NONUNIFORM NONRESONANCE CONDITIONS AT THE TWO FIRST EIGENVALUES FOR PERIODIC SOLUTIONS OF FORCED LIENARD AND DUFFING EQUATIONS

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

1. Introduction. This paper is devoted to the study of the periodic boundary value problem for the Lienard differential equation

$$(1.1) \qquad x''(t) + f(x(t))x'(t) + g(t, x(t)) = 0$$

and its special case, the Duffing equation

$$(1.2) \qquad x''(t) + cx'(t) + g(t, x(t)) = 0.$$

Without loss of generality, we can assume that the boundary conditions are taken on the interval $[0, 2\pi]$, namely,

$$(1.3) \qquad x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0.$$

Our results are only significant for the so-called forced case, i.e., when $g(t, 0) \not\equiv 0$.

There is a vast literature dealing with problems (1.1–1.3) and (1.2–1.3) and we refer to [10] and its bibliography for further references. For f continuous and g of the form $g(t, x) = h(x) - e(t)$ with h and e continuous and e 2π-periodic, Reissig [9] has proved that problem (1.1–1.3) has at least one solution if

$$0 < \liminf_{|x|\to\infty} \frac{h(x)}{x} \leq \limsup_{|x|\to\infty} \frac{h(x)}{x} < 1.$$

On the other hand, Amaral and Pera [1] have proved that problem (1.2–1.3) has at least one solution for the case where $c = 0$, g is 2π-periodic in t and continuous,

$$\alpha \leq \gamma_-(t) = \liminf_{|x|\to\infty} \frac{g(t, x)}{x} \leq \limsup_{|x|\to\infty} \frac{g(t, x)}{x} = \gamma_+(t) \leq \beta < 1$$

uniformly in $t \in [0, 2\pi]$, for some $\alpha \in \mathbf{R}$, and $\int_0^{2\pi} \gamma_-(t)dt > 0$.

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Both results are based on Leray-Schauder's type techniques and differ in the way of getting the required a priori estimates for the possible solutions. The aim of this paper is to adapt the approach introduced in [8] for semi-linear elliptic Dirichlet problems to prove existence theorems for (1.1–1.3) and (1.2–1.3) which unify the results of Reissig and of Amaral and Pera and extend them in various directions. Theorem 1 in §3 will prove the existence of a solution for (1.1–1.3) when f is continuous, g satisfies the Caratheodory conditions (see §3), and when the inequalities

$$(1.4) \qquad \gamma(t) \leq \liminf_{|x| \to \infty} \frac{g(t, x)}{x} \leq \limsup_{|x| \to \infty} \frac{g(t, x)}{x} \leq \Gamma(t)$$

hold uniformly a.e. in $t \in [0, 2\pi]$ and γ and Γ are measurable functions such that

$$(1.5) \qquad \gamma \in L^1(0, 2\pi), \ \int_0^{2\pi} \gamma(t)dt > 0, \ \Gamma(t) \leq 1,$$

for a.e. $t \in [0, 2\pi]$, with the strict inequality

$$(1.6) \qquad\qquad\qquad \Gamma(t) < 1$$

on a subset of $[0, 2\pi]$ of positive measure.

Theorem 2 in §4 will show that the existence result still holds for problem (1.2–1.3) with arbitrary $c \in \mathbf{R}$ when one has $\int_0^{2\pi} \gamma(t)dt = 0$ as soon as $\gamma(t) \neq 0$ on a subset of $[0, 2\pi]$ of positive measure, or, equivalently, as soon as $\int_0^{2\pi} \gamma^+(t)dt > 0$, where we define, as usual, γ^+ by $\max(\gamma, 0)$ and γ^- by $\max(-\gamma, 0)$. All the conditions are in particular satisfied for $\gamma(t) = \sin t$.

The approach used in this paper requires a preliminary study of problems of the form $x''(t) + f(x(t))x'(t) + p(t)x(t) = 0$ or $x''(t) + cx'(t) + p(t)x(t) = 0$ together with the boundary conditions (1.3). Results in this line are given in Lemmas 1 to 4 of §2 and §4, which can be of independent interest because the obtained results seem to be new even for the linear case.

