ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 12, Number 4, Fall 1982

ON REGULAR SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

ABSTRACT. The system (1) $A dx/dt + Bx = f$, where A and B are $n \times n$ over **C**, x and f are $n \times 1$ over **C**, is termed "regular" if $\det(sA + B) \not\equiv 0$, $s \in \mathbf{C}$. Various results in the literature pertaining to regular systems (1) when A is singular are derived here by exploiting basic properties of the Laurent series for $(A + zB)^{-1}$ on the domain $0 < |z| < \delta$ in **C**. The results for (1) are then used to get analogous results for (2) $(A_q D^q + \cdots + A_0)y = g$ where $D = d/dt$ and the coefficient matrix A_q may be singular. These include a procedure for drawing valid conclusions regarding solutions of (2) when $\det(s^q A_q + \cdots + A_0) \not\equiv 0$, $s \in \mathbf{C}$, by formal application of the Laplace transform.

1. We consider systems of the form

$$A\dot{x} + Bx = f \qquad (\dot{x} = dx/dt) \tag{1.1}$$

in which A and B are constant $n \times n$ matrices over **C**, the complex numbers, and x and f are $\mathbf{C}^n$-valued functions of a real variable t. We will say the system is *regular* if

$$\Delta(s) = \det(sA + B) \not\equiv 0,\ s \in \mathbf{C}, \tag{1.2}$$

that is, $sA + B$ is invertible for some $s \in \mathbf{C}$. This terminology is adopted because the pencil of matrices $sA + B$ is called regular when A and B are square and (1.2) holds; see [**4**, p. 25, Vol. II].

If A is non-singular, condition (1.2) clearly holds and in this case the formulas and results presented below reduce to the familiar ones for the equation $\dot{x} + A^{-1}Bx = A^{-1}f$ equivalent to (1.1). Thus the interest here lies in the case when $\det A = 0$. This case has been treated elsewhere (see, for example, [**1**], [**4**, Vol. II], [**9**], [**10**]) and arises in some applications (see, for example,[**2**], [**3**]). Our treatment unifies much of this earlier work; moreover, it generally involves only well known mathematical tools and

Received by the editors on November 1, 1980.

provides methods which often should be relatively more straightforward to apply in analyzing any particular example of equation (1.1).

It should be pointed out that in recent years several investigations on systems of the type considered here have been published in the engineering literature. Often the interest there is in the impulsive type of solution which such systems can have, a topic which we do not address. (See **[11]** which contains an extensive list of references.)

The regularity condition (1.2) is a natural one to impose when trying to apply Laplace transform methods to equation (1.1). Specifically, if $x: [0, \infty) \to \mathbf{C}^n$ is a solution of (1.1), with f continuous on $[0, \infty)$, if x and f have Laplace transforms $\hat{x}$ and $\hat{f}$, and if $e^{-st}x(t) \to 0$ as $t \to \infty$ when $\mathrm{Re}(s)$ is sufficiently large, then it follows from (1.1) that

$$(sA + B)\hat{x}(s) = \hat{f}(s) + A\zeta$$

for $\zeta = x(0)$. If (1.1) is regular, then $(sA + B)^{-1}$ exists as a matrix of rational functions of s and

$$\hat{x}(s) = (sA + B)^{-1}(\hat{f}(s) + A\zeta). \tag{1.3}$$

If A is singular, the elements of $(sA + B)^{-1}$ need not tend to 0 as $\mathrm{Re}(s) \to \infty$; hence for some $\zeta \in \mathbf{C}^n$ the term $(sA + B)^{-1}A\zeta$ in (1.3) may not be the Laplace transform of a function. For such ζ the initial value problem (1.1) with $f(t) \equiv 0$ and $x(0) = \zeta$ will have no solution.

In §2 a Laurent series for $(A + zB)^{-1}$ is used to obtain a series for $(sA + B)^{-1}$ when $\Delta(s) \not\equiv 0$, and various relations are developed involving A and B and the coefficients in this expansion. These are used in §3 to obtain complementary projections P_0 and P_1 acting on $\mathbf{C}^n$ in terms of which we describe the conditions for existence and uniqueness of solutions of the initial value problems

$$A\dot{x} + Bx = f, \quad x(0) = \zeta \tag{1.4}$$

$$A\dot{x} + Bx = 0, \quad x(0) = \zeta \tag{1.5}$$

and an explicit representation for the solutions. Such conditions, the values ζ for which solutions exist, and an explicit representation were developed in **[1]** and were derived also by Rose **[9]** using Laplace transform methods. However, in both these instances equation (1.1) was first modified to the equivalent

$$A_c\dot{x} + B_cx = (cA + B)^{-1}f \tag{1.6}$$

where

$$A_c = (cA + B)^{-1}A, \; B_c = (cA + B)^{-1}B \tag{1.7}$$

for some $c \in \mathbf{C}$ such that $\Delta(c) \neq 0$. The resulting formulas and results then involve the Drazin inverses A_c^D and B_c^D of A_c and B_c. Our results are equivalent, of course, but our development avoids explicit use of Drazin inverses by direct consideration of the matrix function $(sA + B)^{-1}$ which seems to us more fundamental in light of the (so far purely formal) relation (1.3).

In §4 we show how the eigenvalues and related Jordan chains for $A\lambda + B$ characterize the ranges of the projections P_0 and P_1. The finite eigenvalues $\lambda \in \mathbf{C}$ for which $\det(A\lambda + B) = 0$ and the corresponding Jordan chains are used to give a basis for the solution space of $A\dot{x} + Bx = 0$. These results are then extended to higher order systems

$$(A_q D^q + \cdots + A_1 D + A_0)y = g \qquad (D = d/dt) \tag{1.8}$$

in the case $g = 0$. Here the A_k are $n \times n$ and A_q may be singular.

An expression for the Laplace transform of any solution of a regular equation (1.1) is developed in §5. This was done in [9] using Drazin inverses. We extend our result to obtain the transform of solutions of (1.8) and determine conditions therefrom for the initial values $y(0)$, $Dy(0)$, $\ldots$, $D^{q-1}y(0)$ for which solutions exist.

2. Hereafter, unless stated otherwise, we assume that condition (1.2) holds. For $s \neq 0$ we may then write

$$(sA + B)^{-1} = s^{-1}(A + s^{-1}B)^{-1}.$$

Since the elements of $(A + zB)^{-1}$ are rational functions of $z \in \mathbf{C}$ we have a unique Laurent expansion

$$(A + zB)^{-1} = \sum_{k=-\mu}^{\infty} z^k Q_k, \qquad \mu \geqq 0, \tag{2.1}$$

valid in some set $0 < |z| < \delta$, $\delta > 0$. This gives

$$(sA + B)^{-1} = p(s) + q(s), \qquad |s| > 1/\delta, \tag{2.2}$$

where

$$p(s) = \sum_{k=1}^{\mu} s^{k-1} Q_{-k} \tag{2.3}$$

is an $n \times n$ matrix polynomial (or polynomial matrix) and

$$q(s) = \sum_{k=0}^{\infty} s^{-k-1} Q_k, \qquad |s| > 1/\delta, \tag{2.4}$$

which tends to 0 as $s \to \infty$. The uniquely determined coefficients Q_k may by computed by routine methods at least for moderate size n. As we show later all Q_k are generated readily from Q_{-1} and Q_0 so it suffices to compute the residues at $z = 0$ of the matrix functions $(A + zB)^{-1}$ and $z^{-1}(A + zB)^{-1}$.

