ_0 *be a subset of* $\{1, 2, \ldots, k\}$. *Let* $\lambda_0 \in \bigcap_{i \in \sigma_0} \Gamma_i$. *Let* $\mathscr{C}_{\lambda_0}^i$ *be the continuum of nontrivial solutions emerging from* $\lambda_0 \times \{0\}$ *in* $\mathbf{R}^k \times \{0\} \times \cdots \times E_i \times \cdots \times \{0\}$. *Let* $W = \bigcap_{i \in \sigma_0} \Pi(\mathscr{C}_{\lambda_0}^i)$. *Then nontrivial solutions emerge from* $(\lambda_0, 0, \ldots, 0)$ *in* $\mathbf{R}^k \times K_1 \times \cdots \times K_k$, *where* $K_i = E_i$ *if* $i \in \sigma_0$ *and* $K_i = \{0\}$ *if* $i \notin \sigma_0$ *precisely for the parameter values in W.*

Thus generically one gets "m-dimensional" bifurcation in $\mathbf{R}^k \times K_1 \times \cdots \times K_k$, where $m = k + 1 - |\sigma_0|$ and $|\sigma_0|$ denotes the number of elements in the set σ_0.

REMARK 4.19. Using Lyapunov-Schmidt techniques and a generalization of the method of **[12]**, Browne and Sleeman (**[1]**, **[2]**) have shown the existence of one-dimensional global branches emanating from Σ_A' in $\mathbf{R}^k \times E_1 \times \cdots \times E_k$. However, they make no mention of the remainder of the bifurcation structure for this problem.

REFERENCES

1. P. J. Browne and B. D. Sleeman, *Nonlinear multiparameter eigenvalue problems for ordinary differential equations*, preprint.

2. ———, *Nonlinear multiparameter Sturm-Liouville problems*, J. Differential Equations **34** (1979), 139–146.

3. R. S. Cantrell, *Multiparameter bifurcation problems and topological degree*, preprint.

4. ———, Ph.D. Thesis, University of Utah, 1981.

5. M. G. Crandall and P. H. Rabinowitz, *Nonlinear Sturm-Liouville eigenvalue problems and topological degree*, J. Math. Mech. **19** (1970), 1083–1102.

6. M. Faierman, Ph.D. Thesis, University of Toronto, 1966.

7. E. L. Ince, *Ordinary Differential Equations*, Longmans, Greene and Co., London, 1927.

8. J. Leray and J. Schauder, *Topologie et équations fonctionelles*, Ann. Sci. Ecole. Norm. Sup. **51** (1934), 45–78.

9. N. G. Lloyd, *Degree Theory*, Cambridge Univ. Press, Cambridge, 1977.

10. P. H. Rabinowitz, *Nonlinear Sturm-Liouville problems for second order ordinary differential equations*, Comm. Pure Appl. Math. **23** (1970), 939–961.

11. M. Schechter, *Principles of Functional Analysis*, Academic Press, New York, 1971.

12. R. E. L. Turner, *Nonlinear Sturm-Liouville problems*, J. Differential Equations **10** (1971), 141–146.

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