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A COINCIDENCE THEOREM IN CONVEX SETS WITH APPLICATIONS TO PERIODIC SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

1. Introduction. Variations of the expansion and compression theorems of Krasnosel'skii have been used frequently (see, for example, [1], [3], [4], [5], [6], [8], [9]) to obtain existence of solutions to various problems involving ordinary and functional differential equations where the solution is required to lie in some cone. These theorems apply to operator equations of the form $x = Ax$.

In §2 of this paper we establish a more general framework to treat equations of the form $Lx = Nx$ where L is not necessarily invertible; we obtain Theorem 2.3 as a very simple consequence of invariance under homotopy of the Leray-Schauder degree. This theorem is in the same spirit as the continuation theorem of Mawhin [7] for coincidence degree.

In §3 we illustrate the use of this theorem by giving conditions (Theorems 3.1 and 3.2) under which the problem $\dot{x}(t) = f(t, x(t))$, $x(0) = x(1)$, has a nonzero solution $x(t)$ satisfying $x(t) \geqq 0$.

Finally, in §4 we discuss the problem $\ddot{x}(t) = f(t, x(t))$, $x(0) - x(1) = \dot{x}(0) - \dot{x}(1) = 0$ where we again seek non-negative solutions.

2. A coincidence theorem for convex sets. Let X and Z be real Banach spaces. We will consider a linear mapping L: dom $L \subset X \to Z$ and a not necessarily linear mapping $N: X \to Z$ with the following properties.

a) L is Fredholm of index 0. This entails that Im L be closed and that dim Ker L = codim Im L. As a consequence of this property there exist continuous projection mappings $P: X \to X$ and $Q: Z \to Z$ such that Im P = Ker L and Ker Q = Im L. These projections induce a decomposition of X and Z into corresponding subspaces as indicated in the following diagram.

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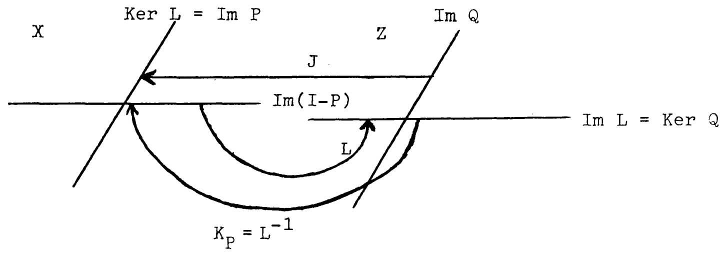

Since dim Im Q = codim Im L = dim Ker L, there is an isomorphism J: Im $Q \to$ Ker L. Moreover, $L|\mathrm{Im}(I - P)$ has an inverse which we shall denote by K_P. It is well known (see [7]) that $Lx = Nx$ is equivalent to

$$x = Mx \equiv Px + JQNx + K_P(I - Q)Nx.$$

(b) N is L-completely continuous. This entails that the mappings QN: $X \to X$ and $K_P(I - Q)N$: $X \to X$ are compact on every bounded subset of X.

We will let C denote a nonempty closed convex subset of X and γ: $X \to C$ will denote a continuous retraction, i.e., $\gamma|C = I$. Finally, Ω will denote an open bounded subset of X and we assume that γ maps bounded subsets of $\bar{\Omega}$ into bounded subsets. Let $\tilde{M} \equiv M \circ \gamma$. It then follows that $\tilde{M}$ is completely continuous.

PROPOSITION 2.1. *If* $Lx \neq Nx$ *for* $x \in C \cap \partial\Omega \cap$ dom L *and* $\tilde{M}(\bar{\Omega}) \subset C$, *then* $d[I - \tilde{M}, \Omega, 0]$ *is defined* (*where* $d[I - \Phi, \Omega, 0]$ *denotes the Leray-Schauder degree of the compact perturbation* Φ *of the identity on the set* Ω *with respect to* 0).