We may assume $Q_{-\mu} \neq 0$ in (2.1). Then if $\mu \geqq 1$, $(A + zB)^{-1}$ has a pole of order μ at $z = 0$. In any case it is convenient to define $Q_k = 0$ for $k < -\mu$ and write (2.1) as

$$(A + zB)^{-1} = \sum_{k=-\infty}^{\infty} z^k Q_k, \qquad 0 < |z| < \delta. \tag{2.5}$$

Substituting this into the relations

$$(A + zB)^{-1}(A + zB) \equiv I$$
$$(A + zB)(A + zB)^{-1} \equiv I$$

which are valid for $0 < |z| < \delta$, and equating coefficients of like powers of z, we find

$$Q_k A = -Q_{k-1}B,\ AQ_k = -BQ_{k-1}, \quad k \neq 0 \tag{2.6}$$

$$Q_0 A + Q_{-1}B = I,\ AQ_0 + BQ_{-1} = I. \tag{2.7}$$

Here I is the $n \times n$ identity matrix. From (2.6) one easily obtains $AQ_kB = BQ_kA$ if $k \neq 0$ and from (2.7) one gets $AQ_0B = BQ_0A$. Thus

$$AQ_kB = BQ_kA, \text{ all integers } k. \tag{2.8}$$

Suppose now that $k \neq 0$ and $j \neq -1$. Then by (2.6) $Q_kAQ_j = -Q_{k-1}BQ_j = Q_{k-1}AQ_{j+1}$. A similar manipulation gives a like result for Q_jAQ_k and we have

$$Q_kAQ_j = Q_{k-1}AQ_{j+1},\ Q_jAQ_k = Q_{j+1}AQ_{k-1},\ k \neq 0, j \neq -1. \tag{2.9}$$

Repeated application of the first of these in case $k \leqq -1$ and $j \geqq 0$ gives $Q_kAQ_j = Q_{k-r}AQ_{j+r}$ for all $r \geqq 0$. Since $Q_k = 0$ for $k < -\mu$, then $Q_kAQ_j = 0$. A similar argument applies to Q_jAQ_k, so we have

$$Q_kAQ_j = Q_jAQ_k = 0 \text{ if } k \leqq -1, j \geqq 0. \tag{2.10}$$

If $k \geqq 0$, $j \geqq 0$, then repeated use of (2.9) and then use of (2.7) gives $Q_kAQ_j = Q_0AQ_{j+k} = Q_{j+k} - Q_{-1}BQ_{j+k} = Q_{j+k}$ by virtue of (2.6) and (2.10). That is,

$$Q_kAQ_j = Q_{k+j}, \text{ if } k \geqq 0, j \geqq 0 \tag{2.11}$$

and a similar argument gives

$$Q_kAQ_j = -Q_{k+j}, \text{ if } k \leqq -1, j \leqq -1. \tag{2.12}$$

These relations enable us to establish the following theorem.

THEOREM 2.1. *If A and B are $n \times n$ over $\mathbf{C}$ and $A + zB$ is invertible for* *some* $z \in \mathbf{C}$, *then the coefficient matrices in* (2.5) *satisfy the following relations*:

$$Q_0AQ_0 = Q_0, \qquad Q_{-1}BQ_{-1} = Q_{-1}, \tag{2.13}$$

$$Q_{-k} = (-Q_{-1}A)^{k-1}Q_{-1} = Q_{-1}(-AQ_{-1})^{k-1}, \qquad k \geqq 1, \tag{2.14}$$

and

$$Q_k = (-Q_0B)^kQ_0 = Q_0(-BQ_0)^k, \qquad k \geqq 0. \tag{2.15}$$

Proof. The first part of (2.13) is (2.11) for $k = j = 0$. From (2.7) and (2.10) we get $Q_{-1}BQ_{-1} = Q_{-1} - Q_0AQ_{-1} - Q_{-1}$. Relation (2.14) can be proved inductively using (2.12) and a similar induction using (2.11) and (2.6) establishes (2.15).

Corollary 2.1. *Under the hypotheses of Theorem* 2.1, *if A is non-singular, then $Q_k = 0$ for $k \leqq -1$. If A is singular, then $Q_k = 0$ for $k < -\mu$ for some $\mu \geqq 1$ but $Q_{-\mu} \neq 0$ and, moreover,*

$$(Q_{-1}A)^\mu = (AQ_{-1})^\mu = 0 \tag{2.16}$$

and

$$(Q_{-1}A)^{\mu-1} \neq 0, \qquad (AQ_{-1})^{\mu-1} \neq 0. \tag{2.17}$$

Proof. If A is non-singular, then $(A + zB)^{-1} = A^{-1}(I + zBA^{-1})^{-1}$ which is analytic at $z = 0$ so $Q_k = 0$ for $k \leqq -1$. If A is singular, then $Q_k = 0$ for $k \leqq -1$ implies $Q_0A = AQ_0 = I$ by (2.7), a contradiction. Thus if A is singular, some element of $(A + zB)^{-1}$ has a pole at $z = 0$ and there is some $\mu \geqq 1$ such that $Q_{-\mu} \neq 0$ and $Q_k = 0$ for $k < -\mu$. Then (2.17) follows immediately from (2.14). Also from (2.14) we get $(-Q_{-1}A)^\mu = -Q_{-\mu}A$. By (2.6) this equals $Q_{-\mu-1}B$ and hence is the zero matrix. Similarly $(-AQ_{-1})^\mu = 0$ so (2.16) holds.

We might note here that in [7] the relations (2.14) and (2.15) along with the property (2.16) were used to define the Q_k, $k \neq 0, -1$, in (2.1). The development here is thus a sort of converse of that in [7].

3. Here we derive an explicit representation for solutions of a regular system (1.1). Our derivation is similar to that in [1] but we avoid a direct use of the Drazin inverse and the intermediate step of treating instead the equivalent equation (1.6). The equivalence of our representation to that in [1] is shown later in this section.

To simplify the notation in what follows let us define

$$P_0 = Q_0A, \qquad P_1 = Q_{-1}B, \qquad M_0 = AQ_0, \qquad M_1 = BQ_{-1}. \tag{3.1}$$

By virtue of (2.7) and (2.13) we see that P_0 and P_1 are complementary projections, as are M_0 and M_1. That is, (3.2) $$P_0 + P_1 = I, \qquad P_i^2 = P_i, \qquad i = 0, 1,$$

(3.3) $$M_0 + M_1 = I, \qquad M_i^2 = M_i, \qquad i = 0, 1.$$

Then one also has $P_iP_j = M_iM_j = 0$ if $i \neq j$. We will make frequent use of these and other relations obtained from those in §2; for example, $P_0Q_0B = Q_0BP_0$ and $P_1Q_{-1}A = Q_{-1}AP_1$, by (2.8), and $P_0Q_0 = Q_0$ by (2.13).

LEMMA 3.1. *The vector function x is a solution of the regular system* (1.1) *if and only if*

(3.4) $$AP_0\dot{x} + BP_0x = M_0f$$

and

(3.5) $$AP_1\dot{x} + BP_1x = M_1f.$$

PROOF. Equations (3.4) and (3.5) imply (1.1) by virtue of (3.2) and (3.3). Conversely, if x satisfies (1.1), then

(3.6) $$M_0A\dot{x} + M_0Bx = M_0f.$$

But $M_0A = AP_0$ and $M_0B = AQ_0B = BQ_0A = BP_0$ by (2.8); hence, (3.6) is (3.4). Similarly, multiplication of (1.1) by M_1 gives (3.5).

LEMMA 3.2. *Suppose* (1.1) *is regular and f is continuous. Then* (3.4) *holds for differentiable x if and only if*

(3.7) $$P_0x(t) = e^{-Q_0Bt}P_0x(0) + \int_0^t e^{-Q_0B(t-\tau)}Q_0f(\tau)d\tau.$$

PROOF. Equation (3.7) implies

(3.8) $$P_0\dot{x} + Q_0BP_0x = Q_0f.$$

Multiplication of this by A gives (3.4) since $AQ_0BP_0 = BQ_0AP_0 = BP_0^2 = BP_0$. Conversely, multiplication of (3.4) by Q_0 gives (3.8) since $Q_0M_0 = Q_0$ by (2.13), and $Q_0AP_0 = P_0^2 = P_0$. But (3.8) says that $u = P_0x$ is a solution of $\dot{u} + Q_0Bu = Q_0f$ for which (3.7) is the familiar variation of parameters formula.

LEMMA 3.3. *Suppose* (1.1) *is regular, A is singular and the pole of $(A + zB)^{-1}$ at $z = 0$ has order $\mu\,(\geqq 1)$. If f has continuous derivatives through order μ, then* (3.5) *holds for differentiable x if and only if*

(3.9) $$P_1x = \sum_{k=0}^{\mu-1}(-Q_{-1}A)^kQ_{-1}f^{(k)} = \sum_{k=0}^{\mu-1}Q_{-1}(-AQ_{-1})^kf^{(k)}.$$

PROOF. Equation (3.9) implies

$$AP_1\dot{x} + BP_1x = \sum_{k=0}^{\mu-1}\{-(-AQ_{-1})^{k+1}f^{(k+1)} + BQ_{-1}(-AQ_{-1})^k f^{(k)}\}. \tag{3.10}$$

But $BQ_{-1} = I - AQ_0$, $Q_0AQ_{-1} = 0$ and $(AQ_{-1})^\mu = 0$ by (2.7), (2.10) and (2.16). Applying these in (3.10) and recalling $M_1 = BQ_{-1}$, we obtain (3.5). Conversely, if (3.5) holds, we multiply by Q_{-1}, use (3.1), (3.2) and (2.13) and obtain

$$P_1x = Q_{-1}f - Q_{-1}AP_1\dot{x}. \tag{3.11}$$

Proceeding inductively, we may use (3.11) to establish

$$P_1x = \sum_{k=0}^{j-1}(-Q_{-1}A)^kQ_{-1}f^{(k)} + (-Q_{-1}A)^jP_1x^{(j)} \tag{3.12}$$

for $j = 1, \ldots, \mu$. For $j = \mu$ this is (3.9), since $(Q_{-1}A)^\mu = 0$ by virtue of Corollary 2.1.