The conditions (1.5)–(1.6) relate the asymptotic behavior of $g(t, x)/x$ to the two first eigenvalues 0 and 1 of the periodic boundary value problem on $[0, 2\pi]$ for the linear operator $-(d^2/dt^2)$. One shall notice that a crossing of the zero eigenvalue is allowed on subsets of $[0, 2\pi]$ of positive measure. Of course other results for (1.1)–(1.3) or (1.2)–(1.3) have been obtained or can be obtained when $g(t, x)/x$ is compared for $|x|$ large with other eigenvalues. In particular, it has been shown in [6] (see also [7]) that problem (1.1)–(1.3) has a solution for every continuous f and Caratheodory g if

$$\gamma_+(t) = \limsup_{|x| \to \infty} \frac{g(t, x)}{x} \leq 0$$

uniformly a.e. in $t \in [0, 2\pi]$ and

(1.7) $$\gamma_+ \in L^1(0, 2\pi), \quad \int_0^{2\pi} \gamma_+(t)dt < 0.$$

More generally, it is proved in [4] for (1.2)–(1.3) with $c = 0$ (but the result is easily extended to (1.1)–(1.3)), that existence still holds if (1.7) is replaced by

$$\gamma_+ \in L^1(0, 2\pi), \quad \int_0^{2\pi} (\gamma_+)^+(t)dt < 3/2\pi,$$

$$\int_0^{2\pi} (\gamma_+)^+(t)dt - [1 - ((2\pi/3) \int_0^{2\pi} (\gamma_+)^+(u)du)^{1/2}]^2 \int_0^{2\pi} (\gamma_+)^-(t)dt < 0.$$

Those results, in which $g(t, x)/x$ is compared for large $|x|$ with the first eigenvalue 0 can also be obtained by the approach of the present paper. For problem (1.2–1.3), an easy adaptation of the method used in [8] furnishes the existence of at least one solution if for some $k \in \mathbf{N}^*$ the functions γ_- and γ_+ defined in (1.4), are such that, for a.e. $t \in [0, 2\pi]$, one has $k^2 \leq \gamma_-(t), \gamma_+(t) \leq (k + 1)^2$, where in each relation, the strict inequality holds on a subset of $[0, 2\pi]$ of positive measure.

We also mention that, in contrast with the techniques used for getting a priori bounds in [1] and [9], our approach allows the study of similar problems for some systems of Lienard and Duffing equations to which a future paper will be devoted.

We end this introduction by mentioning that besides the classical spaces $C([0, 2\pi])$, $C^k([0, 2\pi])$ and $L^k(0, 2\pi)$ of continuous, k-times continuously differentiable or measurable real functions whose k-th power of the absolute value is Lebesgue integrable, we shall make use in what follows of the Sobolev spaces $W^{2,1}(0, 2\pi)$ and $H^1(0, 2\pi)$ respectively defined by $W^{2,1}(0, 2\pi) = \{x: [0, 2\pi] \to \mathbf{R} | x$ and x' are absolutely continuous on $[0, 2\pi]\}$, with norm

$$|x|_{W^{2,1}} = \int_0^{2\pi} |x(t)|dt + \int_0^{2\pi} |x'(t)|dt + \int_0^{2\pi} |x''(t)|dt,$$

and $H^1(0, 2\pi) = \{x: [0, 2\pi] \to \mathbf{R} | x$ is absolutely continuous on $[0, 2\pi]$ and $x' \in L^2(0, 2\pi)\}$, with norm

$$|x|_{H^1} = \left[\left((2\pi)^{-1} \int_0^{2\pi} x(t)dt \right)^2 + (2\pi)^{-1} \int_0^{2\pi} x'^2(t)dt \right]^{1/2}.$$

In any used normed space, the strong and the weak convergence of sequences will be denoted respectively by $\to$ and $\rightharpoonup$, and we shall use the fact (see, e.g., [2]) that $H^1(0, 2\pi)$ is compactly imbedded into $C([0, 2\pi])$ and is a Hilbert space with inner product defined by

$$(x, y)_{H^1} = \left((2\pi)^{-1} \int_0^{2\pi} x(t)dt\right)\left((2\pi)^{-1} \int_0^{2\pi} y(t)dt\right) + (2\pi)^{-1} \int_0^{2\pi} x'(t)y'(t)dt.$$

2. An inequality for some Lienard operators with periodic boundary conditions. If $x \in L^1(0, 2\pi)$, we shall write $\bar{x} = (2\pi)^{-1}\int_0^{2\pi}x(t)dt$, $\tilde{x}(t) = x(t) - \bar{x}$, so that, $\int_0^{2\pi} \tilde{x}(t)dt = 0$.