PROOF. Suppose $\tilde{M}x = x$ for $x \in \partial\Omega$. Then $x \in C \cap$ dom L since $\tilde{M}(\bar{\Omega}) \subset C \cap$ dom L Thus $\tilde{M}x = (M \circ \gamma)x = Mx = x$. Since $Mx = x$ is equivalent to $Lx = Nx$, we have a contradiction. Thus $\tilde{M}x \neq x$ for $x \in \partial\Omega$ and the Leray-Schauder degree is defined.

PROPOSITION 2.2. *If* $d[I - \tilde{M}, \Omega, 0] \neq 0$ *and* $\tilde{M}(\bar{\Omega}) \subset C$, *then* $Lx = Nx$ *has a solution in* $C \cap \Omega$.

PROOF. This follows immediately from the proof of Proposition 2.1.

REMARK. An alternative to Proposition 2.2 is to take a set $\Omega \subset C$. With this approach it would be unnecessary to require $\tilde{M}(\bar{\Omega}) \subset C$. On the other hand, we would need $Lx \neq Nx$ on the entire boundary $\partial\Omega$. Thus Proposition 2.2 provides a trade-off. Note further that the compression and expansion theorems of Krasnosel'skii may be obtained by using Proposition 2.2 and some clever homotopy arguments (see [8]).

Next we prove our main result.

THEOREM 2.3. *If*

A) $(P + JQN)\gamma(\bar{\Omega}) \subset C$ *and* $\tilde{M}(\bar{\Omega}) \subset C$,

B) $Lx \neq \lambda Nx$, *for* $x \in C \cap \partial\Omega \cap \operatorname{dom} L$ *and* $\lambda \in (0, 1]$, *and*

C) $d_B[I - (P + JQN)\gamma|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0] \neq 0$ (d_B *denotes the Brouwer degree*),

then $Lx = Nx$ *has a solution* $x \in C \cap \Omega$.

PROOF. Consider the family of mappings

$$\tilde{M}(x, \lambda) = (P + JQN)\gamma x + \lambda K_P(I - Q)N\gamma x$$

for $\lambda \in [0, 1]$. By standard arguments, $Lx = \lambda Nx$ is equivalent to $x = M(x, \lambda)$ where $M(x, \lambda) = Px + JQNx + \lambda K_P(I - Q)Nx$. We first show that $\tilde{M}(x, \lambda) \neq x$ for $x \in \partial\Omega$ and $\lambda \in (0, 1]$. If $x \in \partial\Omega$, then $x \in \bar{\Omega}$ and A) implies that $(P + JQN)\gamma x \in C$ and $\tilde{M}x \in C$. But then

$$(1 - \lambda)(P + JQN)\gamma x + \lambda\tilde{M}x = \tilde{M}(x, \lambda) \in C.$$

Hence, if $x = \tilde{M}(x, \lambda)$, then $x \in C \cap \partial\Omega$. Thus B) gives $Lx \neq \lambda Nx$, which implies $M(x, \lambda) = \tilde{M}(x, \lambda) \neq x$, a contradiction.

For $\lambda = 0$ the statement follows from the implicit assumption in C) that the Brouwer degree is well defined. Thus by the property of invariance under homotopy

$$d[I - \tilde{M}(\cdot, 1), \Omega, 0] = d[I - \tilde{M}(\cdot, 0), \Omega, 0].$$

But $\tilde{M}(\cdot, 0) = (P + JQN)\gamma$. Since the range of this mapping is in the finite-dimensional space Ker L,

$$d[I - \tilde{M}(\cdot, 0), \Omega, 0] = d_B[I - (P + JQN)\,\gamma|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega, 0].$$

The conclusion of the theorem then follows from C) and Proposition 2.2.

