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COMPARISON TECHNIQUES AND THE METHOD OF LINES FOR A PARABOLIC FUNCTIONAL EQUATION

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

1. Introduction. In a recent paper [3], a detailed mathematical analysis for the implicit integro-differential equation

$$\text{(I)} \qquad \theta_t - \Delta\theta = \delta e^{\theta} + ((\gamma - 1)/\gamma)(1/\text{vol}\,\Omega)\int_{\Omega} \theta_t dy$$

was given. Equation (I) is the model for the induction period for the thermal explosion process of a compressible reactive gas in a bounded container.

In particular in [3], it was shown that the solution of (I) is always dominated by the solution of the explicit integro-differential equation

$$\text{(E)} \qquad u_t - \Delta u = \delta e^{u} + ((\gamma - 1)/\text{vol}\,\Omega)\,\delta\int_{\Omega} e^{u} dy$$

on their common interval of existence, if $\Omega = \mathscr{B}$, a ball in $\mathbf{R}^n$.

The purpose of this paper is to analyse initial-boundary value problems for a class of explicit integro-differential equations (see IBVP (1)–(2)) which include (E) (see IBVP (13)–(14)) as a special case.

2. Known existence results. Consider the scalar integro-partial differential equation

$$\text{(1)} \qquad u_t - \Delta u = f(t, u) + \int_{\Omega} g(t, u) dx$$

with the initial-boundary conditions

$$\text{(2)} \qquad \begin{aligned} u(x, t) &= u_0(x), \quad (x, t) \in \Omega \times \{0\}, \\ u(x, t) &= 0, \qquad (x, t) \in \partial\Omega \times [0, \infty), \end{aligned}$$

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where Ω is a bounded domain in $\mathbf{R}^n$, f, g are continuous on $[0, \infty) \times \mathbf{R}^n$, locally Lipschitz with respect to u, are convex functions of u, $f(t, 0) > 0$, $g(t, 0) > 0$, and g is increasing in u.

We use the following three theorems.

THEOREM 1. *If* $u_0 \in L^2(\Omega)$, $\sup_{x\in\Omega} u_0(x) < \infty$, *then* IBVP (1)–(2) *has a unique classical solution on* $\Omega \times [0, \sigma)$, *where either* $\sigma = +\infty$ *or* $\sigma < +\infty$ *and*

$$\limsup_{t\to\sigma^-} \sup_{x\in\Omega} u(x, t) = +\infty.$$

THEOREM 2. *If* $u_0(x) \equiv 0$ *for* $x \in \bar{\Omega}$, *then the solution* $u(x, t)$ *of* IBVP (1)–(2) *is nonnegative and nondecreasing as a function of t on* $\Omega \times [0, \sigma)$, *provided f, g are independent of t, and* f', g' *are Lipschitz continuous.*

THEOREM 3. *If* $\Omega = \mathscr{B} \equiv \{x: \|x\| < 1\} \subset \mathbf{R}^n$ and $u_0(x) \equiv 0$ *for* $x \in \mathscr{B}$, *then the solution* $u(x, t)$ *is radially symmetric in x for each* $t \in [0, \sigma)$.

Theorems 1 and 3 can be proven as in [3], as can Theorem 2 for $\Omega = \mathscr{B}$. But Theorem 2 also holds for arbitrary Ω, using known comparison techniques [7].

3. Extending Kaplan's theorem. In order to obtain more precise information concerning the blow-up time σ for the solution of IBVP (1)–(2), we utilize the following extensions of known comparison theorems. The first theorem is an easy extension of the classical Nagumo-Westphal Theorem. Let $\Pi_T = \Omega \times (0, T)$ and $\Gamma_T = (\partial\Omega \times [0, T]) \cup (\Omega \times \{0\})$.

THEOREM 4. *Let* $u, v \in C^{2,1}(\Pi_T)$ *satisfy*

$$v_t - \Delta v \geqq f(t, v) + \int_\Omega g(t, v)dy,$$

$$u_t - \Delta u \leqq f(t, u) + \int_\Omega g(t, u)dy$$

with $v(x, t) \geqq u(x, t)$ *on* Γ_T. *Then* $v(x, t) \geqq u(x, t)$ *on* Π_T.