LEMMA 1. *Let $\gamma \in L^1(0, 2\pi)$ and $\Gamma \in L^1(0, 2\pi)$ be such that*

$$(2.1) \qquad\qquad \bar{\gamma} = (2\pi)^{-1} \int_0^{2\pi} \gamma(t)dt > 0$$

and such that for a.e. *$t \in [0, 2\pi]$, one has*

$$(2.2) \qquad\qquad \Gamma(t) \leqq 1$$

with the strict inequality on a subset of $[0, 2\pi]$ of positive measure. Then there exists $\delta = \delta(\bar{\gamma}, \Gamma) > 0$ such that for all $x \in H^1(0, 2\pi)$, one has

$$(2.3) \quad B_{\gamma,\Gamma}(x) = \bar{\gamma}\bar{x}^2 + (2\pi)^{-1} \int_0^{2\pi} [(\tilde{x}'(t))^2 - \Gamma(t)\tilde{x}^2(t)]dt \geqq \delta|x|_{H^1}^2.$$

PROOF. We first show that $B_{\gamma,\Gamma}(x) = 0$ if and only if $x = 0$, the first part being trivial. Using (2.2) and Wirtinger's inequality [7], we see that, for all $x \in H^1(0, 2\pi)$, we have

$$(2.4) \quad (2\pi)^{-1} \int_0^{2\pi} [(\tilde{x}'(t))^2 - \Gamma(t)\tilde{x}^2(t)]dt$$

$$\geqq (2\pi)^{-1} \int_0^{2\pi} [(\tilde{x}'(t))^2 - \tilde{x}^2(t)]dt \geqq 0$$

with moreover $\int_0^{2\pi} [(\tilde{x}'(t))^2 - \tilde{x}^2(t)]dt = 0$ it and only if

$$(2.5) \qquad\qquad \tilde{x}(t) = A \sin(t + \phi)$$

for some $A \geqq 0$ and $\phi \in \mathbf{R}$. Thus both terms in the definition (2.3) of $B_{\gamma,\Gamma}(x)$ are nonnegative, so that, for all $x \in H^1(0, 2\pi)$, we have

$$(2.6) \qquad\qquad B_{\gamma,\Gamma}(x) \geqq 0.$$

Moreover, if $B_{\gamma,\Gamma}(x)$ vanishes, then, by (2.1), we must have $\bar{x} = 0$ and by (2.4), we must have

$$\int_0^{2\pi} [(\tilde{x}'^2(t))^2 - \Gamma(t)\tilde{x}^2(t)]dt = \int_0^{2\pi} [(\tilde{x}'(t))^2 - \tilde{x}^2(t)]dt = 0.$$

Consequently, $\tilde{x}$ is of the form (2.5) and

$$0 = \int_0^{2\pi} (1 - \Gamma(t))\tilde{x}^2(t)dt = A^2 \int_0^{2\pi} (1 - \Gamma(t))\sin^2(t + \phi)dt,$$

which implies $A = 0$, hence $\bar{x} = 0$ and thus $x = 0$. Assume now that the conclusion of the lemma is not true. We can then find a sequence (x_n) in $H^1(0, 2\pi)$ and $x \in H^1(0, 2\pi)$ such that (see, e.g., [2]),

$$(2.7) \qquad |x_n|_{H^1} = 1, \; x_n \to x \text{ in } C([0, 2\pi]), \; x_n \rightharpoonup x \text{ in } H^1(0, 2\pi)$$

and

$$(2.8) \qquad 0 \leqq B_{\gamma,\Gamma}(x_n) \leqq 1/n, \; n \in \mathbf{N}^*.$$

From the Schwarz inequality in $H^1(0, 2\pi)$, we have

$$(\bar{x}_n, \bar{x})^2_{H^1} \leqq |\bar{x}_n|^2_{H^1} |\bar{x}|^2_{H^1}, \; n \in \mathbf{N}^*$$

and hence

$$|\bar{x}|^2_{H^1} \leqq \liminf_{n \to \infty} |\bar{x}_n|^2_{H^1}.$$

By (2.7) and (2.8), we obtain

$$(2.9) \qquad \lim_{n \to \infty} |\bar{x}_n|^2_{H^1} = -\bar{\gamma}\bar{x}^2 + (2\pi)^{-1} \int_0^{2\pi} \Gamma(t)\bar{x}^2(t)dt,$$

and hence

$$|\bar{x}|^2_{H^1} \leqq -\bar{\gamma}\bar{x}^2 + (2\pi)^{-1} \int_0^{2\pi} \Gamma(t)\bar{x}^2(t)dt,$$

i.e., $B_{\gamma,\Gamma}(x) \leqq 0$. By (2.6) and the first part of the proof, this implies that $x = 0$ and hence, by (2.7) and (2.9), $\bar{x}_n \to 0$, $|\bar{x}_n|_{H^1} \to 0$ for $n \to \infty$, i.e., $|x_n|_{H^1} \to 0$, a contradiction with the first equality in (2.7), which completes the proof.

