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ON EXISTENCE OF LIMIT CYCLES IN SYSTEMS WITH DISCONTINUOUS NONLINEAR TERMS

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

ABSTRACT. In this paper we establish conditions which insure the existence of self-excited oscillations (limit cycles) in nonlinear feedback systems whose nonlinearity can be discontinuous or can exhibit hysteresis. The applicability of the result is demonstrated by means of a specific example.

I. Introduction. We study the existence of self excited oscillations (limit cycles) in a class of nonlinear feedback systems using the sinosoidal describing function. The nonlinear element in this feedback system may be discontinuous or may exhibit hysteresis. We establish reasonable and easily verified conditions which insure the existence of a limit cycle. We give an example which demonstrates how the conditions are verified and our theory applied.

A good source for background material on the sinosoidal describing function and on equivalent linearization is Gelb and Van der Velde [5]. The first attempt known to the authors to provide mathematical justification for the describing function is the work of Bass [1]. The results in [1] are rather special. A quite different local analysis is used in the work of Holtzman [6] and of Cesari [2, 3]. The nonlinear term in these papers must be smoothly differentiable. Sandberg [10, 11] studied general problems, which include analysis of equivalent linearization as a special case, using generalized contraction mapping techniques. In his work, the nonlinear terms are required to be globally Lipschitz continuous and the systems are periodically forced. Somewhat similar results, but for large scale systems with several nonlinearities are proved by Miller and Michel [8]. Mees and Bergen [7] prove the existence of limit cycles for feedback systems with Lipschitz continuous nonlinearities. Their analysis is extremely interesting in that they study autonomous (i.e., nonforced) systems

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under conditions too weak to guarantee stability (a difficult case) and in that they prove some nonexistence theorems. Skar, Miller and Michel **[12, 13]** greatly modify the results of [7] and generalize them to large scale systems which contain several nonlinearities. In [9] Michel and Miller study existence of periodic solutions of forced feedback systems whose nonlinear term may be discontinuous or may exhibit hysteresis.

The results of the present paper can be thought of as a generalization of the existence results for autonomous equations in [7] to include a more general class of nonlinearities such as those allowed in [9]. The proof of the result uses some analysis similar to [7] and a Schauder fixed point theorem argument as in [9]. However the type of solution obtained is different than in either of these papers.

The remainder of this paper is organized as follows. §II contains necessary background material. In §III we prove some preliminary lemmas and then state and prove our main result. In §IV we discuss the meaning of the assumptions needed for our main result and we study in detail an example to see how the various assumptions can be verified.

II. Preliminaries. Let $\mathbf{R}$ be the real line and $\mathbf{R}^+$ the nonnegative real numbers. The symbols $L^2(0, T)$ will denote Lebesgue measurable functions f defined on $(0, T)$ such that f is square integrable. We define $H(w)$ to be the set of all square integrable functions ϕ: $[0, 2\pi/w] \to \mathbf{R}$ which satisfy the conditions (i) $\phi(t + 2\pi/w) = \phi(t)$ a.e. on $\mathbf{R}$, and (ii) $\phi(t + \pi/w) = -\phi(t)$ a.e. on $\mathbf{R}$. Property (i) states that ϕ is $2\pi/w$ – periodic. Property (ii) is called *π-symmetry* [7].

Given $\phi \in H(w)$, ϕ can be expanded in a Fourier series

$$\phi(t) \sim \frac{1}{2} \sum_{n\, odd} \hat{\phi}_n \exp(inwt)$$

with modified Fourier coefficients

$$\hat{\phi}_n = \frac{w}{\pi} \int_0^{2\pi/w} e^{-inwt}\, \phi(t) dt. \tag{1}$$

Note that $\hat{\phi}_n = 0$ when n is even by (ii). Also $\hat{\phi}_{-n} = \hat{\phi}_n$ since ϕ is real valued. Also note that by the Parseval equation

$$\|\phi\|_w^2 = \frac{w}{\pi} \int_0^{2\pi/w} [\phi(t)]^2\, dt = \frac{1}{2} \sum_{n\, odd} |\hat{\phi}_n|^2.$$

This expression determines a norm on $H(w)$ which is equivalent to the L^2-norm but modified so that if $\phi(t) = a \cos(wt + b)$, then $|\hat{\phi}_1| = |\hat{\phi}_{-1}| = a = \|\phi\|_w$.

Define $H_c(w) = \{\phi \in H(w)$: ϕ is continuous on $\mathbf{R}\}$. This subset of $H(w)$ is a Banach space under the uniform norm, i.e., under $\sup\{|\phi(t)| : 0 \leqq t \leqq 2\pi/w\}$.

We define a projection P on $H(w)$ by

$$(P\phi)(t) = \frac{1}{2}\hat{\phi}_1 e^{iwt} + \frac{1}{2}\hat{\phi}_{-1} e^{-iwt}$$

for each $\phi \in H(w)$. $P^* = I - P$ will denote the complementary projection.

The results of this paper are meant to apply to single valued nonlinear functions $n(x)$ and also to hysteresis operators. Hysteresis is a very complex phenomena which involves both multivaluedness and also memory. To characterize hysteresis operators and their action on functions, we make the following definitions.