REMARK 3.1. If A is non-singular, then the conclusion of Lemma 3.3 is trivially valid since then $Q_{-1} = P_1 = M_1 = 0$.

THEOREM 3.1. *Under the hypotheses of Lemma* 3.3 *the initial value problem* (1.4) *has a solution if and only if*

$$P_1\zeta = \sum_{k=0}^{\mu-1}(-Q_{-1}A)^kQ_{-1}f^{(k)}(0). \tag{3.13}$$

If a solution exists, it is unique and is given by

$$x(t) = u(t) + v(t) \tag{3.14}$$

where

$$u(t) = e^{-Q_0Bt}P_0\zeta + \int_0^t e^{-Q_0B(t-\tau)}Q_0f(\tau)d\tau \tag{3.15}$$

$$v(t) = \sum_{k=0}^{\mu-1}(-Q_{-1}A)^kQ_{-1}f^{(k)}(t). \tag{3.16}$$

PROOF. If x is a solution of (1.4), then, since $P_0 + P_1 = I$, we have $x(t) = P_0x(t) + P_1x(t) = u(t) + v(t)$ where, by Lemmas 3.1–3.3, $u(t) = P_0x(t)$ and $v(t) = P_1x(t)$ are given in (3.15) and (3.16). Then $P_1\zeta = P_1x(0) = v(0)$ which is (3.13). Conversely, suppose ζ satisfies (3.13) and we define $x(t)$ as in (3.14)–(3.16). Then $P_1\zeta = v(0)$ and $x(0) = u(0) + v(0) = P_0\zeta + P_1\zeta = \zeta$. Since P_0 commutes with Q_0B and $P_0Q_0 = Q_0$ and $P_0Q_{-1} = 0$, then $P_0v(t) = 0$ and $P_0x(t) = P_0u(t) = u(t)$. Similarly P_1 commutes with Q_0B, $P_1P_0 = 0$, $P_1Q_0 = Q_{-1}BQ_0 = -Q_{-1}AQ_1 = 0$ and

$P_1Q_{-1} = Q_{-1}$ so $P_1x(t) = P_1v(t) = v(t)$. It follows by Lemmas 3.1–3.3 that $A\dot{x} + Bx = f$, and since $x(0) = \zeta$, then x is a solution of (1.4).

THEOREM 3.2. *Under the hypotheses of Lemma* 3.3 *the initial value problem* (1.5) *has a solution if and only if*

$$\zeta = P_0\omega, \text{ some } \omega \in \mathbf{C}^n, \tag{3.17}$$

or, equivalently, if and only if

$$P_0\zeta = \zeta. \tag{3.18}$$

In this case the unique solution is given by

$$x(t) = e^{-Q_0Bt}\zeta. \tag{3.19}$$

PROOF. Condition (3.13) when $f(t) \equiv 0$ becomes $P_1\zeta = 0$ or $P_0\zeta = \zeta$ since $P_1 = I - P_0$. Conversely (3.18) implies $P_1\zeta = 0$ since $P_1P_0 = 0$. Relations (3.17) and (3.18) are equivalent since $P_0^2 = P_0$. When $f(t) \equiv 0$ equation (3.14) becomes (3.19) in view of (3.18).

It follows from Theorem 3.2 that the dimension of the solution space of $A\dot{x} + Bx = 0$ equals the rank of P_0. We show later in this section that it also equals the degree of $\det(sA + B)$. Before doing this, however, we shall discuss the relationship between the preceding results and those of Rose in [9] and of Campbell, Meyer and Rose in [1]. In particular we show that our Theorem 3.1 gives the solutions of (1.4) contained in Theorems 5 and 7 in [1].

We note first that in [9] Rose gives a representation of the Laurent expansion (2.2) for $(sA + B)^{-1}$ on $|s| > 1/\delta$. From equation (12) in [9] it follows that

$$Q_0 = A_c^D(cA + B)^{-1} \tag{3.20}$$

$$Q_{-1} = (I - A_cA_c^D)B_c^D(cA + B)^{-1}. \tag{3.21}$$

(Incidentally, formula (12) in [9] is misprinted; the term involving z^k there should be summed on k from 0 to $\nu - 1$.) We note also that A_c, B_c, A_c^D and B_c^D all commute. From (3.20) and (3.21) we get

$$P_0 = Q_0A = A_c^DA_c = A_cA_c^D \tag{3.22}$$

$$P_1 = Q_{-1}B = (I - A_cA_c^D)B_c^DB_c = P_1B_c^DB_c \tag{3.23}$$

since $I - P_0 = P_1$. Similarly,

$$Q_0B = A_c^DB_c \tag{3.24}$$

$$Q_{-1}A = P_1B_c^DA_c. \tag{3.25}$$

From (3.25) we obtain for all $k \geqq 1$

$$(3.26)\qquad \begin{aligned}(Q_{-1}A)^k &= P_1^k(B_c^DA_c)^k = P_1(B_c^DA_c)^k\\ &= (I - A_cA_c^D)(B_c^DA_c)^k.\end{aligned}$$

If one defines $f_c = (cA + B)^{-1}f$, then

$$(3.27)\qquad f = (cA + B)f_c$$

and by (3.20) and (3.21) one gets

$$(3.28)\qquad Q_0f = A_c^Df_c$$

$$(3.29)\qquad Q_{-1}f = (I - A_cA_c^D)B_c^Df_c.$$

Substituting from (3.22), (3.24), (3.26), (3.28) and (3.29) into (3.13), (3.15) and (3.16), one finds that our solution formula (3.14) and condition (3.13) are transformed into essentially the form in which they appear in Theorem 7 of **[1]**.

In Theorem 5 of **[1]** it was assumed that $AB = BA$ and that $\mathscr{N}(A) \cap \mathscr{N}(B) = \{0\}$ where $\mathscr{N}(A)$ and $\mathscr{N}(B)$ are the null spaces of A and B. These conditions in fact imply that $A\dot{x} + Bx = f$ is regular. (It is easy to see that regularity implies $\mathscr{N}(A) \cap \mathscr{N}(B) = \{0\}$.) Indeed, under these conditions it was shown in the proof of Lemma 1 of **[1]** that for some non-singular T one has

$$(3.30)\qquad A = T\begin{bmatrix} J & 0\\ 0 & N\end{bmatrix}T^{-1}, \qquad B = T\begin{bmatrix} B_1 & 0\\ 0 & B_4\end{bmatrix}T^{-1}$$

where J and B_4 are non-singular, $JB_1 = B_1J$, $NB_4 = B_4N$, and N is nilpotent (or absent if A is non-singular). But then $J + zB_1$ is invertible for z near zero and $N + zB_4 = zB_4(I + z^{-1}B_4^{-1}N)$ is also invertible for z near zero since $B_4^{-1}N$ is nilpotent. Thus $A + zB$ is invertible near zero so $\det(sA + B) \not\equiv 0$. One can use the representations in (3.30) to compute the Laurent series for $(A + zB)^{-1}$ on $0 < |z| < \delta$. When $AB = BA$ and $\mathscr{N}(A) \cap \mathscr{N}(B) = \{0\}$, then the hypotheses in Theorem 2.1 of **[9]** hold and we infer from the expansion given there by Rose in equation (5) that

$$(3.31)\qquad Q_0 = A^D, \qquad Q_{-1} = B^D(I - AA^D).$$

These may now be substituted in (3.13), (3.15) and (3.16) to give the formulas of Theorem 5 and Corollary 2 in **[1]**. We note, however, that when A and B do not commute but $\det(sA + B) \not\equiv 0$, then, in contrast to (3.31), our Q_0 need not be the Drazin inverse of A.