3. Application to periodic solutions of first-order systems. In this section we consider the problem

$$\dot{x}(t) = f(t, x(t)) \tag{3.1}$$

$$x(0) = x(1) \tag{3.2}$$

where f: $[0, 1] \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous and $f(0, \cdot) = f(1, \cdot)$. We seek nonzero solutions satisfying $x(t) \geqq 0$ on $[0, 1]$.

In order to apply Theorem 2.3 we define appropriate operators associated with (3.1)–(3.2). Let $X = \{x: [0, 1] \to \mathbf{R}^n | x \text{ is continuous and } x(0) = x(1)\}$, ($X$ is equipped with the norm $\|x\|_0 = \max_{[0,1]} \|x(t)\|$) $Z = X$, dom $L = \{x \in X | \dot{x} \text{ is continuous on } [0, 1]\}$, L: dom $L \to Z$, $x \mapsto \dot{x}$, and $N: X \to Z$, $x \mapsto f(\cdot, x(\cdot))$. We note that $\operatorname{Ker} L = \{x \in \operatorname{dom} L | x(t) = c \in \mathbf{R}^n \text{ for all } t \in [0, 1]\}$, $\operatorname{Im} L = \{z \in Z | \int_0^1 z(s)ds = 0\}$ (Note that Im L is closed), and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = n$. Thus L is linear and Fredholm of index 0. We define $P: X \to \operatorname{Ker} L$, $x \mapsto \int_0^1 x(s)ds$, and $Q: Z \to Z$, $z \mapsto \int_0^1 z(s)ds$. We note that P and Q induce the structure described in §2. In particular, for $z \in \operatorname{Im} L$,

$$K_P z = \int_0^1 G(s, t)z(s)ds$$

where

$$G(s, t) = \begin{cases} s, & 0 \leqq s < t \\ s - 1, & t \leqq s \leqq 1. \end{cases}$$

It is easily verified that N is L-completely continuous. We will let $\|x\|^2 = x \cdot x$ for $x \in \mathbf{R}^n$ and J will be taken to be βI where I is essentially the identity mapping.

THEOREM 3.1. *Let* $0 < r < R$ *be constants. Suppose*
i) $f(t, x) \cdot x > 0$, *for* $x \geqq 0$ *and* $\|x\| = R$,
ii) $f(t, x) \cdot x < 0$, *for* $x \geqq 0$ *and* $\|x\| = r$, *and*
iii) $f(t, x) \geqq -x$, *for* $x \geqq 0$ *and* $r \leqq \|x\| \leqq R$.
Then (3.1)–(3.2) *has a solution satisfying* $r < \|x(t)\| < R$ *and* $x(t) \geqq 0$.

PROOF. Let $\Omega = \{x \in X: r < \|x(t)\| < R \text{ for } t \in [0, 1]\}$. We define $C = \{x \in X: x(t) \geqq 0 \text{ on } [0, 1]\}$ and $\gamma: X \to C$ by $(x_1(t), x_2(t), \ldots, x_n(t)) \mapsto (|x_1(t)|, |x_2(t)|, \ldots, |x_n(t)|)$. For convenience we will use the notation $\gamma(x)(t) = \gamma_x(t)$.

A) We first show that $\tilde{M}(\bar{\Omega}) \subset C$. We have

$$\tilde{M}x = \int_0^1 \gamma_x(s)ds + \beta \int_0^1 f(s, \gamma_x(s))ds + \int_0^1 G(s, t)[f(s, \gamma_x(s)) - \int_0^1 f(\tau, \gamma_x(\tau))d\tau]ds.$$