As a corollary, we have the following result.

COROLLARY. If $\beta(t)$ *is the solution of*

$$\begin{aligned} y' &= f(t, y) + (\text{vol } \Omega)g(t, y), \\ y(0) &= y_0 \geqq \sup_\Omega u(x) \text{ on } [0, T), \end{aligned} \tag{3}$$

and if $u(x, t)$ *is the solution of* IBVP (1)–(2) *then* $\beta(t) \geqq u(x, t)$ *on* $[0, T)$ *and* $\sigma > T$.

The next theorem extends a result of Kaplan [4] to the class of integro-partial differential equations considered here.

THEOREM 5. *Let* $u_0(x) \equiv 0$ *and let* $u(x, t)$ *be the nonnegative solution of* IBVP (1)–(2) *on* $\Omega \times [0, T)$. *Let* $\phi(t)$ *satisfy*

$$\text{(4)} \qquad \begin{aligned} \phi' &= f(t, \phi) - \lambda_1\phi + (\text{vol } \Omega)g(t, \phi/M), \\ \phi(0) &= 0 \end{aligned}$$

on $[0, T)$ *where* λ_1 *is the first eigenvalue of*

$$\text{(5)} \qquad \begin{aligned} -\Delta\psi &= \lambda\psi, \quad x \in \Omega, \\ \psi &= 0, \quad x \in \partial\Omega \end{aligned}$$

and $M = (\text{vol } \Omega)\sup_\Omega \psi_1(x)$, $\psi_1(x) \geqq 0$ *is the eigenfunction of* (5) *associated with* λ_1 *normalized by* $\int_\Omega \psi_1(x)dx = 1$. *Then* $\sup_{x\in\bar\Omega} u(x, t) \geqq \phi(t)$, $t \in [0, T)$.

PROOF. Define $v(t) \equiv \langle u(x, t), \psi_1(x)\rangle = \int_\Omega u(x, t)\psi_1(x)dx$. Multiply (1) by $\psi_1(x)$ and integrate over Ω. Then we have

$$\text{(6)} \qquad \begin{aligned} v_t = \int_\Omega \Delta u\psi_1(x)dx &+ \int_\Omega \psi_1(x)f(t, u(x, t))dx \\ &+ \int_\Omega \psi_1(x)\left[\int_\Omega g(t, u(x, t))dx\right]dx. \end{aligned}$$

Inspecting each of the three integrals on the right hand side, we have

$$\text{(7)} \qquad \begin{aligned} \int_\Omega \psi_1(x)\, \Delta u dx &= \int_\Omega u\Delta\psi_1(x)dx + \int_{\partial\Omega}(\psi_1(x)(\partial u/\partial n) - u(\partial\psi_1/\partial x)dx \\ &= \int_\Omega u[-\lambda_1\,\psi_1]\, dx + 0 \\ &= -\lambda_1 v(t) \end{aligned}$$

by Stokes' Theorem;

$$\text{(8)} \qquad \begin{aligned} \int_\Omega \psi_1(x)f(t, u)dx &\geqq f\left(t, \int_\Omega \psi_1(x)u(x, t)dx\right) \\ &= f(t, v) \end{aligned}$$

by Jensen's inequality since f is convex in u and $\psi_1(x)$ has mass 1; and

$$\text{(9)} \qquad \begin{aligned} \int_\Omega \psi_1(x)\left[\int_\Omega g(t, u)dx\right]dx &= \int_\Omega g(t, u)dx \\ &= \text{vol}\,\Omega \int_\Omega g(t, u)(1/\text{vol}\,\Omega)dx \\ &\geqq \text{vol } \Omega g\left(t, \int_\Omega (u(x,t)/\text{vol } \Omega)dx\right) \end{aligned}$$

again by Jensen's inequality.