From relations (3.20), (3.22), and (3.24) we see immediately that $A_c^D(cA + B)^{-1}$, $A_c^DA_c$ and $A_c^DB_c$, respectively, are independent of c for those c for which $(cA + B)^{-1}$ exists. This independence was pointed out for these expressions as well as for $B_c^D(cA + B)^{-1}$ and $A_cB_c^D$ in Theorem 8 in **[1]**. For these last two it does not seem to follow directly from our representation. However, as noted in Corollary 2.2 of [9], the index of A_c is our parameter μ which is determined by the behavior of $(A + zB)^{-1}$ at $z = 0$ and hence not dependent on c.

Finally, it is of interest to observe from (3.20) that A_c^D is a linear polynomial matrix in c;

$$A_c^D = Q_0(cA + B) = cQ_0A + Q_0B. \tag{3.32}$$

Hence $\lim_{c\to 0} A_c^D = Q_0B$ and $\lim_{c\to\infty}(1/c)A_c^D = Q_0A = P_0$ which limits were given in equation (20) of [1] in the notation there.

We turn now to a characterization of the dimension of the solution space of a regular system $A\dot{x} + Bx = 0$. In the next section we describe a basis for this space as an alternative to the description in Theorem 3.2.

THEOREM 3.3. *If* $A\dot{x} + Bx = 0$ *is regular and* A *is singular, then the dimension of the solution space, the rank of* P_0 *and the degree of* $\Delta(s) = \det(sA + B)$ *are all equal.*

PROOF. Suppose the rank of P_0 is r. That the dimension of the solution space of $A\dot{x} + Bx = 0$ is r is clear from Theorem 3.2. Indeed, suppose X is $n \times r$ and its columns are a basis for the column space of P_0. Then the columns of $\exp(-Q_0Bt)X$ are a basis for the solution space of $A\dot{x} + Bx = 0$. Now let Y be $n \times (n - r)$ and let its columns be a basis for the null space of P_0. Then, since P_0 and P_1 are complementary projections, we have

$$P_0X = X, \qquad P_0Y = 0, \qquad P_1X = 0, \qquad P_1Y = Y \tag{3.33}$$

and the $n \times n$ matrix

$$T = [X, Y] \tag{3.34}$$

is non-singular. Let

$$T^{-1} = \begin{bmatrix} U \\ V \end{bmatrix} \tag{3.35}$$

where U is $r \times n$ and V is $(n - r) \times n$. From (3.33), (3.34) and (3.35) we get

$$P_0 = P_0[X, Y]T^{-1} = [X, 0]T^{-1} = XU \tag{3.36}$$

and similarly

$$P_1 = YV. \tag{3.37}$$

Moreover, the partitioning of T^{-1} in (3.35) implies $UX = I_r$ and $VY = I_{n-r}$ where I_k denotes the $k \times k$ identity. Now, applying (2.13) and (2.8), we get

$$Q_{-1}A = P_1^2 Q_{-1}A = P_1 Q_{-1} A P_1 = YVQ_{-1}AYV \tag{3.38}$$

$$Q_0 B = P_0^2 Q_0 B = P_0 Q_0 B P_0 = XUQ_0 BXU. \tag{3.39}$$

If we define

$$\alpha = -VQ_{-1}AY, \qquad \beta = -UQ_0 BX, \tag{3.40}$$

then $Q_0 A + Q_{-1}A = XU - Y\alpha V$ and $Q_0 B + Q_{-1}B = YV - X\beta U$ from which it follows that

$$(Q_0 + Q_{-1})(sA + B) = [X, Y]\begin{bmatrix} sI_r - \beta & 0 \\ 0 & I_{n-r} - s\alpha \end{bmatrix}\begin{bmatrix} U \\ V \end{bmatrix}. \tag{3.41}$$

Since $VY = I_{n-r}$, then $(-Q_{-1}A)^k = Y\alpha^k V$, $k \geqq 1$, by (3.38) and (3.40). Hence $\alpha^k = V(-Q_{-1}A)^k Y$ so α is nilpotent by (2.16). It follows that $\det(I_{n-r} - s\alpha) \equiv 1$. By (3.41) then

$$\det(Q_0 + Q_{-1})\det(sA + B) = \det(sI_r - \beta);$$

that is

$$\Delta(s) = \frac{1}{c}\det(sI_r - \beta) \tag{3.42}$$

for some nonzero constant c and the degree of $\Delta(s)$ is therefore r, the rank of P_0.

Remark 3.2. When $A\dot{x} + Bx = 0$ is regular and A is non-singular, the conclusion of the theorem is trivially true since then $P_0 = I_n$ and $\Delta(s) = \det A \det(sI_n + A^{-1}B)$. We note also that the above theorem is part of Corollary 3.3 in Wong [**10**] where the relation (3.41) is developed, in effect, in somewhat less explicit form. Finally, it should be pointed out that the matrix involving s on the right in (3.41) is the canonical form established by Weierstrass for the regular pencil of matrices $As + B$ (see [**4**, pp. 25–28, Vol. II]).

4. In this section we relate the projections P_0 and P_1 to the eigenvalues and generalized eigenspaces for the problem $(A\lambda + B)\xi = 0$. These concepts have been exploited earlier by Lancaster (see [**5**], for example) in connection with the equation $(A_q D^q + \cdots + A_1 D + A_0)y = g$ $(D = d/dt)$ for which the relevant polynomial matrix is $C(\lambda) = A_q\lambda^q + \cdots + A_1\lambda + A_0$, $\lambda \in \mathbf{C}$. Here we use results from §3 to establish the basic facts regarding eigenvalues and Jordan chains for the simpler form $A\lambda + B$. These provide an alternate description of a basis for the solutions of $A\dot{x} + Bx = 0$. We then derive the corresponding facts about $C(\lambda)$ from those for the simpler form.

Definition 4.1. Given $n \times n$ matrices A and B such that $\Delta(s) =$

$\det(As + B) \not\equiv 0$, the number $\lambda \in \mathbf{C}$ is an eigenvalue for $As + B$ of multiplicity μ if λ is a zero of $\Delta(s)$ of multiplicity μ. The vectors $\xi^1, \ldots, \xi^\sigma$ in $\mathbf{C}^n$ form a Jordan chain of length σ corresponding to λ if $\xi^1 \neq 0$,

$$(A\lambda + B)\xi^1 = 0 \tag{4.1}$$

and

$$(A\lambda + B)\xi^p + A\xi^{p-1} = 0, \qquad p = 2, \ldots, \sigma. \tag{4.2}$$

They form a Jordan chain corresponding to ∞ for $As + B$ if they form a Jordan chain corresponding to the eigenvalue zero for $A + Bs$; that is, if $\xi^1 \neq 0$,

$$A\xi^1 = 0 \tag{4.3}$$

and

$$A\xi^p + B\xi^{p-1} = 0, \qquad p = 2, \ldots, \sigma. \tag{4.4}$$

THEOREM 4.1. *Suppose $\Delta(s) = \det(As + B) \not\equiv 0$ and that $\lambda_1, \ldots, \lambda_m$ are the distinct zeros of $\Delta(s)$ with multiplicities $\mu_1, \ldots, \mu_m$, respectively. Then for each λ_k there is a set of Jordan chains for $As + B$ corresponding to λ_k such that*

(a) *the sum of the lengths of the chains corresponding to λ_k is μ_k, $k = 1, \ldots, m$;*

(b) *the collection of all these chains for all λ_k is a linearly independent set of r elements where r is the degree of $\Delta(s)$; and*

(c) *the collection of all these chains is a basis for the range of P_0.*

PROOF. Using the notation in the proof of Theorem 3.3, we get from (3.41) that

$$A(Q_0 + Q_{-1})(As + B)X = AX(sI_r - \beta).$$

Since $X = P_0X = Q_0AX$ and $Q_{-1}AQ_0 = 0$, this may be written as

$$(As + B)X = AX(sI - \beta). \tag{4.5}$$

Now we note from (3.42) that an eigenvalue for $As + B$ is an eigenvalue of β of the same multiplicity. Thus if λ is an eigenvalue for $sA + B$ and $u^1, \ldots, u^\sigma \in \mathbf{C}^r$ a corresponding Jordan chain for β, we have

$$(\beta - \lambda I)u^1 = 0, \qquad u^1 \neq 0, \tag{4.6}$$

$$(\beta - \lambda I)u^p = u^{p-1}, \qquad p = 2, \ldots, \sigma. \tag{4.7}$$

Define $\xi^1, \ldots, \xi^\sigma \in \mathbf{C}^n$ by

$$\xi^p = Xu^p, \qquad p = 1, \ldots, \sigma. \tag{4.8}$$

Then from (4.5)–(4.8) we get $(A\lambda + B)\xi^1 = AX(\lambda I - \beta)u^1 = 0$ and