LEMMA 2. *Let γ and Γ be as in Lemma 1, let $\delta > 0$ be associated to γ and Γ by that lemma and let $\varepsilon > 0$. Then for all measurable real functions p on $[0, 2\pi]$ satisfying*

$$(2.10) \qquad \gamma(t) - \varepsilon \leqq p(t) \leqq \Gamma(t) + \varepsilon$$

a.e. on $[0, 2\pi]$, all continuous functions $f: \mathbf{R} \to \mathbf{R}$ and all $x \in W^{2,1}(0, 2\pi)$ with

$$(2.11) \qquad x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0,$$

one has

$$(2\pi)^{-1} \int_0^{2\pi} (\bar{x} - \bar{x}(t))(x''(t) + f(x(t))x'(t) + p(t)x(t))dt \geqq (\delta - \varepsilon)|x|^2_{H^1}.$$

PROOF. If $x \in W^{2,1}(0, 2\pi)$ and satisfies (2.11), then we obtain easily, using integration by parts and Lemma 1,

$$(2\pi)^{-1} \int_0^{2\pi} (\bar{x} - \bar{x}(t))(x''(t) + f(x(t))x'(t) + p(t)x(t))dt$$

$$= \bar{p}\bar{x}^2 + (2\pi)^{-1} \int_0^{2\pi} [\bar{x}'(t))^2 - p(t)\bar{x}^2(t)]dt$$

$$\geqq \bar{\gamma}\bar{x}^2 + (2\pi)^{-1} \int_0^{2\pi} [(\bar{x}'(t))^2 - \Gamma(t)\bar{x}^2(t)]dt$$

$$- \varepsilon(\bar{x}^2 + (2\pi)^{-1} \int_0^{2\pi} \bar{x}^2(t)dt)$$

$$\geqq B_{\gamma,\Gamma}(x) - \varepsilon |x|_{H^1}^2 \geqq (\delta - \varepsilon) |x|_{H^1}^2,$$

and the proof is complete.

3. Existence of periodic solutions for some forced Lienard equations. Let $f: \mathbf{R} \to \mathbf{R}$ be continuous and let $g: [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$, $(t, x) \mapsto g(t, x)$ be such that $g(\cdot, x)$ is measurable on $[0, 2\pi]$ for each $x \in \mathbf{R}$ and $g(t, \cdot)$ is continuous on $\mathbf{R}$ for almost each $t \in [0, 2\pi]$. Assume moreover that for each $r > 0$ there exists $\gamma_r \in L^1(0, 2\pi)$ such that $|g(t, x)| \leqq \gamma_r(t)$ for a.e. $t \in [0, 2\pi]$ and all $x \in [-r, r]$. Such a function g will be said to satisfy the Caratheodory conditions. Consider the following periodic boundary-value problem for the Lienard equation

$$(3.1) \qquad \begin{aligned} x''(t) + f(x(t))x'(t) + g(t, x(t)) &= 0, \, t \in [0, 2\pi], \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) &= 0. \end{aligned}$$

We prove the following existence result for (3.1).

THEOREM 1. *Assume that the inequalities*

$$(3.2) \qquad \gamma(t) \leqq \liminf_{|x| \to \infty} x^{-1}g(t, x) \leqq \limsup_{|x| \to \infty} x^{-1}g(t, x) \leqq \Gamma(t)$$

hold uniformly a.e. in $t \in [0, 2\pi]$ and that γ and Γ satisfy the following conditions:

a. *$\gamma \in L^1(0, 2\pi)$ and $\int_0^{2\pi} \gamma(t)dt > 0$; and*
b. *$\Gamma(t) \leqq 1$ with the strict inequality on a subset of $[0, 2\pi]$ of positive measure.*
Then problem (3.1) has at least one solution.