Furthermore, since g is increasing in its second argument and $M = \text{vol}\,\Omega \sup_{\bar\Omega} \psi(x)$,

$$\int_\Omega (u(x, t)/\mathrm{vol}\,\Omega)dx = (\sup \psi_1(x)/M) \int_\Omega u(x, t)\, dx$$

(10)
$$\geqq (1/M) \int \psi_1(x) u(x, t) dx$$
$$= v(t)/M.$$

From (9) and (10), we have

$$\int_\Omega \psi_1(x) \left[\int_\Omega g(t, u) dx \right] dx \geqq \mathrm{vol}\,\Omega\, g(t, v(t)/M).$$

Thus, $v(t)$ satisfies the differential inequality

(11) $$v'(t) \geqq f(t, v) - \lambda_1 v(t) + \mathrm{vol}\,\Omega\, g(t, v/M),$$

with initial condition

(12) $$v(0) = 0.$$

Since $\phi(t)$ satisfies (4) on $[0, T)$, $\phi(t) \leqq v(t)$ on $[0, T)$. But $v(t) = \int_\Omega \psi_1(x) u(x, t) dx \leqq \sup_{x \in \Omega} u(x, t)$, and the conclusion

$$\phi(t) \leqq \sup_{x \in \Omega} u(x, t) \text{ on } [0, T)$$

then follows.

4. Conclusions for an important special case. The particular initial boundary problem which is of special interest in our previous analyses of the thermal behavior of a reactive gas in a bounded container Ω is the following:

(13) $$u_t - \Delta u = \delta e^u + ((\gamma - 1)/\mathrm{vol}\,\Omega)\, \delta \int_\Omega e^u dy,\ \Pi$$

(14) $$u(x, t) = 0,\ \Gamma.$$

By the results of the two previous sections, we can immediately make the following observations.

By Theorem 1 and 2, IBVP (13)–(14) has a unique classical solution $u(x, t)$ which is nonnegative and nondecreasing as a function of t on $\Omega \times [0, \sigma)$ where either $\sigma = +\infty$ or $\sigma < +\infty$ and $\lim_{t \to \sigma^-} \sup_{x \in \bar{\Omega}} u(x, t) = +\infty$.

By Theorem 4, since $\beta(t) = \ln(1 - \gamma\delta t)^{-1}$, the solution of the IVP: $v' = \varepsilon \delta v$, $v(0) = 0$, is an upper solution relative to IBVP(13)–(14), $\beta(t) \geqq u(x, t)$ on $\Omega \times [0, 1/\gamma\delta)$ and $\sigma \geqq 1/\gamma\delta$ for any $\gamma \geqq 1$, $\delta > 0$.

Again by Theorem 4, if $\psi(x)$ is any solution of the steady state inequality

(15)
$$-\Delta\psi \geqq \delta e^\psi + ((\gamma - 1)/\mathrm{vol}\,\Omega)\, \delta \int_\Omega e^\psi dx,$$
$$\psi(x) = 0,$$

then $\psi(x) \geqq u(x, t)$ on $\bar{\Omega} \times [0, \infty)$. If $\Omega = \mathscr{B} = \{x: \|x\| < 1\} \subset \mathbf{R}^n$, then $\psi(x) = 1 - \|x\|^2$ is a solution of (15) provided

$$\delta \leqq (2n/e)(1/(c\gamma + (1 - c))) \equiv \delta$$

where

$$c \equiv (\text{vol}\,\Omega)^{-1} \int_{\Omega} e^{-\|x\|^2}\, dx < 1.$$

These observations can be summarized as follows.

THEOREM 6. (a) *For any* $\delta > 0$, $\gamma \geqq 1$, *the solution* $u(x, t)$ *of* IBVP(13)–(14) *exist on* $\bar{\Omega} \times [0, \sigma)$ *where* $\sigma > 1/\gamma\delta$ *and* $0 \leqq u(x, t) \leqq ln(1 - \delta\gamma t)^{-1}$ on $\bar{\Omega} \times [0, 1/\gamma\delta)$.

(b) *If* $\Omega = \mathscr{B} \subset \mathbf{R}^n$ *and if* $\delta \leqq (2n/e)(1/(c\gamma + (1 - c)))$, *then the solution* $u(x, t)$ *of* IBVP (13)–(14) *exists on* $\Omega \times [0, \infty)$ *and* $0 \leqq u(x, t) \leqq \psi(x) \leqq 1$ *for* $(x, t) \in \bar{\Omega} \times [0, \infty)$.