$(A\lambda + B)\xi^p = AX(\lambda I - \beta)u^p = -A\xi^{p-1}$, $p = 2, \ldots, \sigma$. Hence $\xi^1, \ldots, \xi^\sigma$ is a Jordan chain for $As + B$ corresponding to λ. Now for each λ_k, $k = 1, \ldots, m$, there are Jordan chains $u^p(\lambda_k, h)$, $p = 1, \ldots, \sigma_k(h)$, $h = 1, \ldots, \tau_k$ such that $\sigma_k(1) + \cdots + \sigma_k(\tau_k) = \mu_k$ which constitute a basis for the generalized eigenspace of β corresponding to λ_k. Moreover, the set of r vectors made up of all these chains (for all λ_k) forms a basis for $\mathbf{C}^r$. The vectors $\xi^p(\lambda_k, h) = Xu^p(\lambda_k, h)$ form Jordan chains for $As + B$ having property a). The total collection also has property b) since the columns of X, being a basis for the range of P_0, are linearly independent. Also c) is valid since the r vectors $\xi^p(\lambda_k, h)$ are in the range of P_0 which has dimension r.

When $r < n$, then the range of P_1 is determined by the Jordan chains of $As + B$ corresponding to ∞.

THEOREM 4.2. *Suppose $\Delta(s) = \det(As + B) \not\equiv 0$ and $r = \deg \Delta(s) < n$. Then there is a set of Jordan chains for $As + B$ corresponding to ∞ which forms a basis for the range of P_1.*

PROOF. From (3.34) an-(3.41) we get

$$B(Q_0 + Q_{-1})(As + B)Y = BY(I_{n-r} - s\alpha).$$

Since $Y = P_1Y = Q_{-1}BY$, $Q_0AQ_{-1} = 0$ and $Q_0BQ_{-1} = -Q_1AQ_{-1} = 0$, this may be written as $(As + B)Y = BY(I - s\alpha)$. Hence

$$AY = -BY\alpha. \tag{4.9}$$

All eigenvalues of α are zero since α is nilpotent. If $v^1, \ldots, v^\sigma \in \mathbf{C}^{n-r}$ is a Jordan chain for α, then

$$\alpha v^1 = 0, \qquad v^1 \neq 0 \tag{4.10}$$

and

$$\alpha v^p = v^{p-1}, \qquad p = 2, \ldots, \sigma. \tag{4.11}$$

Define $\xi^1, \ldots, \xi^\sigma \in \mathbf{C}^n$ by

$$\xi^p = Yv^p, \qquad p = 1, \ldots, \sigma. \tag{4.12}$$

Then from (4.9)–(4.12) we get $A\xi^1 = -BY\alpha v^1 = 0$ and $A\xi^p = -BY\alpha v^p = -B\xi^{p-1}$, $p = 2, \ldots, \sigma$. Thus $\xi^1, \ldots, \xi^\sigma$ is a Jordan chain for $As + B$ corresponding to ∞. Now there is a set of Jordan chains $v^p(h) \in \mathbf{C}^{n-r}$, $p = 1, \ldots, \sigma(h)$, $h = 1, \ldots, \tau$, for α which forms a basis for $\mathbf{C}^{n-r}$. The corresponding set of Jordan chains $\xi^p(h) = Yv^{p(h)} \in \mathbf{C}^{n-r}$, $p = 1, \ldots, \sigma(h)$, $h = 1, \ldots, \tau$, for $As + B$ corresponding to ∞ has the property claimed inasmuch as the columns of Y are a basis for the range of P_1.

Theorems 4.1 and 4.2 assert that the ranges of P_0 and P_1 are spanned by certain Jordan chains for $As + B$. The next theorem asserts that any

Jordan chain for $As + B$ must be in one of these spaces; thus the generalized eigenspaces for $As + B$ corresponding to the finite eigenvalues and to ∞ give the direct sum decomposition of $\mathbf{C}^n$ effected by the projections P_0 and P_1.

THEOREM 4.3. *If $\xi^1, \ldots, \xi^\sigma$ is a Jordan chain for $As + B$ corresponding to the eigenvalue $\lambda \in \mathbf{C}$* (to ∞), *then $P_0\xi^i = \xi^i (P_1\xi^i = \xi^i)$, $i = 1, \ldots, \sigma$.*

PROOF. Suppose first that (4.1) and (4.2) hold. Multiplication of these by $Q_{-1} = Q_{-1}BQ_{-1}$ yields

$$P_1\xi^1 = -\lambda(Q_{-1}A)P_1\xi^1 \tag{4.13}$$

$$P_1\xi^p = -\lambda(Q_{-1}A)P_1\xi^p - Q_{-1}AP_1\xi^{p-1}, \qquad p = 2, \ldots, \sigma. \tag{4.14}$$

Repeated substitution into the right side of (4.13) gives $P_1\xi^1 = 0$ since $(Q_{-1}A)^\mu = 0$. Proceeding inductively using (4.14), we conclude that $P_1\xi^i = 0$, $i = 1, \ldots, \sigma$, so $P_0\xi^i = \xi^i$ as claimed. In the alternative case equations (4.3) and (4.4) hold. Multiplication of these by $Q_0 = Q_0AQ_0$ gives $P_0\xi^1 = 0$, $P_0\xi^p = -Q_0BP_0\xi^{p-1}$, $p = 2, \ldots, \sigma$. It follows inductively that $P_0\xi^i = 0$, $i = 1, \ldots, \sigma$, whence $P_1\xi^i = \xi^i$ as claimed.

REMARK 4.1. Theorems 4.1–4.3 generalize some of the relations in Lemma 2.1 of Wong [**10**] for the case $m = n$. The matrices T and S in Wong's notation are our A and B; his space H_1 is the range of our P_0; his P and U are our P_0 and β, respectively.

The linear independence property stated in b) of Theorem 4.1 follows from a seemingly weaker property. In the first place, the Jordan chains $u^p(\lambda_k, h)$, $p = 1, \ldots, \sigma_k(h)$ corresponding to different eigenvalues are necessarily linearly independent. Secondly, for each k the Jordan chains $u^p(\lambda_k, h)$, $p = 1, \ldots, \sigma_k(h)$, $h = 1, \ldots, \tau_k$, form a linearly independent set provided the initial vectors $u^1(\lambda_k, h)$, $h = 1, \ldots, \tau_k$, are linearly independent. These properties are inherited by the Jordan chains $\xi^p(\lambda_k, h)$, $p = 1, \ldots, \sigma_k(h)$, for $As + B$ corresponding to the finite eigenvalues λ_k and a like property holds for those chains corresponding to ∞. Accordingly, we introduce the following definition.

DEFINITION 4.2. Suppose $\Delta(s) = \det(As + B) \not\equiv 0$ and that $\lambda_1, \ldots, \lambda_m \in \mathbf{C}$ are the distinct eigenvalues of $As + B$ with multiplicities $\mu_1, \ldots, \mu_m$. The Jordan chains $\xi^i(\lambda_k, h)$, $i = 1, \ldots, \sigma_k(h)$, $h = 1, \ldots, \tau_k$, for $As + B$ corresponding to λ_k, $k = 1, \ldots, m$, will be called a complete set of finite-value chains for $As + B$ if property a) of Theorem 4.1 holds (that is, $\sigma_k(1) + \cdots + \sigma_k(\tau_k) = \mu_k$, $k = 1, \ldots, m$) and for each $k = 1, \ldots, m$, the initial vectors $\xi^1(\lambda_k, h)$, $h = 1, \ldots, \tau_k$, are linearly independent. The Jordan chains $\xi^i(h)$, $i = 1, \ldots, \sigma(h)$, $h = 1, \ldots, \tau$, for $As + B$ corresponding to ∞ will be called a complete set of ∞-value chains for $As + B$

if $\sigma(1) + \cdots + \sigma(\tau) = n - r$ where $r = \mu_1 + \cdots + \mu_m$ and $\xi^1(1), \ldots, \xi^1(\tau)$ are linearly independent.