DEFINITION 2.1. A hysteresis operator n is a triple of (single-valued) functions $n(x, j), j = 0, \pm 1$ where $n(\cdot, j)\colon \mathbf{R} \to \mathbf{R}$ and $n(x, 0) = (n(x, 1) + n(x, -1))/2$. Moreover $n(x, -1) = n(x, 1) = n(x, 0)$ for all x sufficiently large.

Let $D_1(n)$ be the closure of the set $\{x \in \mathbf{R}\colon n(x, 1) \neq n(x, -1)\}$. This set is bounded. Figure 1 illustrates these definitions. The nonlinearity depicted in this figure is a simple idealized relay with hysteresis. The set $D_1(n)$ is an interval.

DEFINITION 2.2. A continuously differentiable function $F\colon \mathbf{R} \to \mathbf{R}$ is of class $\mathscr{F}(\zeta, \eta)$ where $0 < \zeta < \eta$ if

(i) $F'(t) \neq 0$ for $|F(t)| < \zeta$, and

(ii) $|F(t)| < \eta$ for all $t \in \mathbf{R}$.

DEFINITION 2.3. A hysteresis operator n is of class N if

(i) there exist constants c and $d > 0$ such that $|n(x, j) - cx| \leqq d$ for $j = 0, \pm 1$ and for all $x \in \mathbf{R}$,

(ii) $n(x, j) = -n(-x, -j)$ for $j = 0, \pm 1$ and all $x \in \mathbf{R}$, and

(iii) for $j = \pm 1$ the functions $n(x, j)$ is piecewise continuous with jump discontinuities at $\{\alpha_{jk}\}$ where $-\infty < \alpha_{0j} < \alpha_{1j} < \cdots < \alpha_{\ell j} < \infty$.

Note that $-\alpha_{jk}$ is a point of discontinuity of $n(\cdot, -j)$. For any $n \in N$ define $D(n) = D_1(n) \cup \{\alpha_{jk}\colon j = \pm 1 \text{ and } k = 0, 1, \ldots, \ell\}$. The domain of the hysteresis operator n will consist of all functions $F \in \mathscr{F}(\zeta, \eta)$ where $[\zeta, \eta]$ is chosen so that it does not intersect $D(n)$. If F is in $\mathscr{F}(\zeta, \eta)$, then $(nF)(t) = n(F(t), \operatorname{sgn} F'(t))$, $t \in \mathbf{R}$. Notice that $F'(t) \neq 0$ at points of discontinuity of the hysteresis operator (where $F(t) = \alpha_{jk}$). Moreover $F(t)$ will move monotonically across any hysteresis interval (where $n(x, 1) \neq n(x, -1)$). Hence there is never any ambiguity in the definition of $(nF)(t)$ and, indeed, (nF) will be a piecewise continuous function of $t \in \mathbf{R}$.

Given $n \in N$, the describing function of n is defined by

$$N(a) = \frac{1}{\pi a}\int_0^{2\pi} e^{-i\theta} n\,(a \cos \theta)d\theta, \; a \geqq 0.$$

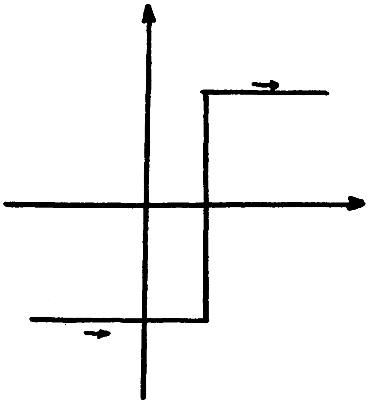

Fig. 1(a) Graph for $n(x, +1)$

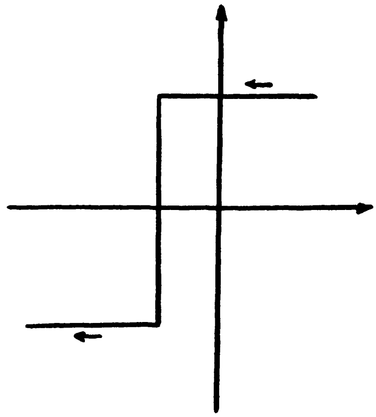

Fig. 1(b) Graph for $n(x, -1)$

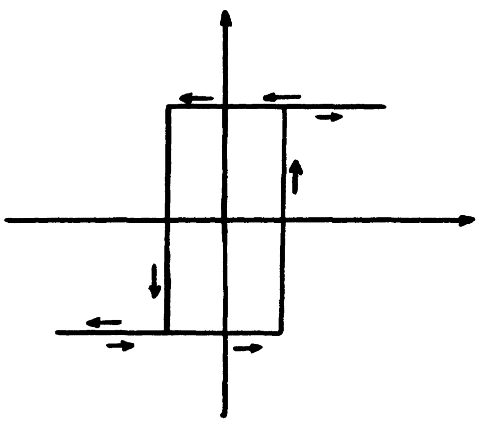

Fig. 1(c) Graph for $n(x)$

The describing function has the property that if $\phi(t) = a\cos(wt + b)$ and $\psi(t) = Pn(\phi(t))$, then $\hat{\psi}_1 = N(a)\hat{\phi}_1 = aN(a)e^{ib}$. The describing function will be real if $n(x)$ is single valued but will generally be complex valued if $n(x)$ exhibits hysteresis.