We now can use Theorem 5 to determine a range of parameter values for δ and γ which will force $\sigma < \infty$ and hence forces the solution $u(x, t)$ of (13)–(14) to become unbounded in finite time. Recall that λ_1 is the first eigenvalue of (5), $\psi_1(x) \geqq 0$ is the eigenfunction of (5) associated with λ_1, $\int_{\Omega} \psi_1(x)dx = 1$, and $M = \text{vol}\,\Omega \sup_{x \in \Omega} \psi_1(x)$.

THEOREM 7. (a) *The solution* $\phi(t)$ *of*

$$\begin{aligned} \phi' &= \delta e^{\phi} - \lambda_1 \phi + (\gamma - 1)\delta e^{\phi/M}, \\ \phi(0) &= 0 \end{aligned} \tag{16}$$

exists on $[0, T)$ *where*

$$T = \int_0^{\infty} dz/(\delta e^z - \lambda_1 z + (\gamma - 1)\delta e^{z/M}),$$

(b) $T < \infty$ if and only if $\delta[e^z + (\gamma - 1)e^{z/M}] > \lambda_1 z$ *for all* $z > 0$,

(c) *if* $T < \infty$, $\lim_{t \to T^-} \phi(t) = +\infty$.

The above theorem is easily proven since the IVP (16) is autonomous.

COROLLARY. *The solution* $u(x, t)$ *of* IBVP(13)–(14) *exists on* $\bar{\Omega} \times [0, \sigma)$ *where* $1/\gamma\delta < \sigma \leqq T = \int_0^{\infty} dz/(\delta e^z - \lambda_1 z + (\gamma - 1)\delta e^{z/M})$.

In order to determine the range of values of γ, δ for which T is finite, observe that the limiting case occurs when $\delta[e^z + (\gamma - 1)e^{z/M}] \geqq \lambda_1 z$ for all $z > 0$ and $\delta[e^{z_0} + (\gamma - 1)e^{z_0/M}] = \lambda_1 z_0$ for some z_0. We note that if $(\gamma - 1)/M$ is small, then $z_0 = 1 - \eta$, $\eta > 0$ small, and hence the critical value $\bar{\delta}$ for δ is approximately

$$\bar{\delta} = \lambda_1/(e + (\gamma - 1)e^{1/M}). \tag{17}$$

Thus for $\delta > \bar{\delta}$, the solution $u(x, t)$ of IBVP(13)–(14) blows up in finite time $\sigma < T$.

For the standard container geometries, we can make the following comparison. Let Ω be an infinite slab S of half-width 1 in $\mathbf{R}^3$ (or equivalently a bounded interval in $\mathbf{R}^1$), an infinite right circular cylinder C of radius 1 in $\mathbf{R}^3$ (or, equivalently, a bounded circle in $\mathbf{R}^2$), or a ball B of radius 1 in $\mathbf{R}^3$. Let $\bar{\delta}$ be the critical value defined by (17), and let δ_{CRIT} be the numerically computed critical value for (13)–(14). Then, for $\gamma = 1.4$, we have:

	$\underline{\delta} \equiv \dfrac{2n}{e[c\gamma + (1-c)]}$	δ_{CRIT}	$\bar{\delta}$
S	.562	.65	.71
C	1.175	1.53	1.73
B	1.777	2.61	3.03

5. Convergence of method of lines. In this section, we prove that the method of lines as developed by Walter [8] can be used to construct solutions to approximating systems of ordinary differential equations which converge to the solution $u(x, t)$ of IBVP (13)–(14). We choose to give the proof for the special case with $\Omega = S$ and $n = 1$. The method of proof extends to IBVP (1)–(2) with $\Omega = \mathscr{B} \subset \mathbf{R}^n$.