Theorem 4.1 asserts the existence of a complete set of finite-value Jordan chains for $As + B$ which form a basis for the range of P_0. Theorem 4.2 asserts the existence of a complete set of ∞-value chains which form a basis for the range of P_1. From such complete sets one can construct P_0 and P_1. Indeed, each vector in $\mathbf{C}^n$ is the sum of a unique vector in the range of P_0 and a unique vector in the range of P_1. Since $P_0 + P_1 = I$, for any j the j-th columns of P_0 and P_1 are determined uniquely by this decomposition of the j-th column of I. From a complete collection of finite-value Jordan chains for $As + B$ one also obtains a basis for the solution space of $A\dot{x} + Bx = 0$.

THEOREM 4.4. *Suppose* $\Delta(s) = \det(As + B) \not\equiv 0$ *and that*

(4.15) $$\xi^i(\lambda_k, h),\ i = 1, \ldots, \sigma_k(h),\ h = 1, \ldots, \tau_k,\ k = 1, \ldots, m,$$

is a complete set of finite-value Jordan chains for $As + B$. *Then the functions of* t

(4.16) $$x(t; i, k, h) = \sum_{p=0}^{i-1} \frac{1}{p!} t^p e^{\lambda_k t} \xi^{i-p}(\lambda_k, h)$$

with i, k, h *ranging as in* (4.15) *form a basis for the solution space of* $A\dot{x} + Bx = 0$.

PROOF. Using (4.1) and (4.2), one can readily verify that $x(t; i, k, h)$ is a solution of $A\dot{x} + Bx = 0$. There are $r = \deg \Delta(s)$ solutions given in (4.16) and the dimension of the solution space of $A\dot{x} + Bx = 0$ is r. Note that $x(0; i, k, h) = \xi^i(\lambda_k, h)$, so any $\zeta = P_0\omega$ is a linear combination of the vectors $x(0; i, k, h)$. The same linear combination of the solutions $x(t; i, k, h)$ gives the solution $x(t)$ such that $x(0) = \zeta$. Thus the set of solutions in (4.16) is a spanning set and hence a basis for the solution space of $A\dot{x} + Bx = 0$.

We now indicate how Theorems 4.1 and 4.4 can be used to analyze the solution space of

(4.17) $$(A_q D^q + \cdots + A_1 D + A_0)y = g \qquad (D = d/dt)$$

in which the A_j are $n \times n$ matrices over $\mathbf{C}$ and y and g are $\mathbf{C}^n$-valued functions. (An interesting application of systems (4.17) is given in [2].) The results we obtain in the homogeneous case, $g = 0$, are the same as given in [5] where it was assumed that A_q is non-singular. The more general case was treated in [6] using the Smith normal form of the polynomial matrix

(4.18) $$C(s) = A_q s^q + \cdots + A_1 s + A_0, \qquad s \in \mathbf{C}.$$

We give an alternative treatment here.

We will say the system (4.17) is *regular* if $\Delta(s) = \det C(s) \not\equiv 0$. The system (4.17) is equivalent to the first order system

$$Ax' + Bx = f \tag{4.19}$$

in which x and f are the $nq \times 1$ vector functions

$$x = \begin{bmatrix} y \\ Dy \\ \vdots \\ D^{q-1}y \end{bmatrix}, \quad f = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g \end{bmatrix} \tag{4.20}$$

and A and B are the $nq \times nq$ matrices

$$A = \begin{bmatrix} I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \\ 0 & \cdots & 0 & A_q \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -I \\ A_0 & A_1 & \cdots & A_{q-1} \end{bmatrix}. \tag{4.21}$$

Note that $\det A = \det A_q$, so A is singular if and only if A_q is singular.

From the form of A and B in (4.21) one can verify that

$$As + B = \begin{bmatrix} 0 & -I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -I \\ C_0(s) & C_1(s) & \cdots & C_{q-1}(s) \end{bmatrix} \begin{bmatrix} I & 0 & \cdots & 0 & 0 \\ -sI & I & \cdots & 0 & 0 \\ 0 & -sI & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -sI & I \end{bmatrix} \tag{4.22}$$

where $C_k(s) = A_k + A_{k+1}s + \cdots + A_q s^{q-k}$, $k = 0, 1, \ldots, q-1$. Noting that $C_0(s) = C(s)$, we easily get from (4.22) that

$$\det(As + B) = \det C(s). \tag{4.23}$$

Hence (4.17) is regular if and only if (4.19) is regular.

The dimension of the solution space of $A\dot{x} + Bx = 0$ is the degree of $\det(As + B)$ by Theorem 3.3. Since the relation (4.20) sets up a one-to-one linear correspondence between the solution spaces of (4.17) and (4.19), it is clear from (4.23) that the dimension of the solution space of (4.17) in the homogeneous case, $g = 0$, equals the degree of $\det C(s)$. This fact is known as Chrystal's Theorem (see, for example, [3] or p. 327 of [6]).

A basis for the solution space of (4.17) when $g = 0$ can be obtained from the Jordan chains for $C(s)$.

DEFINITION 4.3. Given the $n \times n$ matrices in (4.18) suppose that det $C(s) \not\equiv 0$. Let $\lambda_1, \ldots, \lambda_m \in \mathbf{C}$ be the distinct zeros of det $C(s)$ with multiplicities $\mu_1, \ldots, \mu_m$, respectively. The vectors $\eta^i(\lambda_k, h)$, $i = 1, \ldots, \sigma_k(h)$, form a Jordan chain for $C(s)$ corresponding to λ_k if $\eta^1(\lambda_k, h) \neq 0$ and

$$(4.24) \qquad \sum_{i=0}^{p-1} \frac{1}{i!} C^{(i)}(\lambda_k)\eta^{p-i}(\lambda_k, h) = 0, \qquad p = 1, \ldots, \sigma_k(h),$$

where $C^{(i)}(s)$ is the i-th derivative of $C(s)$. A collection of such chains, $h = 1, \ldots, \tau_k$, $k = 1, \ldots, m$, is complete if

a) for each k, the vectors $\eta^1(\lambda_k, h)$, $h = 1, \ldots, \tau_k$, are linearly independent, and

b) for each $k = 1, \ldots, m$,

$$(4.25) \qquad \sigma_k(1) + \cdots + \sigma_k(\lambda_k) = \mu_k.$$

We sketch now how the existence of such a complete set follows from our Theorem 4.1. First note that (4.1) and (4.2) can be combined to a single relation

$$(4.26) \qquad (A\lambda + B)\xi^p + A\xi^{p-1} = 0, \qquad p \leqq \sigma,$$

with the requirement $\xi^1 \neq 0$ and the convention $\xi^i = 0$ for $i \leqq 0$. In the present context with A and B as in (4.21) we write $\xi^p \in \mathbf{C}^{nq}$ in the form

$$(4.27) \qquad \xi^p = \begin{bmatrix} \xi_1^p \\ \vdots \\ \xi_q^p \end{bmatrix}$$

with each $\xi_j^p \in \mathbf{C}^n$. The equation (4.26) is then equivalent to

$$(4.28) \qquad \xi_{j+1}^p = \lambda\xi_j^p + \xi_j^{p-1}, \qquad j = 1, \ldots, h-1,$$

and

$$(4.29) \qquad \sum_{j=0}^{q-1} A_j\xi_{j+1}^p + A_h(\lambda\xi_h^p + \xi_h^{p-1}) = 0.$$

From (4.28) it follows inductively that

$$(4.30) \qquad \xi_{j+1}^p = \sum_{i=0}^{j} \binom{j}{i}\lambda^{j-i}\xi_1^{p-i}, \qquad j = 0, 1, \ldots, q-1.$$

Using this with $j = q - 1$, we also obtain

$$(4.31) \qquad \lambda\xi_q^p + \xi_q^{p-1} = \sum_{i=0}^{q} \binom{h}{i}\lambda^{q-i}\xi_1^{p-1}.$$

With (4.30) and (4.31) we see that (4.29) is then equivalent to

$$\sum_{j=0}^{q}\sum_{i=0}^{j} \binom{j}{i}\lambda^{j-i}A_j\xi_1^{p-i} = 0$$

which in turn may be written as

$$\sum_{i=0}^{q} \frac{1}{i!} C^i(\lambda)\xi_1^{p-i} = 0.$$

This is the form (4.24) inasmuch as $\xi^i = 0$ for $i \leqq 0$.