PROOF. We shall first write the equation in an equivalent more suitable form. Let $\delta > 0$ be associated to γ and Γ by Lemma 2; then, by (3.2) we can find $r > 0$ such that for a.e. $t \in [0, 2\pi]$ and all x with $|x| \geqq r$, we have

$$(3.3) \qquad \gamma(t) - (\delta/2) \leqq x^{-1} g(t, x) \leqq \Gamma(t) + (\delta/2).$$

Define $\tilde{g}$ on $[0, 2\pi] \times \mathbf{R}$ by $\tilde{g}(t, x) = \tilde{\gamma}(t, x)x$, where

$$\tilde{\gamma}(t, x) = \begin{cases} x^{-1} g(t, x) & \text{if } |x| \geq r \\ r^{-1} g(t, r) & \text{if } 0 < x < r \\ -r^{-1} g(t, -r) & \text{if } -r < x < 0 \\ \Gamma(t) & \text{if } x = 0. \end{cases}$$

By construction, $\tilde{g}$ satisfies the Caratheodory conditions and, by (3.3), one has

(3.4) $$\gamma(t) - (\delta/2) \leq \tilde{\gamma}(t, x) \leq \Gamma(t) + (\delta/2)$$

for a.e. $t \in [0, 2\pi]$ and all $x \in \mathbf{R}$. Moreover, if $h: [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$ is defined by $h(t, x) = g(t, x) - \tilde{g}(t, x)$, then, for a.e. $t \in [0, 2\pi]$ and all $t \in \mathbf{R}$, we have

(3.5) $$|h(t, x)| \leq \sup_{|x| \leq r} |g(t, x) - \tilde{g}(t, x)| \leq \alpha(t)$$

where $\alpha \in L^1(0, 2\pi)$ only depends on γ, Γ and γ_r. The equation in (3.1) is thus equivalent to

$$x''(t) + f(x(t))x'(t) + \tilde{\gamma}(t, x(t))x(t) + h(t, x(t)) = 0$$

to which we shall apply coincidence degree theory [3, 7]. (Leray-Schauder's degree [5], could also be used at the expense of a reformulation as an integral equation). Let $X = C^1([0, 2\pi])$, $Z = L^1(0, 2\pi)$, dom $L = \{x \in X: x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$ and x' is absolutely continuous on $[0, 2\pi]\}$.

$$L: \text{dom } L \subset X \to Z, \quad x \mapsto x''$$
$$F: X \to Z, \quad x \mapsto f(x(\cdot))x'(\cdot),$$
$$G: X \to Z, \quad x \mapsto \tilde{\gamma}(\cdot, x(\cdot))x(\cdot),$$
$$H: X \to Z, \quad x \mapsto h(\cdot, x(\cdot)),$$
$$A: X \mapsto Z, \quad x \mapsto \tilde{\gamma}(\cdot, 0)x(\cdot) = \Gamma(\cdot)x(\cdot).$$

It is routine to check that F, G, H and A are well defined and L-compact on bounded subsets of X, and that L is a linear Fredholm mapping of index zero. We consider the homotopy $\Phi: \text{dom } L \times [0, 1] \to Z$ defined by $\Phi(x, \lambda) \equiv Lx + \lambda F x + (1 - \lambda)Ax + \lambda Gx + \lambda Hx$, $\lambda \in [0, 1]$, $x \in \text{dom } L$, and, in order to apply Theorem IV.5 of [7] with $\Omega = \{x \in X: |x|_{C^1} < R\}$, we have only to show that there exists $R > 0$ for which $\Phi(x, \lambda) \neq 0$ when $\lambda \in [0, 1]$ and $x \in \text{dom } L$ with $|x| \geq R$. By construction, we have, for all $x \in \text{dom } L$,

$$\gamma(t) - (\delta/2) \leq (1 - \lambda)\Gamma(t) + \lambda\tilde{\gamma}(t, x(t)) \leq \Gamma(t) + (\delta/2),$$

and hence, by Lemma 2, denoting by $\langle\,,\,\rangle$ the inner product in $L^2(0, 2\pi)$ defined by

$$\langle u, v \rangle = \frac{1}{2\pi} \int_0^{2\pi} u(t)v(t)dt,$$

we obtain

$$(3.6) \quad \langle \bar{x} - \tilde{x}, Lx + \lambda Fx + (1 - \lambda)Ax + \lambda Gx \rangle \geq (\delta/2)|x|_{H^1}^2$$

for all $x \in \text{dom } L$ and $\lambda \in [0, 1]$. Consequently, by (3.5) and the continuous imbedding of $H^1(0, 2\pi)$ into $C([0, 2\pi])$, we get, for all $x \in \text{dom } L$ and $\lambda \in [0, 1]$,

$$\langle \bar{x} - \tilde{x}, \Phi(x, \lambda) \rangle \geq (\delta/2)|x|_{H^1}^2 - \lambda|\langle \bar{x} - \tilde{x}, Hx \rangle|$$
$$\geq (\delta/2)|x|_{H^1}^2 - |\alpha|_{L^1}|\bar{x} - \tilde{x}|_C$$
$$\geq (\delta/2)|x|_{H^1}^2 - \beta|x|_{H^1}.$$