Now consider a real valued integrodifferential operator L defined by

$$(Ly)(t) = y^{(J)}(t) + \sum_{j=0}^{J-1}\left(\sum_{k=1}^{\infty} b_{jk}\, y^{(j)}(t - t_k) + \int_{-\infty}^{t} C_j(t - s)y^{(j)}(s)\, ds\right)$$

where $J \geqq 2$, $\{t_k\}$ is an increasing sequence with $t_1 \geqq 0$ and $t_k \to \infty$. Let $C_j^*(s)$ denote the Laplace transform of $C_j(t)$ and let

$$A(s) = s^J + \sum_{j=0}^{J-1} s^j\Big(\sum_{k=1}^{\infty} b_{jk}\exp(-st_k) + C_j^*(s)\Big)$$

be the transfer function for L. We shall assume the following for L and A.

(A1) $$\int_0^\infty |C_j(t)|\,dt + \sum_{k=1}^{\infty} |b_{jk}| < \infty,\ j = 0, 1, 2, \ldots, J-1.$$

(A2) In the half plane $\operatorname{Re} s \geqq 0$ the characteristic equation $A(s) = 0$ has at most finitely many roots s_j with $\operatorname{Re} s_j > 0$ and no roots with $\operatorname{Re} s = 0$.

(These assumptions can be weakened slightly to allow purely imaginary roots, see [**8**, assumption A7, A8 and A9].) Under these assumptions we can define $G(iv) = A(iv)^{-1}$ for all real v. It was shown in [**8**, Theorem 2] that for any $f \in H(w)$, $Ly = f$ has a unique solution $y \in H(w)$ and $\hat{y}_n = G(inw)\hat{f}_n$ for all integers n. In particular this means that for any fixed $n \in N$ the equation

(2) $$Lx + n(x) = 0,$$

is equivalent on the set $H(w)$ to the operator equation

(E) $$x + gn(x) = 0,$$

where g is the operator defined by $y = gf$ means

(3) $$\hat{y}_n = G(inw)\hat{f}_n,\ n = \pm 1, \pm 3, \pm 5, \ldots$$

The describing function method, as applied to (2), may be summarized as follows. Replace equation (2) by its operator form (E). In (E) replace $n(x)$ by $Pn(x)$ where P is the projection defined above. There results the approximation

(4) $$u + Pgn(u) = 0.$$

This equation is solved, using the describing function $N(a)$ for the given nonlinearity $n(x)$. Solving (4) is equivalent to finding an $a \geqq 0$ and an $w \geqq 0$ such that

$$(1 + G(iw)N(a))a = 0.$$

Of if $a > 0$,

(5) $$N(a) + G(iw)^{-1} = 0.$$

Equation (5) is usually solved graphically by plotting, in the complex plane, the loci $\{N(a)\colon a \geqq 0\}$ and $\{-G(iw)^{-1}\colon w \geqq 0\}$. The points of intersection of the two loci determine values of a and w which satisfy (5). (Thus we see that the describing function method is a Galerkin approximation technique on $H(w)$ with the number of terms in the approximation fixed at two.)

In the next section we shall assume that (5) can be solved for a pair a_0 and w_0. We assume that a_0 is not near one of the discontinuities of $n(x)$ nor is a_0 in a hysteresis interval of $n(x)$. We also require some estimates similar to those in [7]. If the assumptions are satisfied, we show that (E) has a periodic solution near the approximate solution $u(t) = a_0 \cos wt$ obtained from (4).

III. Main result. Consider the operator equation

$$x + gn(x) = 0 \tag{E}$$

where $n \in N$ with constants c and d and g is a continuous linear operator on $H(w)$ satisfying the following condition.

(A3) There is a continuous complex valued function $G(iv) = \overline{G(-iv)} \neq 0$ for $v \in R$ such that if $u \in H(w)$ for $w > 0$ and $y = gu$, then $\hat{y}_k = G(ikw)\hat{u}_k$ for $k = \pm 1, \pm 3, \ldots$. Furthermore, there is a continuous bounded function $M(w) > 0$ such that

$$\gamma_k(w) = \left| \frac{G(ikw)}{1 + c\, G(ikw)} \right| \leqq M(w)/k^2$$

for $k = \pm 1, \pm 3, \ldots$.

We have seen that a large class of integro-differential equations of the form (2) with $J \geqq 2$ will satisfy (A3). Define

$$b(w) = \left[\sum_{\substack{k \text{ odd} \\ |k|>1}} \gamma_k(w)^2 \right]^{1/2}$$

and

$$e(w) = \left[\sum_{\substack{k \text{ odd} \\ |k|>1}} k^2 w^2 \gamma_k(w)^2 \right]^{1/2}.$$

Let $\Omega_2(w)$ be the set of all elements $v \in P^*H(w)$ such that $\hat{v}_1 = 0$ and $\hat{v}_k = \gamma_k(w)\hat{y}_k$ for some $y \in H(w)$ with $\|y\|_w \leqq \sqrt{2}\, d$ and for $k = \pm 3, \pm 5, \ldots$. We will assume that

(A4) there are constants w_i and a_i with $0 < a_1 < a_0 < a_2$ and $0 < w_1 < w_0 < w_2$, such that