We see then that a Jordan chain $\xi^1, \ldots, \xi^\sigma \in \mathbf{C}^{nq}$ for $As + B$ corresponding to λ determines a Jordan chain $\eta^p = \xi_1^p \in \mathbf{C}^n$, $p = 1, \ldots, \sigma$, for $C(s)$ corresponding to λ. From (4.30) we have $\xi_{j+1}^1 = \lambda^j \xi_1^1$. It follows that if $\xi^p(\lambda, h)$, $p = 1, \ldots, \sigma(h)$, $h = 1, \ldots, \tau$, is a set of Jordan chains for $As + B$ corresponding to λ such that $\xi^1(\lambda, 1), \ldots, \xi^1(\lambda, \tau)$ are linearly independent, then for the related Jordan chains for $C(s)$ given by $\eta^p(\lambda, h) = \xi_1^p(\lambda, h)$ the initial vectors $\eta^1(\lambda, 1), \ldots, \eta^1(\lambda, \tau)$ are linearly independent.

Using the relation (4.20) and the result in Theorem 4.4, we easily get the following theorem.

THEOREM 4.5. *Suppose* det $C(s) \not\equiv 0$ *and that* $\eta^i(\lambda_k, h)$, $i = 1, \ldots, \sigma_k(h)$, $h = 1, \ldots, \tau_k$, $k = 1, \ldots, m$, *is a complete set of Jordan chains for* $C(s)$. *Then the functions of t given by*

$$y(t; i, k, h) = \sum_{p=0}^{i-1} \frac{1}{p!} t^p e^{\lambda_k t} \eta^{i-p}(\lambda_k, h), \tag{4.32}$$

with i, k, h ranging as above, form a basis for the solution space of (4.17) *when* $g = 0$.

5. As noted in §1, if (1.1) is regular and the Laplace transforms $\hat{x}$ and $\hat{f}$ exist and $e^{-st}x(t) \to 0$ as $t \to \infty$ for sufficiently large Re(s), then

$$\hat{x}(s) = (sA + B)^{-1}(\hat{f}(s) + A\zeta) \tag{5.1}$$

with $\zeta = x(0)$. In (2.2)–(2.4) we gave a representation for $(sA + B)^{-1}$. If we substitute for Q_{-k} and Q_k using (2.14) and (2.15), we find

$$(sA + B)^{-1} = (I + sQ_{-1}A)^{-1}Q_{-1} + (sI + Q_0B)^{-1}Q_0 \tag{5.2}$$

since $Q_{-1}A$ is nilpotent by (2.16). The representation (5.2) is valid for $|s| > 1/\delta$ and hence, by analytic continuation, for all finite s except where $sI + Q_0B$ is singular. Substituting from (5.2) into (5.1), we find that

$$\begin{aligned} \hat{x}(s) &= (I + sQ_{-1}A)^{-1}Q_{-1}(\hat{f}(s) + A\zeta) \\ &\quad + (sI + Q_0B)^{-1}Q_0(\hat{f}(s) + A\zeta). \end{aligned} \tag{5.3}$$

Regardless of ζ, the second term on the right in (5.3) is the Laplace transform of $u(t)$ as defined in (3.15). The first term must then be the transform of $v(t)$ as defined in (3.16).

Let us now adopt the point of view in [9] and ask the question: For what vectors ζ is the right hand side of (5.3) the Laplace transform of a function

x, and is such a function a solution of (1.1)? It is clear that not all ζ are permissible inasmuch as the expression $(I + sQ_{-1}A)^{-1}Q_{-1}A\zeta$ (corresponding to the homogeneous case $f(t) \equiv 0$) is a polynomial and would not be the Laplace transform of a function unless it were identically zero.

We now impose the assumption that the derivatives $f^{(k)}$ are continuous on $[0, \infty)$, $k = 0, 1, \ldots, \mu - 1$, and

$$|f^{(k)}(t)| \leqq Ke^{\gamma t}, \qquad t \geqq 0,\ k = 0, 1, \ldots, \mu - 1, \tag{5.4}$$

for some real constants K, γ with $K > 0$. Here $|\cdot|$ denotes some norm on $n \times 1$ vectors. Then by a standard result for Laplace transforms we have for $k = 0, 1, \ldots, \mu - 1$, where $\mathscr{L}g = \hat{g}$,

$$s^k \hat{f}(s) = \mathscr{L}f^{(k)}(s) + \sum_{h=0}^{k-1} s^h f^{(k-h-1)}(0), \tag{5.5}$$

(the sum being zero for $k = 0$) and, moreover,

$$\mathscr{L}f^{(k)}(s) \to 0 \text{ as } \mathrm{Re}(s) \to +\infty. \tag{5.6}$$

Using (5.5), we have

$$(I + sQ_{-1}A)^{-1}Q_{-1}\hat{f}(s) = \sum_{k=0}^{\mu-1}(-Q_{-1}A)^k Q_{-1}\,\mathscr{L}f^{(k)}(s) + \sum_{k=1}^{\mu-1}\sum_{h=0}^{k-1}(-Q_{-1}A)^k Q_{-1} s^h f^{(k-h-1)}(0)$$

since $(Q_{-1}A)^k = 0$ for $k \geqq \mu$. Interchanging the order of summation in the double sum and combining the result with the polynomial $(I + sQ_{-1}A)^{-1}Q_{-1}A\zeta$, we find

$$\begin{aligned}(I + sQ_{-1}A)^{-1}&Q_{-1}(\hat{f}(s) + A\zeta)\\ &= \sum_{k=0}^{\mu-1}(-Q_{-1}A)^k Q_{-1}\,\mathscr{L}f^{(k)}(s)\\ &\quad - \sum_{h=0}^{\mu-1} s^h(-Q_{-1}A)^{h+1}\left[\zeta - \sum_{j=0}^{\mu-1}(-Q_{-1}A)^j Q_{-1} f^{(j)}(0)\right].\end{aligned} \tag{5.7}$$

The sum on k in (5.7) is the Laplace transform of $v(t)$ as defined in (3.16) and tends to 0 as $\mathrm{Re}(s) \to +\infty$. The sum on h is a polynomial and hence not the Laplace transform of a function unless it is identically zero. Clearly this polynomial is identically zero if and only if

$$Q_{-1}A\left[\zeta - \sum_{j=0}^{\mu-1}(-Q_{-1}A)^j Q_{-1} f^{(j)}(0)\right] = 0. \tag{5.8}$$

THEOREM 5.1. *Suppose A is singular, that* (1.1) *is regular, that $f^{(k)}$ is continuous on* $[0, \infty)$ *for $k = 0, 1, \ldots, \mu - 1$, that* (5.4) *holds and $\zeta \in \mathbf{C}^n$. Then $\hat{x}(s)$ defined in* (5.1) *is the Laplace transform of a function $x(t)$ if and* *only if* ζ *satisfies* (5.8). *If* ζ *satisfies* (5.8), *then the corresponding function* $x(t)$ *is a solution of* (1.1) *and*

$$x(0) = P_0\zeta + v(0) \tag{5.9}$$

where v *is defined in* (3.16). *Moreover, every solution of* (1.1) *is obtained from the Laplace transform* $\hat{x}(s)$ *for some* ζ *satisfying* (5.8).

PROOF. From the development preceding the statement of the theorem it should be clear that $\hat{x}(s)$ given in (5.1) is the Laplace transform of a function if and only if ζ satisfies (5.8). Moreover, if ζ satisfies (5.8), then $\hat{x}(s)$ is the Laplace transform of $x(t) = u(t) + v(t)$ where $u(t)$ and $v(t)$ are defined in (3.15) and (3.16). But then $u(t) = P_0x(t)$ and $v(t) = P_1x(t)$ and it follows from Lemmas 3.1–3.3 that x satisfies (1.1). Moreover, then $x(0) = u(0) + v(0) = P_0\zeta + v(0)$ by (3.15). Finally, suppose $x(t)$ is a solution of (1.1) and let $\tilde{\zeta} = x(0)$. By Theorem 3.1 we have then that $P_1\tilde{\zeta} = v(0)$ where v is given in (3.16). But then $\tilde{\zeta} = P_0\tilde{\zeta} + P_1\tilde{\zeta} = P_0\tilde{\zeta} + v(0)$ and since $Q_{-1}AP_0 = Q_{-1}AQ_0A = 0$ by (2.10) we see that $\zeta = \tilde{\zeta}$ satisfies (5.8).