Consider

$$\theta_t = \theta_{xx} + \delta e^{\theta} + ((\gamma - 1)/2)\,\delta \int_{-1}^{1} e^{\theta} dx, \tag{18}$$

$$\begin{aligned} &\theta(x, 0) = 0, \\ &\theta(-1, t) = \theta(1, t) = 0. \end{aligned} \tag{19}$$

Since the initial-boundary conditions are not compatible with (18) at the corner points $(0, -1)$ and $(0, +1)$, we replace the boundary values by an approximating initial-boundary function which is compatible with (18) on the parabolic boundary. Let $\eta(t)$ denote such a boundary function. We see that η must satisfy

$$\eta(0) = 0 \quad \text{and} \quad \eta'(0) = \delta\gamma. \tag{20}$$

For a given $\varepsilon > 0$, let $\eta_\varepsilon(t)$ be a C^∞-smooth function satisfying

$$\eta_\varepsilon(0) = 0, \quad \eta_\varepsilon'(0) = \gamma\delta, \quad |\eta_\varepsilon(t)| \leqq \varepsilon, \quad t \in [0, \infty). \tag{21}$$

Consider the approximating IBVP:

$$(P_\varepsilon)\qquad \begin{aligned} &\theta_t = \theta_{xx} + \delta e^\theta + ((\gamma - 1)/2)\,\delta \int_{-1}^{1} e^\theta dx \\ &\theta(x, 0) = 0 \\ &\theta(-1, t) = \theta(1, t) = \eta_\varepsilon(t). \end{aligned}$$

Next we replace e^θ in (P_ε) by the function $g_N(\theta)$, where

$$g_N(\theta) = \begin{cases} e^\theta, & \theta \leqq N, \\ e^N, & \theta > N. \end{cases}$$

Then the reaction terms in the right hand side of (18) are uniformly Lipschitz in θ for all $\theta \in \mathbf{R}$. In the following, we suppress the subscript N.

Consider

$$(\bar{P}_\varepsilon)\qquad \begin{aligned} &\theta_t = \theta_{xx} + \delta g(\theta) + ((\gamma-1)/2)\,\delta \int_{-1}^{1} g(\theta) dx \\ &\theta(x, 0) = 0 \\ &\theta(-1, t) = \theta(1, t) = \eta_\varepsilon(t). \end{aligned}$$

We will first apply the methods of lines of $(\bar{P}_\varepsilon)$. Approximate $(\bar{P}_\varepsilon)$ by the following system of $m - 1$ first order ordinary differential equations

$$(\bar{P}_m)\qquad \begin{aligned} &\frac{d}{dt} v_k^m = \frac{v_{k+1}^m - 2v_k^m + v_{k-1}^m}{h^2} + \delta g(v_k^m) + \left(\frac{\gamma - 1}{2}\right)\delta h \sum_{i=1}^{m} g(v_i^m), \\ &v_k^m(0) = 0,\ k = 1, \ldots, m - 1,\ h = 2/m \end{aligned}$$

and define $v_0^m(t)$, $v_m^m(t)$ by the boundary values

$$v_0^m(t) = v_m^m(t) = \eta_\varepsilon(t).$$

Denote the solution of $(\bar{P}_m)$ by $v^m = (v_0^m, \ldots, v_m^m)$. Let $\theta(x, t)$ be the solution of $(\bar{P}_\varepsilon)$ and let $\theta_\kappa(t) = \theta(-1 + kh, t)$, $k = 0, 1, \ldots, m$. Let θ^m denote the $m + 1$ vector $\theta^m = (\theta_0, \theta_1, \ldots, \theta_m)$. For $\omega \in \mathbf{R}^{m+1}$, define

$$\|\omega\| = \max_{i=0,\ldots,m} |\omega_i|.$$

We will first prove that if $J = [0, a]$ is a common t-interval of existence for the solutions of $(\bar{P}_\varepsilon)$ and $(\bar{P}_m)$, $m = 1, 2, \ldots$, then $\|\theta^m - v^m\| \to 0$ uniformly for $t \in [0, a]$ as $m \to \infty$. The superscript m will be dropped in the next discussion.

Define

$$f_i(t, z, r_i) \equiv r_i + \delta g(z_i) + \left(\frac{\gamma - 1}{2}\right)\delta h \sum_{j=1}^{m} g(z_j)$$

where z is an $m + 1$ vector. Then

$$f_i(t, z, r_i) - f_i(t, \bar{z}, \bar{r}_i) \leqq (r_i - \bar{r}_i) + \delta L|z_i - \bar{z}_i| + (\gamma - 1)\delta L\|z - \bar{z}\|$$
$$\equiv \omega(t, |z_i - \bar{z}_i|, \|z - \bar{z}\|, r_i - \bar{r}_i),$$

where L is the Lipschitz constant for g and $\omega(t, q, p, r) \equiv r + \delta Lp + (\gamma - 1)\delta Lq$.