If $x \in \bar{\Omega}$, then $r \leqq \|x(t)\| \leqq R$ which implies that $r \leqq \|\gamma_x(t)\| \leqq R$ for $t \in [0, 1]$. Thus, by iii) we have $f(s, \gamma_x(s)) \geqq -\gamma_x(s)$. Thus,

$$\begin{aligned} \tilde{M}x &= \int_0^1 \gamma_x(s)ds + \int_0^1 [\beta + G(s, t) - \int_0^1 G(\tau, t)d\tau] f(s, \gamma_x(s))ds \\ &= \int_0^1 \gamma_x(s)ds + \int_0^1 H(s, t) f(s, \gamma_x(s))ds, \end{aligned}$$

where

$$H(s, t) = \begin{cases} \beta + 1/2 - (t - s), & 0 \leqq s < t \\ \beta - 1/2 + (s - t), & t \leqq s \leqq 1. \end{cases}$$

Note that $H(s, t) \geqq 0$ if $\beta \geqq 1/2$. Thus, we take $\beta = 1/2$. Then

$$(\tilde{M}x)(t) \geqq (1 - 1)\int_0^1 \gamma_x(s)ds = 0.$$

Second, we assert that $(P + JQN)\gamma(\bar{\Omega}) \subset C$. We have

$$(P + JQN)\gamma x = \int_0^1 \gamma_x(s)ds + \frac{1}{2}\int_0^1 f(s, \gamma_x(s))ds.$$

If $x \in \bar{\Omega}$, then

$$(P + JQN)\gamma x \geqq \int_0^1 \gamma_x(s)ds - \frac{1}{2}\int_0^1 \gamma_x(s)ds \geqq 0.$$

The assertion follows.

B) $Lx \neq \lambda Nx$, for $x \in C \cap \partial\Omega \cap \operatorname{dom} L$ and $\lambda \in (0, 1]$. Suppose $Lx - \lambda Nx$ for some $x \in C \cap \partial\Omega \cap \operatorname{dom} L$. Then $r \leqq \|x(t)\| \leqq R$ and $x(t) \geqq 0$ on $[0, 1]$. Suppose for definiteness that $\|x(t_0)\| = R$ for some $t_0 \in [0, 1]$. We may assume without loss of generality that $t_0 \in [0, 1)$. Then $\|x(t)\|^2$ has a maximum on $[0, 1]$ at t_0 and we must have

$$\left.\frac{d(\|x(t)\|^2)}{dt}\right|_{t=t_0} \leqq 0.$$

But

$$\left.\frac{d(\|x(t)\|^2)}{dt}\right|_{t=t_0} = 2x(t_0)\cdot\dot{x}(t_0) = 2x(t_0)\cdot\lambda f(t_0, x(t_0)) > 0.$$

Thus we reach a contradiction. A similar contradiction is obtained in the case where $\|x(t_0)\| = r$.

C) $d_B[I - (P + JQN)\gamma|_{\operatorname{Ker} L}, \ker L \cap \Omega, 0] \neq 0$. In this case, $\ker L \cap \Omega = \{c \in \mathbf{R}^n | r < \|c\| < R\}$. We write $\ker L \cap \Omega = B_R - \bar{B}_r$ where B_m denotes the ball of radius m centered at the origin. We have

$$Ic - (P + JQN)\gamma c = c - \gamma(c) - \frac{1}{2}\int_0^1 f(s, \gamma(c))ds = \phi(c).$$

We first show that $d_B[\phi(c), B_R, 0] = 0$. Let $y \in \mathbf{R}^n$ be chosen so that $y_i > R$ for $i = 1, 2, \ldots, n$. We consider the family of mappings for $\lambda \in [0, 1]$

$$c - \lambda\gamma(c) - \frac{\lambda}{2}\int_0^1 f(s, \gamma(c))ds - (1 - \lambda)y.$$

Note that if $c \in \partial B_R$, then $\|c\| = R$. If

$$c - \lambda\gamma(c) - \frac{\lambda}{2}\int_0^1 f(s, \gamma(c))ds - (1 - \lambda)y = 0,$$

then

$$c = \lambda\gamma(c) + \frac{\lambda}{2}\int_0^1 f(s, \gamma(c))ds - (1 - \lambda)y.$$

This implies that $c \geqq 0$. Thus, $c = \lambda c + \lambda/2\int_0^1 f(s, c)ds + (1 - \lambda)y$.