(a) $N(a_0) + G(iw_0)^{-1} = 0$,

(b) $w^2 a_1^2 > (e(w)^2 + w^2 b(w)^2)d^2$ and no point of the interval $[-b(w)d + (a_1^2 - e(w)^2 d^2/w^2)^{1/2}, a_2 + b(w)d]$ is in $D(n)$ for $w_1 \leqq w \leqq w_2$,

(c) the map $J(a, w) = N(a) + G(iw)^{-1}$ is one-to-one and continuous on the rectangle $[a_1, a_2] \times [w_1, w_2]$,

(d) there is a function $r(a, w) > 0$ such that if $(a, w) \in [a_1, a_2] \times [w_1, w_2]$ and $v \in \Omega_2(w)$, then

$$\left| \frac{w}{a\pi} \int_0^{2\pi/w} e^{-iwt} \left(n(a \cos wt) - n(a \cos wt + v(t)) + cv(t)\right)dt \right| < r(a, w),$$

and

(e) $|J(a, w)| > r(a, w)$ for all (a, w) on the boundary of $[a_1, a_2] \times [w_1, w_2]$.

Note that (A4)(a) is the assumption that the describing function approximation has a nontrivial solution. Assumption (A4)(b) ensures that functions of interest will be in the domain of n. If n is a function (i.e., singled valued), then the describing function $N(a)$ for n is real. In that case $J(a, w)$ is one-to-one if $N(a)$ and Im $G(iw)^{-1}$ are one-to-one. Using the Definition 2.3 it is not hard to see that it is possible to define $r(a, w) = 2d/a$ in order to satisfy (A4)(d). However, for a given problem it may be possible to find a better estimate for $r(a, w)$ so that (A4)(e) is easier to satisfy and the intervals $[a_1, a_2]$ and $[w_1, w_2]$ are smaller.

Define $\Omega(w) = \{x \in H(w): \hat{x}_1 = \hat{x}_{-1} = a \in [a_1, a_2] \text{ and } P^*x \in \Omega_2(w)\}$.

LEMMA 1. *If $x \in \Omega(w)$, then $x(t)$ is continuously differentiable on* $\mathbf{R}$.

PROOF. If $x \in \Omega$, then x has the Fourier series

$$x(t) \sim \frac{1}{2} \sum_{k \text{ odd}} \hat{x}_k e^{ikwt}$$

where $|\hat{x}_k| = \gamma_k(w)|\hat{y}_k|$ for $k = \pm 3, \pm 5, \ldots$ and for some $y \in H(w)$ with $\|y\|_w \leqq \sqrt{2}\, d$. The function

$$y(t) = \frac{1}{2} \sum_{k \text{ odd}} (ikw)\, \hat{x}_k e^{ikwt}$$

has an absolutely convergent Fourier series. Indeed by the Schwartz inequality

$$\begin{aligned} \sum |(ikw)\hat{x}_k| &= \sum |(ikw)\gamma_k(w)\hat{y}_k| \\ &\leqq |w| \left(\sum |k\gamma_k(w)|^2\right)^{1/2} \left(\sum |\hat{y}_k|^2\right)^{1/2} \\ &= e(w) \sqrt{2}\, \|y\|_w \leqq e(w)(2d) < \infty. \end{aligned}$$

Hence $y(t)$ is a continuous, $2\pi/w$-periodic function whose integral is $x(t)$.

LEMMA 2. *The set $\Omega(w)$ is closed in $H(w)$ in the weak topology. The weak topology and the uniform topology are equivalent on $\Omega(w)$.*

PROOF. For a definition of the weak topology see [**4**, p. 67]. By [**4**, p. 294, Theorem 5] we see that a sequence $x_m \in H(w)$ converges to x in the weak topology if and only if the k-th Fourier coefficient $(\hat{x}_m)_k$ of x_m converges to the k-th Fourier coefficient $\hat{x}_k$ of x for $k = \pm 1, \pm 3, \pm 5, \ldots$.

Let $\{x_m\}$ be a sequence in $\Omega(w)$ which converges weakly to a limit $x \in H(w)$. For $m = 1, 2, 3, \ldots$ there are functions $y_m \in H(w)$ with $\| y_m\|_w \leqq \sqrt{2}\, d$ such that $(\hat{x}_m)_k = \gamma_k(w)(\hat{y}_m)_k$ for $k = \pm 3, \pm 5, \ldots$.

Now the set $\{y \in H(w): \|y\|_w \leqq \sqrt{2}\, d\}$ is compact in the weak topology. Hence there is a function $y \in H(w)$ with $\|y\|_w \leqq \sqrt{2} d$ such that some subsequence $\{y_{m'}\}$ of $\{y_m\}$ converges to y in the weak topology. Since $\{(\hat{x}_m)_k\}$ is a convergent sequence for $k = \pm 3, \pm 5, \ldots$, we see that $(\hat{x}_m)_k \to \gamma_k(w)\hat{y}_k$ on the whole sequence. But $(\hat{x}_m)_k \to \hat{x}_k$ as well. Thus, $\hat{x}_k = \gamma_k(w)\hat{y}_k$. Also $\hat{x}_1$ is real and $a_1 \leqq \hat{x}_1 = \hat{x}_{-1} = \lim_{m\to\infty}(\hat{x}_m)_1 \leqq a_2$. Thus $x \in \Omega(w)$. Since $\Omega(w)$ contains all its limit points, it must be closed.