Let $d^2\theta_k = (\theta_{k+1} - 2\theta_k + \theta_{k-1})/h^2$ and $I(\theta) = h \sum_{i=1}^{m} g(\theta_i)$.

We first prove an error estimation theorem similar to Theorem III ([**8**], p. 278).

THEOREM 8. *Assume there exist continuous functions $\alpha(t)$, $\beta(t)$ on $[0, a]$ such that*

$$|\theta_{xx}(x_k, t) - d^2\theta_k(t)| < \alpha(t),$$
$$\left|\int_{-1}^{1} \theta(x, t)dx - I(\theta(t))\right| < \beta(t).$$

Let $\rho(t)$ be a continuously differentiable function on $[0, a]$ satisfying $\rho' > \omega(t, \rho, \beta(t), \alpha(t))$, $\rho(0) > 0$, where ω is defined on $J \times \{(p, q, r): q \geqq 0, r \geqq 0\}$, $J = [0, a]$. Then

$$|\theta_k^m(t) - v_k^m(t)| = |\theta(x_k, t) - v_k^m(t)| < \rho(t) \text{ for } t \in [0, a], k = 0, \ldots, m.$$

PROOF. Let $w_k = \theta_k^m + \rho$. We will show that $v_k^m(t) \leqq w_k(t)$ for all $k = 0, \ldots, m$ and $t \in [0, a]$. The fact that $v_k^m(t) \geqq \theta_k^m(t) - \rho(t)$ follows similarly.

$$\begin{aligned} w_k' &= (\theta_k^m)' + \rho' \\ &> \theta_{xx}(x_k, t) + \delta g(\theta(x_k, t)) \\ &\quad + ((\gamma - 1)/2)\delta \int_{-1}^{1} g(\theta)dx + w(t, \rho(t), \beta(t), \alpha(t)) \\ &\geqq f_k(t, \theta + \rho, d^2\theta_k^m) = f_k(t, w, d^2\theta_k^m). \end{aligned}$$

Since the system of ordinary differential equations is quasimonotone, by standard comparison results we will have the desired conclusion if we can show $f_k(t, w, d^2\theta_k^m) \geqq f_k(t, w, d^2w_k)$ for $k = 1, \ldots, m - 1$. Note that they are in fact equal for $k = 2, \ldots, m - 2$. Thus we need only check the ends $k = 1$ and $k = m - 1$. For $k = 1$, we have

$$\begin{aligned} d^2w_1 &= (w_2 + \eta_\varepsilon - 2w_1)/h^2 = (\theta_2 + \rho + \eta_\varepsilon - 2(u_1 + \rho)/h^2 \\ &= d^2\theta_1 - (\rho/h^2) < d^2\theta_1. \end{aligned}$$

Similarly, $d^2\theta_{m-1} > d^2w_{m-1}$. Hence, the desired inequality holds and the theorem is proven.

COROLLARY. *If $J = [0, a]$ is the common t-interval of existence for the solutions $\theta(x, t)$ and $v^m(t)$ of $(\bar{P}_\varepsilon)$ and $(\bar{P}_m)$, respectively, then $\|\theta^m - v^m\| \to 0$ uniformly for $t \in [0, a]$ as $m \to \infty$.*

PROOF. By the theorem, the difference between the solution $v^m(t)$ of $(\bar{P}_m)$ and the solution $\theta^m(t)$ of $(\bar{P}_\varepsilon)$ is bounded by the solution $\rho(t)$ of

$$\rho' = \alpha(t) + (\gamma - 1)\delta L\beta(t) + \delta L\rho + \varepsilon_1,\ \rho(0) = \varepsilon_1.$$

Since this is true for each $\varepsilon_1 > 0$ we have $\rho(t)$ as a bound where ρ solves

$$\rho' = \alpha(t) + (\gamma - 1)\delta L\beta(t) + \delta L\rho,\ \rho(0) = 0.$$

But $\alpha(t)$, $\beta(t) \to 0$ in t as $m \to \infty$. Thus $\rho(t) \to 0$ as $m \to 0$ and we have convergence.