Clearly this relation is not satisfied if $\lambda = 0$. On the other hand, if $\lambda \in (0, 1]$, we would have $0 < (\lambda/2)\int_0^1 c\cdot f(s, c)ds = c\cdot(c - y)(1 - \lambda) \leqq 0$. Thus $d_B[\phi(c), B_R, 0] = d_B[I - y, B_R, 0] = 0$.

Next, we show that $d_B[\phi(c), B_r, 0] = 1$. Consider the family of mappings $c - \lambda\gamma(c) - (\lambda/2)\int_0^1 f(s, \gamma(c))ds$, $\lambda \in [0, 1]$. If $\|c\| = r$ and $c - \lambda\gamma(c) - (\lambda/2)\int_0^1 f(s, \gamma(c))ds = 0$, then $c = \lambda\gamma(c) + (\lambda/2)\int_0^1 f(s, \gamma(c))ds \geqq 0$. Thus $c = \lambda c + (\lambda/2)\int_0^1 f(s, c)ds$, and $0 = (\lambda/2)\int_0^1 f(s, c)ds - (1 - \lambda)c$. But then $0 = (\lambda/2)\int_0^1 f(s, c)\cdot c ds - (1 - \lambda)c\cdot c < 0$. Thus $d_B[\phi(c), B_r, 0] = d_B[I, B_r, 0] = 1$. Thus $d_B[\phi, B_R - \bar{B}_r, 0] = 0 - 1 \neq 0$. Theorem 2.3 then yields the desired conclusion.

REMARK. The verification that $\tilde{M}(\bar{\Omega}) \subset C$ depended heavily on the particular choice for P. To illustrate this we obtain an alternative theorem to Theorem 3.1 by choosing the projector $\tilde{P}$: $X \to \ker L$, $x \mapsto x(0)$. In this case

$$K_{\tilde{P}}z = \int_0^1 \tilde{G}(s, t)z(s)ds = \int_0^t z(s)ds$$

where

$$G(s, t) = \begin{cases} 1, & 0 \leqq s < t \\ 0, & t \leqq s \leqq 1. \end{cases}$$

THEOREM 3.2. *Let* $0 < r < R$. *Suppose* i) *and* ii) *of Theorem* 3.1 *are satisfied and*

iii′) $f(t, x) \geqq g(t)$, *for* $x \geqq 0$ *and* $r \leqq \|x\| \leqq R$ *where* g: $[0, 1] \to \mathbf{R}^n$ *is Lebesgue integrable and*

$$(2 - t)\int_0^t g(s)ds + (1 - t)\int_t^1 g(s)ds \geqq 0 \tag{3.3}$$

for $t \in [0, 1]$.
Then (3.1)–(3.2) *has a solution satisfying* $r < \|x(t)\| < R$ *and* $x(t) \geqq 0$.

PROOF. We define γ, Ω and C as in the proof of Theorem 3.1, but with $\beta = 1$.

A) $\tilde{M}(\bar{\Omega}) \subset C$. In this case we have for $x \in \bar{\Omega}$

$$\begin{aligned}\tilde{M}x &= \gamma_x(0) + \int_0^1 f(s, \gamma_x(s))ds + \int_0^1 \tilde{G}(s, t)[f(s, \gamma_x(s)) - \int_0^1 f(\tau, \gamma_x(\tau))d\tau]ds \\ &\geqq \int_0^1 \tilde{H}(s, t)f(s, \gamma_x(s))ds,\end{aligned}$$

where

$$\tilde{H}(s, t) = \begin{cases} 2 - t, & 0 \leqq s < t \\ 1 - t, & t \leqq s \leqq 1. \end{cases}$$