To see that the uniform and weak topologies are equivalent on $\Omega(w)$, let $\{x_m\}$ be a sequence in $\Omega(w)$. Suppose that $\{x_m\}$ converges uniformly to a limit $x \in H_c(w)$. From equation (1) it is clear that $(\hat{x}_m)_k \to \hat{x}_k$ as $m \to \infty$ for $k = \pm 1, \pm 3, \ldots$. Thus x_m converges weakly to $x \in \Omega(w)$. Conversely, suppose that $\{x_m\}$ converges weakly to $x \in \Omega(w)$. Let

$$\Gamma_w(N) = [\sum_{\substack{k\ odd \\ |k|>N}} \gamma_k(w)^2]^{1/2}.$$

Then $\Gamma_w(N) \to 0$ as $N \to \infty$. Then for all t

$$\begin{aligned}
|x_m(t) - x(t)| &\leqq \frac{1}{2} \sum_{\substack{k\ odd \\ |k|\leqq N}} |(\hat{x}_m)_k - \hat{x}_k| + \frac{1}{2} \sum_{\substack{k\ odd \\ |k|>N}} |(\hat{x}_m)_k - \hat{x}_k| \\
&\leqq \frac{1}{2} \sum_{\substack{k\ odd \\ |k|\leqq N}} |(\hat{x}_m)_k - \hat{x}_k| + \frac{1}{2} \sum_{\substack{k\ odd \\ |k|>N}} \gamma_k(w)|(\hat{y}_m)_k - \hat{y}_k| \\
&\leqq \frac{1}{2} \sum_{\substack{k\ odd \\ |k|\leqq N}} |(\hat{x}_m)_k - \hat{x}_k| + \frac{\sqrt{2}}{2} \Gamma_w(N)(\|y_m\|_w + \|y\|_w) \\
&\leqq \frac{1}{2} \sum_{\substack{k\ odd \\ |k|\leqq N}} |(\hat{x}_m)_k - \hat{x}_k| + 2\,\Gamma_w(N)d
\end{aligned}$$

where we have defined y_m and y in the obvious way and used the Schwartz inequality and the Parseval equality. In the final estimate given above, the last term may be made small independently of m and, for a given N, the first term may be made arbitrarily small by choosing m sufficiently large. Thus $x_m \to x$ uniformly.

LEMMA 3. *The hysteresis operator n is a continuous map from $\Omega(w)$ (in the weak topology) to $H(w)$ (in the L_2-norm topology).*

PROOF. Let $\{x_m\} \subset \Omega(w)$ and let x_m converge to x in the weak topology (and hence, by Lemma 2, in the uniform topology). Now x can be written in the form $x(t) = a \cos wt + v(t)$, $v \in \Omega_2(w)$ so that $x'(t) = -wa \sin wt + v'(t)$ Now if $\hat{v}_k = \gamma_k(w)\hat{y}_k$ for $\|y\|_w < \sqrt{2}\, d$ and $y \in H(w)$, then

$$\begin{aligned}|v'(t)| &\leqq \frac{1}{2}\sum_{\substack{k\ odd\\ |k|>1}} |kw\hat{v}_k| \leqq \frac{1}{2}\sum_{\substack{k\ odd\\ |k|>1}} |kw\gamma_k(w)|\,|\hat{y}_k| \\ &\leqq \frac{1}{2}\left[\sum_{\substack{k\ odd\\ |k|>1}} k^2w^2\gamma_k^2(w)\right]^{1/2}\left[\sum_{\substack{k\ odd\\ |k|>1}} |\hat{y}_k|^2\right]^{1/2} \\ &\leqq \frac{1}{2}e(w)\sqrt{2}\,\|y\|_w \leqq e(w)d\end{aligned}$$

and by a similar argument $|v(t)| \leqq b(w)d$.

We wish to show that x is in the continuous domain of n. First, we note that $|x(t)| \leqq a + |v(t)| \leqq a_2 + b(w)d = \eta$. Let $\zeta = -b(w)d + (a_1^2 - e^2(w)\,d^2/w^2)^{1/2}$. Then $\zeta > 0$ by (A4)(b). Let $|\alpha| < \zeta$. Using the odd symmetry in the problem, we may assume that $\alpha \geqq 0$. Suppose that $x(t_\alpha) = \alpha$ for some t_α and that $x'(t_\alpha) = 0$. Then $a\cos wt_\alpha = \alpha - v(t_\alpha)$ and

$$\begin{aligned}w(a^2 - (\alpha - v(t_\alpha))^2)^{1/2} &= |w\,a\sin wt_\alpha| \\ &= |v'(t_\alpha)| \leqq e(w)d.\end{aligned}$$

Thus

$$\begin{aligned}a_1^2 \leqq a^2 &\leqq e^2d^2/w^2 + (\alpha - v(t_\alpha))^2 \\ &\leqq e^2d^2/w^2 + (\alpha + bd)^2.\end{aligned}$$

Hence

$$0 \leqq e^2d^2/w^2 - a_1^2 + b^2d^2 + 2\,bd\alpha + \alpha^2 = p(\alpha).$$

Now $p(\alpha)$ has zeros $-bd \pm (a_1^2 - e^2d^2/w^2)^{1/2}$. Since

$$0 \leqq \alpha < \zeta = -bd + (a_1^2 - e^2d^2/w^2)^{1/2},$$

we see that $p(\alpha) < 0$ which is a contradiction. Thus, if $|x(t)| < \zeta$, then $x'(t) \neq 0$. By definition 2.2, x is of class $\mathscr{F}(\zeta, \eta)$; and by (A4)(b), x is in the continuous domain of n.