Thus, if $\Phi(x, \lambda) = 0$ for some $(x, \lambda) \in \text{dom } L \times [0, 1]$, we have

$$(3.7) \qquad\qquad |x|_{H^1} \leq 2\beta/\delta = \delta_1,$$

and hence

$$(3.8) \qquad\qquad |x|_C \leq \delta_2$$

for some $\delta_2 > 0$. From the relation

$$x'' + \lambda f(x)x' + (1 - \lambda)\Gamma x + \lambda\tilde{\gamma}(t, x)x + \lambda h(t, x) = 0,$$

this implies

$$(3.9) \qquad\qquad |x''|_{L^1} \leq \delta_3$$

where δ_3 depends only on δ_1, δ_2, γ, Γ and α. By (3.7), (3.8) and (3.9) there exists $R > 0$ such that $|x|_{C^1} < R$ for every $\lambda \in [0, 1]$ and every $x \in \text{dom } L$ for which $\Phi(x, \lambda) = 0$, and the proof is complete.

4. A further result in the case of a Duffing equation. An existence result for the periodic boundary value problem (3.1) with f constant can be given in the case where $\int_0^{2\pi}\gamma(t)dt = 0$. Its proof depends upon the following lemma for the linear periodic problem

$$(4.1) \qquad \begin{array}{c} x''(t) + cx'(t) + p(t)x(t) = 0, \\ x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0 \end{array}$$

where $p \in L^1(0, 2\pi)$ and $c \in \mathbf{R}$ is arbitrary.

LEMMA 3. *Assume that p satisfies the following conditions:*
1. $\int_0^{2\pi} p(t)dt = 0$,
2. *$p(t) \neq 0$ on a subset of $[0, 2\pi]$ of positive measure, and*

3. $p(t) \leq 1$ *for* a.e. $t \in [0, 2\pi]$, *with strict inequality on a set of positive measure.*
Then problem (4.1) *has only the trivial solution.*

PROOF. Let x be a solution of (4.1). Then,

$$0 = (2\pi)^{-1} \int_0^{2\pi} (x''(t) + cx'(t) + p(t)x(t))(\bar{x} - \bar{x}(t))dt$$

$$= (2\pi)^{-1} \int_0^{2\pi} [(\bar{x}'(t))^2 - p(t)\bar{x}^2(t)]dt,$$

where we have used assumption 1. By condition 3, this implies, using Wirtinger's inequality,

$$0 \leq (2\pi)^{-1} \int_0^{2\pi} [(\bar{x}'(t))^2 - \bar{x}^2(t)]dt$$

$$\leq (2\pi)^{-1} \int_0^{2\pi} [(\bar{x}'(t))^2 - p(t)\bar{x}^2(t)]dt = 0.$$

Consequently, $\bar{x}(t) = A \sin(t + \phi)$ and

$$A^2 \int_0^{2\pi} [1 - p(t)] \sin^2(t + \phi)dt = 0.$$

Using the second part of assumption 3, this implies $A = 0$, and hence $x(t) = \bar{x}$. Introducing this expression in the equation (4.1), this gives $p(t)\bar{x} = 0$ and then, by condition 2, $\bar{x} = 0$, which completes the proof.