Thus,

$$\tilde{M}x \geqq (2 - t)\int_0^t g(s)ds + (1 - t)\int_t^1 g(s)ds \geqq 0.$$

To complete part A) we show that $(\tilde{P} + JQN)\gamma(\bar{\Omega}) \subset C$. We have, for $x \in \bar{\Omega}$,

$$\begin{aligned}(\tilde{P} + JQN)\gamma(x) &= \gamma_x(0) + \int_0^1 f(s, \gamma_x(s))ds \\ &\geqq \int_0^1 f(s, \gamma_x(s))ds \geqq \int_0^1 g(s)ds \geqq 0,\end{aligned}$$

the last inequality being deduced from (3.3) with $t = 1$.

Parts B) and C) are identical to the proof of Theorem 3.1 and we again apply Theorem 2.3 to obtain existence of a solution in $C \cap \Omega$.

REMARK. In either Theorem 3.1 or Theorem 3.2 the signs in (i) and (ii) may be reversed. If condition (i) is removed, we obtain non-negative solutions.

4. Application to periodic solutions of second-order systems. In this section we consider the problem

$$\ddot{x} = f(t, x) \tag{4.1}$$

$$x(0) - x(1) = \dot{x}(0) - \dot{x}(1) = 0 \tag{4.2}$$

where f: $[0, 1] \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous and $f(0, x) = f(1, x)$. Associated with problem (4.1)–(4.2), define $X = \{x\colon [0, 1] \to \mathbf{R}^n | x$ is continuous and x satisfies $x(0) - x(1) = 0\}$ with the norm $\|x\|_0$, $Z = X$, dom $L = \{x \in X | \ddot{x}$ is continuous on $[0, 1]$ and $\dot{x}(0) - \dot{x}(1) = 0\}$, L: dom $L \to Z$, $x \mapsto \ddot{x}$, and N: $X \to Z$, $x \mapsto f(\cdot, x(\cdot))$. We note that Ker $L = \{x \in$ dom $L | x(t) = c \in \mathbf{R}^n$ for all $t \in [0, 1]\}$, Im $L = \{z \in Z | \int_0^1 z(s)ds = 0\}$ (Im L is closed), and dim ker L = codim Im L. Thus L is linear and Fredholm of index 0. We define P: $X \to$ Ker L, $x \mapsto \int_0^1 x(s)ds$, and Q: $Z \to Z$, $z \mapsto \int_0^1 z(s)ds$. Again, P and Q induce the type of structure described in §2. In particular, for $z \in$ Im L,

$$K_P z = \int_0^1 G(s, t)z(s)ds$$

where

$$G(s,t)=\begin{cases}\dfrac{s(2t-1-s)}{2}, & 0\leqq s<t\\[2mm] \dfrac{(s-1)(2t-2)}{2}, & t\leqq s\leqq 1.\end{cases}$$

It is again easily shown that N is L-completely continuous. In the next two theorems we shall again apply Theorem 2.3.

THEOREM 4.1. *Let* $\mathbf{n}>0$ *be a vector in* $\mathbf{R}^n$. *Suppose there exist constants* α, r *and* R *with* $0\leqq\alpha\leqq 8$ *and* $0<r<R$ *such that*

i) $\mathbf{n}\cdot f(t,x)>0$ *for* $x\geqq 0$ *and* $\mathbf{n}\cdot x=R$,

ii) $\mathbf{n}\cdot f(t,x)<0$ *for* $x\geqq 0$ *and* $\mathbf{n}\cdot x=r$, *and*

iii) $f(t,x)\geqq-\alpha x$ *for* $x\geqq 0$ *and* $r\leqq\mathbf{n}\cdot x\leqq R$.

Then (4.1)–(4.2) *has a nonzero solution* x *satisfying* $x(t)\geqq 0$.