We have shown the $(nx)(t)$ is a piecewise continuous function of $t \in \mathbf{R}$. Similarly, the functions $(nx_m)(t)$ are piecewise continuous and converge pointwise to $(nx)(t)$. Since the sequence $\{(nx_m)(t)\}$ is also uniformly bounded, it follows from the Lebesgue dominated covergence theorem that $(nx_m)(t)$ converges to $(nx)(t)$ in the L_2-norm.

THEOREM 1. *Let* (E) *satisfy* (A3) *and* (A4). *Then* (E) *has a nontrivial solution in* $\Omega(w)$ *for some* $w \in [w_1, w_2]$.

PROOF. We use the projections P and P^* to write (E) in the form

$$(\mathrm{E}_1) \qquad u = -Pgn(u + v)$$

$$(\mathrm{E}_2) \qquad v = -P^*gn(u + v).$$

In (E_2) add cP^*gv to both sides. Notice that (A3) implies that $(I + cP^*g)^{-1}$ exists on $H(w)$ and is a bounded linear map. Since $P^*gu = 0$, then (E_2) is equivalent to

$$(I + cP^*g)v = -P^*g[n(u + v) - c(u + v)],$$

or

$$v = -(I + cP^*g)^{-1}P^*g[n(u + v) - c(u + v)]. \tag{6}$$

For $\phi \in H(1) = H(w)|_{w=1}$ and $w > 0$ define a map $K(w, \phi)$ as follows. The symbols $z^* = K(w, \phi)(t)$ mean

$$z(t) \sim -\frac{1}{2} \sum_{\substack{k \text{ odd} \\ |k|>1}} \frac{\hat{\phi}_n G(ikw)}{1 + cG(ikw)} e^{ikt}.$$

Then (6) is equivalent to

$$v(t) = K(w, n(u(t/w) + v(t/w))(wt). \tag{7}$$

From (E_1) and the fact that $Pgv = 0$ we get $u + Pgn(u) = Pg(n(u) - n(u + v) + cv)$. On taking Fourier coefficients we see that this is equivalent to satisfying

$$G(iw)^{-1} a + N(a)a = \frac{w}{\pi} \int_0^{2\pi/w} e^{-iwt}(n(u(t)) - n(u(t) + v(t)) + cv(t))dt,$$

where $u(t) = a \cos wt$. This expression can be rearranged to

$$\begin{aligned} N(a) + G(iw)^{-1} = \frac{1}{\pi a} \int_0^{2\pi} e^{-is}[&n(a \cos s) \\ &- n(a \cos s + v(s/w)) + cv(s/w)]ds. \end{aligned} \tag{8}$$

Define $J(a, w) = N(a) + G(iw)^{-1}$ and define

$$F(a, \phi) = \frac{1}{\pi a} \int_0^{2\pi} e^{-is} [n(a \cos s) - n(a \cos s + \phi(s)) + c\phi(s)]ds.$$

Then (8) can be written as

$$J(a, w) = F(a, v(t/w)). \tag{9}$$

Since $z = J(a, w)$ is one-to-one by (A4), then there are continuous inverse function $a(z)$ and $w(z)$ defined on the set $\Omega_1 = J([w_1, w_2] \times [a_1, a_2])$. Define $\Omega_3 = \{(z, \phi): z \in \Omega_1 \text{ and } \phi \in \Omega_2(1)\}$. For any $(z, \phi) \in \Omega_3$ define

$$T(z, \phi) = (F(a(z), \phi), K(w(z), n(a(z) \cos t + \phi(t))).$$

By (7) and (9) we see that if (z, ϕ) is a fixed point of T in Ω_3, then

$$u(t) = a(z) \cos w(z)t, \ v(t) = \phi(w(z)t)$$

solves (E_1) and (E_2) and $x = u + v$ solves (E). Moreover we will have $a_1 \leqq a(z) \leqq a_2$ and $w_1 \leqq w(z) \leqq w_2$.

To see that T has a fixed point we first note that Ω_1 is the homeomorphic image of a compact convex subset of $\mathbf{R}^2$ and $\Omega_2(1)$ is a compact and convex subset of $H(1)$. By the definition of T and by Lemma 3 we see that T is continuous on $\Omega_3 = \Omega_1 \times \Omega_2(1)$. Since $|F(a(z), \phi)| < r(a(z), w(z))$, then $F(a(z), \phi) \in \Omega_1$. The k-th Fourier coefficient of $K(w, n(a \cos t + \phi))$ is

$$\frac{G(ikw)}{1 + c\, G(ikw)} y_k$$

where $y = n(a \cos t + \phi) - c(a \cos t + \phi) \in H(1)$. By definition 2.3 property (i) we see that $\|y\|_w < \sqrt{2}\, d$. Let $G(ikw)/(1 + c\, G(ikw)) = \gamma_k(w) \exp(i\delta_k)$. Then the k-th Fourier coefficient of $K(w, n(a \cos t + \phi))$ is $\gamma_k(w)[\hat{y}_k \exp(i\delta_k)]$. Thus we see that

$$K(w(z), n(a(z) \cos t + \phi)) \in \Omega_2(1).$$

We have shown that $T\colon \Omega_3 \to \Omega_3$, T is a continuous function, and Ω_3 is the homeomorphic image of a compact and convex set. By the Schauder Fixed Point Theorem (see [**4**, p. 456]) T has a fixed point. As explained above, this fixed point determines a solution of (E).