REMARK 1. Assumptions 1 and 2 of Lemma 3 are clearly equivalent to $\int_0^{2\pi} p(t)dt = 0$ and $\int_0^{2\pi} p^+(t)dt > 0$

LEMMA 4. *Let* $\gamma \in L^1(0, 2\pi)$ *and* $\Gamma \in L^1(0, 2\pi)$ *be such that* $\gamma(t) \leq \Gamma(t)$ ≤ 1 *a.e. in* $[0, 2\pi]$, $\int_0^{2\pi} \gamma(t)dt \geq 0$, $\int_0^{2\pi} \gamma^+(t)dt > 0$, *and* $\Gamma(t) < 1$ *on a subset of* $[0, 2\pi]$ *of positive measure. Then there exist* $\varepsilon > 0$ *and* $\eta > 0$ *such that for all measurable* p *satisfying* $\gamma(t) - \varepsilon \leq p(t) \leq \Gamma(t) + \varepsilon$ *a.e. on* $[0, 2\pi]$ *and for all* $x \in W^{2,1}(0, 2\pi)$ *such that* $x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$, *one has*

$$|x'' + cx' + px|_{L^1} \geq \eta |x|_{C^1}.$$

PROOF. Suppose that ε and η do not exist. Then there will be a sequence (x_n) in $W^{2,1}(0, 2\pi)$ with $|x_n|_{C^1} = 1$ and $x_n(0) - x_n(2\pi) = x'_n(0) - x'_n(2\pi) = 0$ and a sequence (p_n) in $L^1(0, 2\pi)$ with $\gamma(t) - 1/n \leq p_n(t) \leq \Gamma(t) + 1/n$ for all $n \in \mathbf{N}^*$ and a.e. $t \in [0, 2\pi]$, such that

(4.2) $$|x''_n + cx'_n + p_n x_n|_{L^1} < 1/n.$$

The boundedness of (x''_n) in L^1-norm, of (x_n) in C^1-norm and the fact that, for a.e. $t \in [0, 2\pi]$,

$$|p_n(t)| \leqq \max[2, 1 + \gamma^-(t)] = \beta(t)$$

with $\beta \in L^1(0, 2\pi)$, imply that, by going to subsequences if necessary, we can assume (see, e.g., [2]) $x_n \to x$ in $C([0, 2\pi])$, $x'_n \to x'$ in $C([0, 2\pi])$, $p_n \rightharpoonup p$ in $L^1(0, 2\pi)$, where $|x|_{C^1} = 1$, and

(4.3) $\gamma(t) \leqq p(t) \leqq \Gamma(t)$

for a.e. $t \in [0, 2\pi]$. On the other hand, for every $\phi \in L^\infty(0, 2\pi)$, we have

$$\left| \int_0^{2\pi} (p_n(t)x_n(t) - p(t)x(t))\varphi(t)dt \right|$$

$$\leqq \left| \int_0^{2\pi} p_n(t)(x_n(t) - x(t))\phi(t)dt \right| + \left| \int_0^{2\pi} (p_n(t) - p(t))x(t)\phi(t)dt \right|$$

$$\leqq |\phi|_{L^\infty}|x_n - x|_C|\beta|_{L^1} + \left| \int_0^{2\pi} (p_n(t) - p(t))x(t)\phi(t)dt \right|$$

and hence $p_n x_n \rightharpoonup px$ in $L^1(0, 2\pi)$. By (4.2) we deduce that $x''_n \rightharpoonup -cx' - px$ in $L^1(0, 2\pi)$, and the weak closedness of the graph of the linear operator d^2/dt^2 implies that $x \in W^{2,1}(0, 2\pi)$, $x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0$ and $x''(t) + cx'(t) + p(t)x(t) = 0$ for a.e. $t \in [0, 2\pi]$.

Using (4.3), we then see that $\int_0^{2\pi} p(t)dt \geqq 0$, $\int_0^{2\pi} p^+(t)dt > 0$ and $p(t) \leqq 1$ a.e. in $[0, 2\pi]$ with strict inequality on a set of positive measure. Consequently, Lemma 1 or Lemma 3 according to $\int_0^{2\pi} p(t)dt > 0$ or $\int_0^{2\pi} p(t)dt = 0$ imply that $x = 0$, a contradiction which completes the proof.

REMARK 2. One shall notice that the equation $x''(t) + (\sin t)x(t) = 0$ satisfies the conditions of Lemma 3.

Consider now the periodic boundary-value problem for the Duffing euqation with arbitrary $c \in \mathbf{R}$

(4.4)
$$x''(t) + cx'(t) + g(t, x(t)) = 0$$
$$x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0,$$

where $g: [0, 2\pi] \times \mathbf{R} \to \mathbf{R}$ verifies the Caratheodory conditions.