PROOF. Let us define $\Omega=\{x\in X|r<\mathbf{n}\cdot\gamma(x(t))<R\}$. We assert that condition (B) is satisfied. Suppose $Lx=\lambda Nx$ for some $x\in\partial\Omega\cap C$ and $\lambda\in(0,1]$. We would have $\ddot{x}(t)=\lambda f(t,x(t))$, $\lambda\in(0,1]$, $x(0)-x(1)=\dot{x}(0)-\dot{x}(1)=0$, and $x(t)\geqq 0$. Let $g(t)=\mathbf{n}\cdot x(t)$. Suppose there exists $t_0\in[0,1]$ such that $g(t_0)=\max_{t\in[0,1]}g(t)=R$. Then $g''(t_0)\leqq 0$. On the other hand,

$$g''(t_0)=\mathbf{n}\cdot x''(t_0)=\lambda\mathbf{n}\cdot f(t_0,x(t_0))>0.$$

A similar contradiction is obtained if $g(t_0)=\min_{t\in[0,1]}g(t)=r$. The assertion follows.

Next we show that $\tilde{M}(\bar{\Omega})\subset C$. If $x\in\bar{\Omega}$, then $f(s,\gamma_x(s))\geqq-\alpha\gamma_x(s)$. Hence

$$\begin{aligned}\tilde{M}x&=\int_0^1\gamma_x(s)ds+\beta\int_0^1 f(s,\gamma_x(s))ds+\int_0^1 G(s,t)[f(s,\gamma_x(s))\\&\quad-\int_0^1 f(\tau,\gamma_x(\tau))d\tau]ds=\int_0^1\gamma_x(s)ds+\int_0^1\bar{H}(s,t)f(s,\gamma_x(s))ds,\end{aligned}$$

where

$$\bar{H}(s,t)=\beta+G(s,t)-\int_0^1 G(s,t)ds.$$

If β is taken to be $1/12$, then it can be shown that $0\leqq\bar{H}(s,t)\leqq 1/8$. Thus,

$$\tilde{M}x\geqq(1-\alpha/8)\int_0^1\gamma_x(s)ds\geqq 0.$$

Moreover, it is easily seen that $(P+JQN)\gamma(\bar{\Omega})\subset C$. Therefore, condition A) is satisfied.

Finally we outline the proof of condition C). In this case we have $\operatorname{Ker}L\cap\Omega=\{c\in\mathbf{R}^n|r<\mathbf{n}\cdot\gamma(c)<R\}$. As in the proof of Theorem 3.1,

it can be shown that $d_B[\phi(c), D_R, 0] = 0$ and $d_B[\phi(c), D_r, 0] = 1$ where $D_\rho = \{c \in \mathbf{R}^n | \mathbf{n}\cdot\gamma(c) < \rho\}$ and $\phi(c) = c - \gamma(c) - \int_0^1 f(s, \gamma(c))ds$. Hence,

$$\begin{aligned} &d_B[I - (P + JQN)\gamma|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0] \\ &= d_B[\phi, D_R, 0] - d_B[\phi, D_r, 0] \\ &= 0 - 1 \neq 0. \end{aligned}$$

We conclude this section by using an a priori bound lemma (Lemma 4.4) to obtain nonnegative solutions to (4.1)–(4.2). Our result, Theorem 4.5, is in the spirit of [**2**].

LEMMA 4.2. *There exist $\alpha_0 > 0$ and $\rho_0 > 0$ such that if $0 < \alpha \leqq \alpha_0$ and $M \geqq \rho_0$, the solution $z(t, M)$ to the scalar initial value problem $\ddot{z} = -\alpha z$, $z(0) = M$, $\dot{z}(0) = 0$, satisfies $z(t, M) \geqq R$ on $[-1/2, 1/2]$ where $R > 0$ is given.*

LEMMA 4.3. *Suppose $0 < \alpha \leqq \alpha_0$. Assume that $f(t, x) \geqq -\alpha x$ for $z \geqq 0$. If x is a nonnegative 1-periodic solution to $\ddot{x}(t) = f(t, x(t))$ with $\mathbf{n}\cdot x(t_0) = M$ and $\mathbf{n}\cdot\dot{x}(t_0) = 0$ for some $t_0 \in [0, 1]$, then $\mathbf{n}\cdot x(t) \geqq z(t - t_0, M)$ on $[t_0 - 1/2, t_0 + 1/2]$.*

The above lemmas are proved by simple direct computation, therefore, their proofs are omitted here. It may be shown, in fact, that $\alpha_0 = \pi/2$.