IV. Example. Typically $G(iw)$ is obtained from a differential equation or from a delay-differential equation as explained in §II. For this case the only requirement posed by (A3) is that the differential equation be real valued and of order 2 or more. This corresponds to the physical requirement that sufficient filtering should be present. We further require that the describing function determining equation (5) should have a solution pair (a_0, w_0), a_0 should not be a point of discontinuity of $n(x)$ nor in a hysteresis interval of $n(x)$, and a_0 should not be "too close" to any such point (see (A4)).

Finally we require that $n(x)$ is not "too nonlinear" when x is nearly a sinosoidal, i.e., $x \cong a \cos wt$ (see (A4)(d) and (c)). The assumptions (A4) (c) and the bound on γ_k in (A3) are technical mathematical assumptions. In most examples they are easily satisfied.

As a specific example to illustrate how the hypothesis may be checked consider the system

$$y''' + 2y'' + 4y' + 6y + n(y) = 0 \tag{10}$$

where $n(y)$ is the relay with dead zone depicted in Figure 2. Note that $c = 0$ and $d = 1$ for this nonlinearity. The describing function for $n(y)$ (see [**5**]) is

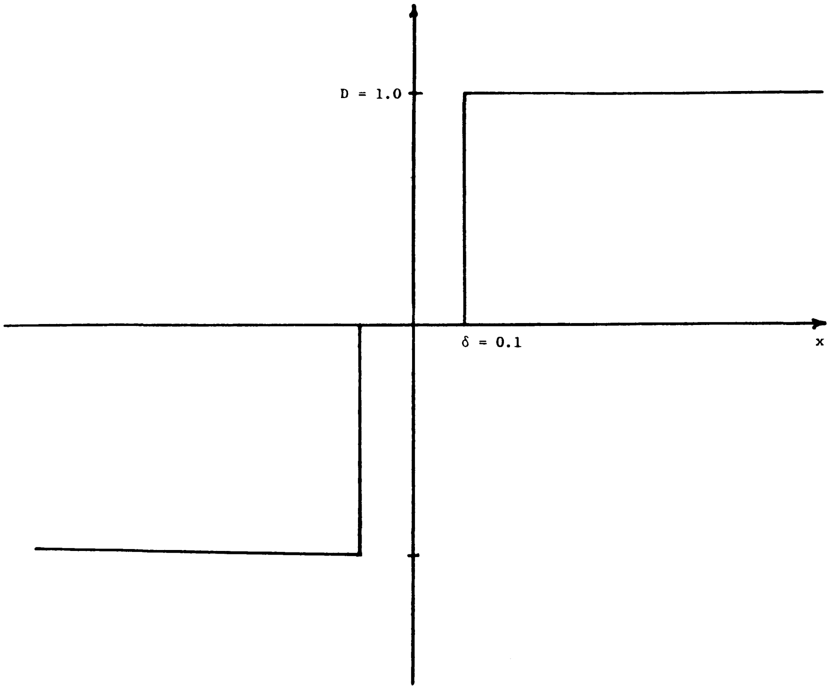

Figure 2. Relay with Dead Zone

$$N(a) = \begin{cases} \dfrac{4}{\pi a}\left(1-\left(\dfrac{1}{10a}\right)^2\right)^{1/2} & \text{if } a \geqq 0.1 \\ 0 & \text{if } 0 \leqq a < 0.1 \end{cases}$$

Equation (5) is, in this case, $iw(4 - w^2) + (6 - 2w^2 + N(a)) = 0$. Hence $w = 2$. There are two solutions for a, namely $a_0 = .62851$ and $a_0 = .10129$. The second of these is too close to the discontinuity $\alpha_1 = 0.1$ of $n(y)$ and must be rejected.

It is easy to compute

$$b(w) = \left\{\sum_{k=1}^{\infty} ([w(2k + 1)]^6 - 4[w(2k + 1)]^4 - 8[w(2k + 1)]^2 + 36)^{-1}\right\}^{1/2}$$

and a similar formula for $e(w)$. These expressions can be evaluated numerically. Typical values are

$$b(2) = .007136, \quad e(2) = .045030$$
$$b(1.98) = .007367, \quad e(1.98) = .046233,$$
$$b(2.02) = .006920, \quad e(2.02) = .044312.$$

After a small amount of numerical experimentation the authors guessed that $w_1 = 1.98$, $w_2 = 2.02$, $a_1 = .56$ and $a_2 = .68$ should work. With a_1, a_2, w_1 and w_2 tentatively assigned these values we start checking hypothesis. First (A3) is automatic since $c = 0$. Assumption (A4)(a) is true by the choice of a_0. It can be verified numerically that (b) is clearly true. For (c) we note that

$$\frac{d}{dw} \operatorname{Im} [G(iw)^{-1}] = 4 - 3w^2 < 0$$

when $|w| > \sqrt{4/3} = 1.1547 \ldots$. Moreover $N'(a) < 0$ on $a_1 \leqq a \leqq a_2$, so

$$\left(\frac{d}{dw} \operatorname{Im} [G(iw)^{-1}]\right) N'(a) \neq 0$$

when $a_1 \leqq a \leqq a_2$ and $w_1 \leqq w \leqq w_2$. Hence (c) is true.