THEOREM 2. *Assume that the inequalities*

$$\gamma(t) = \liminf_{|x| \to \infty} x^{-1} g(t, x) \leqq \limsup_{|x| \to \infty} x^{-1} g(t, x) \leqq \Gamma(t)$$

hold uniformly a.e. in $t \in [0, 2\pi]$ where γ and Γ satisfy the following conditions:

  a. $\gamma \in L^1(0, 2\pi)$; $\int_0^{2\pi} \gamma(t)dt \geqq 0$; $\int_0^{2\pi} \gamma^+(t)dt > 0$; *and*
  b. $\Gamma(t) \leqq 1$ *with strict inequality on a subset of $[0, 2\pi]$ of positive measure.*
Then problem (4.4) has at least one solution.

PROOF. Let $\varepsilon > 0$ be associated to γ and Γ by Lemma 4. Then, using the

approach and notations of the proof of Theorem 1, we can write the equation in (4.4) in the equivalent form

$$(4.5) \qquad x''(t) + cx'(t) + \tilde{\gamma}(t, x(t))x(t) + h(t, x(t)) = 0,$$

where

$$(4.6) \qquad \gamma(t) - \varepsilon \leqq \tilde{\gamma}(t, x) \leqq \Gamma(t) + \varepsilon$$

for a.e. $t \in [0, 2\pi]$ and

$$(4.7) \qquad |h(t, x)| \leqq \alpha(t)$$

for a.e. $t \in [0, 2\pi]$ and all $x \in \mathbf{R}$, where $\alpha \in L^1(0, 2\pi)$.

To apply coincidence degree theory to (4.4) written in the form (4.5), we set $X = C([0, 2\pi])$, $Z = L^1(0, 2\pi)$, dom $L = \{x \in X: x$ is absolutely continuous together with x' on $[0, 2\pi]$, and $x(0) - x(2\pi) = x'(0) - x'(2\pi) = 0\}$ and

$$L: \text{dom } L \subset X \to Z, \ x \mapsto x'' + cx'$$

$$G: X \to Z, \ x \mapsto \tilde{\gamma}(\cdot, x(\cdot))x(\cdot),$$

$$H: X \to Z, \ x \mapsto h(\cdot, x(\cdot)),$$

$$A: X \to Z, \ x \mapsto \tilde{\gamma}(\cdot, 0)x(\cdot) = \Gamma(\cdot)x(\cdot).$$

Again, all those mappings are well defined and such that L is a Fredholm mapping of index zero, G, H and A are L-compact on bounded subsets of X. Moreover, problem (4.4) is equivalent to solving the equation

$$(4.8) \qquad Lx + Gx + Hx = 0$$

in dom L. By Theorem IV.5 in [7] with $\Omega = B(R) = \{x \in C([0, 2\pi]):$ $|x|_C < R\}$, equation (4.8) will have a solution if we can show that for each $\lambda \in [0, 1]$ and each $x \in$ dom L such that

$$(4.9) \qquad Lx + (1 - \lambda)Ax + \lambda Gx + \lambda Hx = 0,$$

one has $|x|_C < R$. Now, if $x \in$ dom L satisfies (4.9) for some $\lambda \in [0, 1]$, then

$$x''(t) + cx'(t) + [(1 - \lambda)\Gamma(t) + \lambda\tilde{\gamma}(t, x(t))] x(t) + \lambda h(t, x(t)) = 0$$

and, by (4.6),

$$\gamma(t) - \varepsilon \leqq (1 - \lambda)\Gamma(t) + \lambda\tilde{\gamma}(t, x(t)) \leqq \Gamma(t) + \varepsilon.$$

Therefore, using Lemma 4 and (4.7), we get

$$(4.10) \qquad \begin{aligned} 0 &= |x'' + cx' + [(1 - \lambda)\Gamma + \lambda\tilde{\gamma}(\cdot, x(\cdot))]x + \lambda h(\cdot, x(\cdot))|_{L^1} \\ &\geqq \eta|x|_{C^1} - |\alpha|_{L^1}, \end{aligned}$$

and hence $|x|_{C^0} \leqq |x|_{C^1} \leqq |\alpha|_{L^1}/\eta$. Thus it suffices to take any $R > |\alpha|_{L^1}/\eta$ to complete the proof.

REMARK 3. It follows easily from Lemma 4 and Theorem 2 that problem (4.4) will have a unique solution if g satisfies the Caratheodory conditions and is such that

$$\gamma(t) \leqq \frac{g(t, x) - g(t, y)}{x - y} \leqq \Gamma(t)$$

for a.e. $t \in [0, 2\pi]$ and all $x \neq y$ in $\mathbf{R}$, where γ and Γ satisfy the conditions of Lemma 4.

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