Let $(\bar{P}_0)$ denote the following IBVP:

$$(\bar{P}_0) \qquad \begin{aligned} &\theta_t = \theta_{xx} + \delta g(\theta) + ((\gamma - 1)/2)\delta \int_{-1}^{1} g(\theta)dx, \\ &\theta(x, 0) = 0, \\ &\theta(-1, t) = 0 = \theta(1, t). \end{aligned}$$

We next show that the solutions of $(\bar{P}_\varepsilon)$ converge to the solutions of (P_0) as $\varepsilon \to 0$.

THEOREM 9. *Let $\theta_\varepsilon(x, t)$ be the solution of $(\bar{P}_\varepsilon)$ and let $\theta(x, t)$ be the solution of $(\bar{P}_0)$. Then $\theta_\varepsilon \to \theta$ uniformly on compact subsets of $\bar{\Omega} \times J$, where J is a common t-interval of existence.*

PROOF. Let $K = L\delta\gamma + 1$. Let $\rho(t) = \varepsilon e^{Kt}$ be the solution of $\rho' = K\rho$ $\rho(0) = \varepsilon$, and set $w = \theta + \rho$. Note that $w(x, t) \geqq \varepsilon \geqq \theta_\varepsilon(x, t)$ for all (x, t) on the parabolic boundary. We wish to prove: $w(x, t) \geqq \theta_\varepsilon(x, t)$ for $(x, t) \in \bar{\Omega} \times \bar{J}$. Set $f(t, u, u_{xx}) = u_{xx} + \delta g(u) + ((\gamma - 1)/2)\cdot \delta \int_{-1}^{1} g(u)dx$. Then $w_t(x, t) = \theta_t + \rho' = f(t, \theta, \theta_{xx}) + L\rho \leqq f(t, w, w_{xx})$. By Theorem 4, $w(x, t) \geqq \theta_\varepsilon(x, t)$. Similarly, $\theta(x, t) - \rho(t) \leqq \theta_\varepsilon(x, t)$. Since $\rho(t) \to 0$ uniformly on compact subintervals as $\varepsilon \to 0$, we have that $\theta_\varepsilon(x, t)$ converges to $\theta(x, t)$ on compact subsets of $\Omega \times \bar{J}$.

Finally, we will show that the system of ordinary differential equations which approximates the IBVP $(\bar{P}_0)$ has a solution which converges uniformly to the solution $\theta(x, t)$ of $(\bar{P}_0)$ as the mesh size tends to zero. Consider

$$(\bar{P}_0^m) \qquad \begin{aligned} &dz_k/dt = d^2 z_k + \delta g(z_k) + ((\gamma - 1)/2)\, h\delta \sum_{i=1}^{m} g(z_i), \\ &z_k(0) = 0,\ k = 1, \ldots, m - 1, \\ &z_0(t) \equiv 0,\ z_m(t) \equiv 0. \end{aligned}$$

THEOREM 10. *Let $\theta(x, t)$ be the solution of* IBVP $(\bar{P}_0)$ *on $\bar{\Omega} \times [0, \sigma)$ and let $z^m(t)$ be the solution of $(\bar{P}_0^m)$ on $[0, \sigma)$, then*

$$\|\theta^m(t) - z^m(t)\| = \max|\theta_i^m(t) - z_i^m(t)| \to 0$$

as $m \to \infty$ *uniformly on compact subsets of* $[0, \sigma)$.

PROOF. Let $\rho(t) = 2\varepsilon e^{Kt}$ where $K = L\gamma\delta + 1$ and $\varepsilon > 0$ is given by the boundary function $\eta_\varepsilon(t)$ for $(\bar{P}_m)$. Set $w_k = z_k(t) + \rho(t)$ for $k = 1, \ldots, m$. We will show $w_k(t) > v_k(t)$ for each k and for each $t \in [0, \sigma)$. Similarly, $v_k(t) > z_k(t) - \rho(t)$ where $v^m(t)$ is the solution of $(\bar{P}_m)$. Since