LEMMA 4.4. *Suppose the hypotheses of Lemma 4.3 are satisfied. Further assume $\mathbf{n}\cdot f(t, x) < 0$ for $\mathbf{n}\cdot x \geqq R$ and $x \geqq 0$. If x is a nonnegative 1-periodic solution to $\ddot{x}(t) = f(t, x(t))$, then $\mathbf{n}\cdot x(t) \leqq \rho_0$.*

PROOF. Let $u(t) = \mathbf{n}\cdot x(t)$. Suppose $\max u(t) = u(t_0) = M > \rho_0$. Then $\dot{u}(t_0) = \mathbf{n}\cdot x(t_0) = 0$ and from Lemma 4.3, $\mathbf{n}\cdot x(t) \geqq z(t - t_0, M)$ on $[t_0 - 1/2, t_0 + 1/2]$. This relation and Lemma 4.2 imply that $\mathbf{n}\cdot x(t) \geqq R$ on $[t_0 - 1/2, t_0 + 1/2]$. Further, there exists $t_1 \in [t_0 - 1/2, t_0 + 1/2]$ such that $u(t_1) = \min u(t)$. On the other hand, $u''(t_1) = \mathbf{n}\cdot x''(t_1) = \mathbf{n}\cdot f(t_1, x(t_1)) < 0$, a contradiction.

REMARK. In Lemmas 4.3 and 4.4, x and f were extended by 1-periodicity when needed.

THEOREM 4.5. *Let $\mathbf{n} \in \mathbf{R}^n$ be positive. Suppose the following conditions are fulfilled*:

i) *there exists $R > 0$ such that $\mathbf{n}\cdot f(t, x) < 0$ for $x \geqq 0$ and $\mathbf{n}\cdot x \geqq R$; and*

ii) *there exists α with $0 < \alpha \leqq 8$ such that $f(t, x) \geqq -\alpha x$ for $x \geqq 0$. Then (4.1)–(4.2) has a nonnegative solution.*

PROOF. We use the notation preceding Theorem 4.1. First we observe that exactly as in Lemma 4.4, any possible solution to $\dot{x}(t) = \lambda f(t, x(t))$, $\lambda \in [0, 1]$, $x(0) - x(1) = \dot{x}(0) - \dot{x}(1) = 0$, $x(t) \geqq 0$, satisfies $\mathbf{n} \cdot x(t) \leqq \rho_0$.

Let $\Omega = \{x \in X | \mathbf{n} \cdot \gamma(x(t)) < \rho_0 + \varepsilon\}$. We apply Theorem 2.3. To verify condition A) we use the same argument as in Theorem 4.1. Condition B) follows immediately from the a priori bounds obtained above.

Finally, we suggest the proof of condition C). In this case we have $\operatorname{Ker} L \cap \Omega = \{c \in \mathbf{R}^n : \mathbf{n} \cdot \gamma(c) < \rho_0 + \varepsilon\}$. As in the proof of Theorem 4.1 it can be shown that $d_B[\phi(c), D_{\rho_0+\varepsilon}, 0] = 1$, where $\phi(c) = c - \gamma(c) - \int_0^1 f(s, \gamma(c))ds$. Hence

$$d_B[I - (P + JQN)\gamma|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0] = d_B[\phi, D_{\rho_0+\varepsilon}, 0] = 1 \neq 0.$$

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