Conditions (d) and (e) are a bit more complicated to verify. First note that $|v(t)| \leqq db(w)$. So by the results in [9, §IV] we can estimate r by $r(a, w) \leqq E(a, b(w))$ where

$$E(a, \varepsilon) = \frac{4}{\pi a}\left(\left(\frac{.2\,\varepsilon + \varepsilon^2}{\sqrt{a^2 - .01} + \sqrt{a^2 - (.1 + \varepsilon)^2}}\right)^2 + \varepsilon^2\right)^{1/2}.$$

With this information (e) can be verified numerically. Typical values satisfy $E(w, a_1) \leqq .154225$ and $E(w, a_2) \leqq .095004$ for $w_1 \leqq w \leqq w_2$.

Since the hypothesis are all true, Theorem 1 applies. By that result there is a solution $y(t) = a \cos wt + v(t)$ where $w \in [1.98, 2.02]$, $a \in [.56, .68]$, $Pv = 0$ and $|v(t)| \leqq .007367$ for all $t \in \mathbf{R}$.

Our theoretical results were checked by numerical simulations of solutions of (10). A periodic solution, which seems to be asymptotically stable, was found. Figure 3 is a graph of the numerical approximation to the solution $y(t)$ using initial conditions $y(0) = .635$, $y'(0) = 0$ and $y''(0) = 2.53$. As can be seen, the solution quickly settles into a nearly sinosoidal oscillation. From the numerical simulations it is possible to estimate that the periodic solution has amplitude $a \cong .6284$ and period $T \cong 3.1438$ so that $w \cong 1.9986$. If $y_p(t)$ is the periodic solution simulated and $u_p(t) = .6284 \cos (1.9986\, t)$, then the numerical data indicates that

$$|y_p(t) - u_p(t)| \leqq .0075,$$

$$|y_p'(t) - u_p'(t)| \leqq .0278,$$

and

$$|y_p''(t) - u_p''(t)| \leqq .0899$$

for all $t \in \mathbf{R}$.

In this example the amplitude of the sinosoidal component of the actual

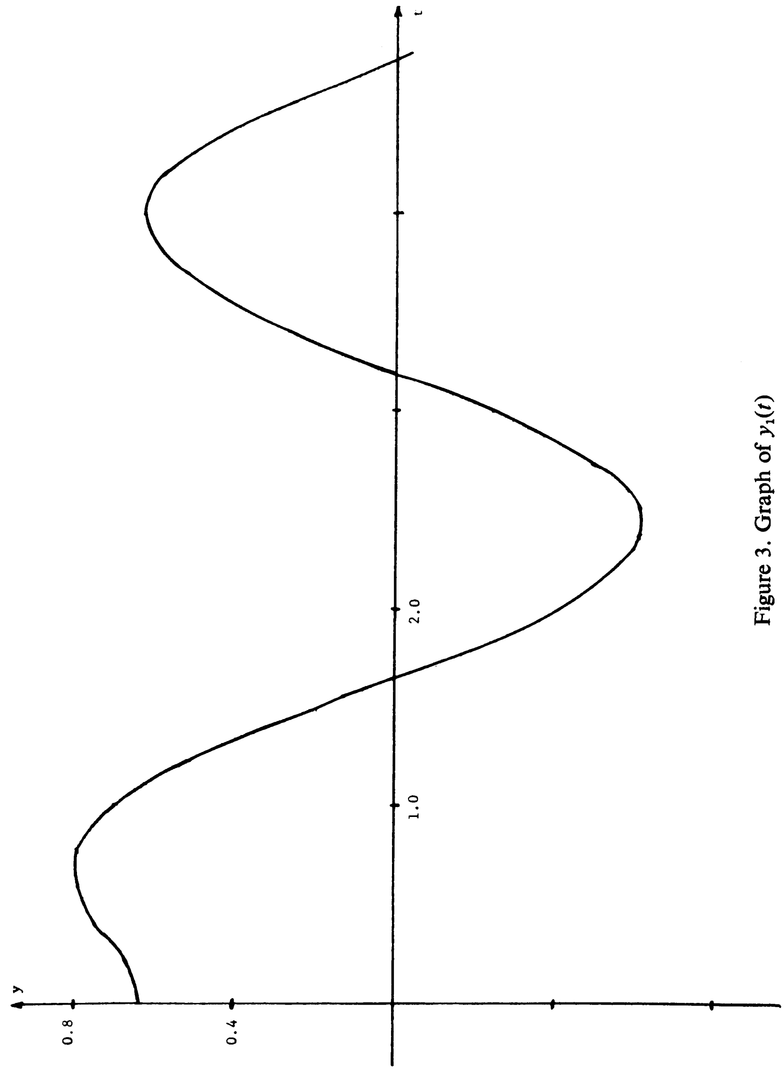

Figure 3. Graph of $y_1(t)$

periodic is not predicted very well, i.e., we can say only that $.56 < a < .68$. However the other theoretical predictions are in excellent agreement with the numerical simulation.

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