$$\begin{aligned} w_k' &= z_k'(t) + \rho'(t) \\ &= d^2 z_k + \delta g(z_k) + \left(\frac{\gamma - 1}{2}\right) h\delta \sum_{i=1}^m g(z_i) + 2K\varepsilon e^{Kt} \\ &> d^2 w_k + \delta g(w_k) + \left(\frac{\gamma - 1}{2}\right) h\delta \sum_{i=1}^m g(w_i). \end{aligned}$$

since the right hand side is quasimonotone, and since $w_k(0) \geqq z_k(0)$ for $k = 1, \ldots, m - 1$, we have that $w_k(t) > z_k(t)$ for $t \in [0, \sigma)$ and $k = 1, \ldots, m - 1$. To see that this last inequality holds for $k = 1$ and $k = m - 1$, observe that $d^2 w_1 < d^2 z_1$ since

$$\begin{aligned} d^2 w_1 = (z_2 + \rho - 2(z_1 + \rho) + \eta_\varepsilon(t))/h^2 &= (z_2 - 2z_1)/h^2 - (\rho - \eta_\varepsilon(t))/h^2 \\ &< d^2 z_1 \end{aligned}$$

and similarly $d^2 w_{m-1} < d^2 z_{m-1}$.

Hence, $\|v^m(t) - z^m(t)\| < 2\varepsilon e^{Kt}$ and $\|v^m(t) - z^m(t)\| \to 0$ uniformly for $t \in [0, \sigma)$ as $\varepsilon \to 0$.

Let $\theta_\varepsilon(x, t)$ be the solution of (P_ε). Then

$$\begin{aligned} \|\theta(x_k, t) - z_k(t)\| &\leqq \|\theta(x_k, t) - \theta_\varepsilon(x_k, t)\| \\ &\quad + \|\theta_\varepsilon(x_k, t) - v_k(t)\| + \|v_k(t) - z_k(t)\|. \end{aligned}$$

Each term on the right hand side tends to zero uniformly on compact subsets of $[0, \sigma)$ as $\varepsilon \to 0$ and $m \to \infty$.

COROLLARY. *The method of lines converges uniformly to the solution* $u(x, t)$ *of* IBVP (18)–(19) *on compact subsets of* $\bar{\Omega} \times [0, \sigma)$.

PROOF. Since N in the definition of $g(\theta)$ is arbitrary and the solution $\theta(x, t)$ of $(\bar{P}_0)$ agrees with the solution $u(x, t)$ of IBVP (18)–(19) for $|\theta(x, t)| < N$, the conclusion is immediate.

In [3], we proved that the solution $u(x, t)$ of the initial boundary value problem

$$\text{(22)} \qquad \begin{aligned} &u - \Delta u = \delta e + ((\gamma - 1)/\text{vol}\,\Omega) \int_\Omega (\Delta u + \delta e)dy, \ \Pi, \\ &u(x, t) = 0, \ \Gamma \end{aligned}$$

satisfies $\chi(x, t) \leqq u(x, t) \leqq \theta(x, t)$ for all $x \in \Omega$ and all $t \geqq 0$ on the common t-interval of existence for χ, u, θ for any $\delta > 0, \gamma \geqq 1$ where $\theta(x, t)$ is the solution of IBVP (13)–(14) and $\chi(x, t)$ is the solution of IBVP:

$$\begin{aligned} &\chi_t - \Delta\chi = \delta e^{\chi}, \Pi, \\ &\chi(x, t) = 0, \Gamma. \end{aligned} \tag{23}$$

By the results of this section, we know that the method of lines converges to the solution χ for IBVP(23) and to the solution θ of IBVP (13)–(14). We have not however succeeded in proving that the method of lines converges for IBVP (22). The table below gives a comparison of blow-up times for the three problems in the one-dimensional case where $\Omega = S = (-1, 1)$ and $\gamma = 1.4$. In each case, the numerical computation employed the method of lines using a grid of 31 points on $[-1, 1]$.

δ	t_θ	t_u	t_χ
.91	1.755	6.123	7.940
1.00	1.401	2.732	3.537
2.00	0.454	0.528	0.680
2.47	0.347	0.390	0.502
20.0	0.037	0.038	0.050
50.0	0.0147	0.0148	0.0200

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