REMARK 5.1. The conclusion in Theorem 5.1 is trivially true when A is nonsingular since then $P_0 = I$, $Q_{-1} = 0$ and $v(t) \equiv 0$, the sum in (3.16) being vacuous in this instance. The utility of Theorem 5.1 is that one may employ Laplace transform methods formally for a regular system and compute the transforms of solutions without explicitly computing Q_{-1} and Q_0: one merely needs to choose ζ so as to assure

$$\lim_{\mathrm{Re}(s)\to+\infty} \hat{x}(s) = 0, \tag{5.10}$$

which is equivalent to (5.8) when f satisfies (5.4). Since $s\hat{x}(s) - x(0) = \mathscr{L}x'(s)$, then if f also satisfies (5.4) with $k = \mu$, we may use the fact that $\mathscr{L}x'(s) \to 0$ as $\mathrm{Re}(s) \to +\infty$ to replace (5.9) by

$$x(0) = \lim_{\mathrm{Re}(s)\to+\infty} s\hat{x}(s) \tag{5.11}$$

for the ζ determined by (5.10).

To illustrate we consider the system

$$\begin{aligned} \dot{x}_1 - \dot{x}_2 + x_2 &= f_1 \\ x_1 + \dot{x}_2 + \dot{x}_3 &= f_2 \\ \dot{x}_1 + 2x_2 + \dot{x}_3 &= f_3. \end{aligned} \tag{5.12}$$

We suppose f satisfies (5.4) for $k = 0, 1$. By an elementary calculation we find for $\hat{x}(s)$ as given in (5.1) that

$$\hat{x}(s) = \begin{bmatrix} -s+2 & -s+1 & s-1 \\ -s+1 & -s & s \\ s-2/s & s+1 & -s-1+1/s \end{bmatrix} \begin{bmatrix} \hat{f}_1(s) + \zeta_1 - \zeta_2 \\ \hat{f}_2(s) + \zeta_2 + \zeta_3 \\ \hat{f}_3(s) + \zeta_1 + \zeta_3 \end{bmatrix}. \tag{5.13}$$

Note that in (5.13) the terms in s multiplying ζ_1, ζ_2 and ζ_3 are identically zero. Thus, since $s\hat{f}(s) \to f(0)$ as $\mathrm{Re}(s) \to +\infty$ equation (5.10) becomes

$$\begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} f(0) + \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \zeta_1 - \zeta_2 \\ \zeta_2 + \zeta_3 \\ \zeta_1 + \zeta_3 \end{bmatrix} = 0. \tag{5.14}$$

This is equivalent to the one condition

$$\zeta_1 - \zeta_2 = f_1(0) + f_2(0) - f_3(0) \tag{5.15}$$

from which we see that we may choose ζ_2 and ζ_3 arbitrarily and determine ζ_1 by (5.15). Condition (5.14), in effect, reduces (5.13) to

$$\begin{aligned} \hat{x}(s) = &\begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \hat{f}'(s) + \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \hat{f}(s) \\ &+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2/s & 0 & 1/s \end{bmatrix} (\hat{f}(s) + A\zeta). \end{aligned} \tag{5.16}$$

We may now read off the solutions of (5.12) from (5.16). Thus

$$\begin{aligned} x(t) = &\begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} f'(t) + \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} f(t) \\ &+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix} \left(\int_0^t f(\sigma)d\sigma + A\zeta\right). \end{aligned} \tag{5.17}$$

The initial value $x(0)$ may be obtained from (5.16) using (5.11) or directly from (5.17). Thus

$$\begin{aligned} x(0) = &\begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} f'(0) + \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} f(0) \\ &+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix} A\zeta. \end{aligned} \tag{5.18}$$

From this we see that $x_1(0)$ and $x_2(0)$ are determined completely by $f(0)$ and $f'(0)$. However, using (5.15) and letting $c = \zeta_2 + \zeta_3$, we get $x_3(0) = f_1'(0) + f_2'(0) - f_3'(0) - f_1(0) + c$. Here c is completely arbitrary. Note that if $f(t) \equiv 0$, then from (5.18) we find $x_1(t) = 0$, $x_2(t) = 0$, $x_3(t) = c$.

This defines a one-dimensional space in keeping with the fact that in this example $\det(sA + B) = -s$ is of degree one.

The Laplace transform technique can be applied to regular systems (4.17) of higher order than the first. Using the relationship to the corresponding first order system

$$A\dot{x} + Bx = f(t) \tag{5.19}$$

with A and B as in (4.21) and f as in (4.20), we prove the following theorem.

THEOREM 5.2. *Suppose that* (4.17) *is regular, that* $g^{(k)}$ *is continuous on* $[0, \infty)$ *and*

$$|g^{(k)}(t)| \leqq Ke^{\gamma t}, \qquad t \geqq 0, \qquad k = 0, 1, \ldots, nq, \tag{5.20}$$

for some real constants K, γ *with* $K > 0$. *For* $\zeta^{(j)} \in \mathbf{C}^n, j = 1, \ldots, q$, *let*

$$\hat{y}(s) = C(s)^{-1}[C_1(s)\zeta^{(1)} + \cdots + C_q(s)\zeta^{(q)} + \hat{g}(s)] \tag{5.21}$$

where $C_k(s)$ *is defined below* (4.22) *and* $C_q(s) = A_q$. *Then* $\hat{y}(s)$ *is the Laplace transform of a function* $y(t)$ *if and only if* $\zeta^{(1)}, \ldots, \xi^{(q)}$ *satisfy the conditions*

$$\lim_{\mathrm{Re}(s)\to+\infty} \hat{y}(s) = 0 \tag{5.22}$$

$$\lim_{\mathrm{Re}(s)\to+\infty} \left[s^k\hat{y}(s) - \sum_{j=1}^{k} s^{k-j}\zeta^{(j)}\right] = 0, \qquad k = 1, \ldots, q-1. \tag{5.23}$$

If $\zeta^{(1)}, \ldots, \zeta^{(q)}$ *satisfy* (5.22) *and* (5.23), *then the corresponding function* $y(t)$ *is a solution of* (4.17) *and*

$$y^{(j-1)}(0) = \zeta^{(j)}, \qquad j = 1, \ldots, q-1, \tag{5.24}$$

$$y^{(q-1)}(0) = \lim_{\mathrm{Re}(s)\to+\infty} \left[s^q\hat{y}(s) - \sum_{j=1}^{q-1} s^{q-j}\zeta^{(j)}\right]. \tag{5.25}$$

PROOF. Define the $nq \times 1$ vector

$$\zeta = \begin{bmatrix} \zeta^{(1)} \\ \vdots \\ \zeta^{(q)} \end{bmatrix}.$$

From the form of A given in (4.21) and of f given in (4.20) the expression $\hat{x}(s) = (sA + B)^{-1}(\hat{f}(s) + A\zeta)$ can be computed using (4.22). After multiplying by the second factor in (4.22), one finds that

$$\begin{bmatrix} I & 0 \cdots & 0 & 0 \\ -sI & I \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 \cdots & -sI & I \end{bmatrix} \hat{x}(s) \tag{5.26}$$

$$= \begin{bmatrix} C(s)^{-1}C_1(s) & \cdots & C(s)^{-1}C_{q-1}(s) & C(s)^{-1} \\ -I & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & -I & 0 \end{bmatrix} \begin{bmatrix} \zeta^{(1)} \\ \vdots \\ \zeta^{(q-1)} \\ \hat{g}(s)+A_q\zeta^{(q)} \end{bmatrix}.$$

If we let

$$\hat{x}(s) = \begin{bmatrix} \phi_1(s) \\ \vdots \\ \phi_q(s) \end{bmatrix}$$

where each $\phi_j(s)$ is $n \times 1$, then (5.26) implies

$$\begin{aligned} \phi_1(s) &= \hat{y}(s) \\ \phi_2(s) &= s\phi_1(s) - \zeta^{(1)} \\ &\vdots \\ \phi_q(s) &= s\phi_{q-1}(s) - \zeta^{(q-1)}. \end{aligned} \tag{5.27}$$

Hence

$$\hat{x}(s) = \begin{bmatrix} \hat{y}(s) \\ s\hat{y}(s) - \zeta^{(1)} \\ s^2\hat{y}(s) - s\zeta^{(1)} - \zeta^{(2)} \\ \vdots \\ s^{q-1}\hat{y}(s) - s^{q-2}\zeta^{(1)} - \cdots - \zeta^{(q-1)} \end{bmatrix}. \tag{5.28}$$

Now appealing to Theorem 5.1, we see that $\hat{x}(s)$ is the transform of a function if and only if the corresponding condition (5.8) is satisfied. As indicated in Remark 5.1, this is equivalent to (5.10) so we get conditions (5.22) and (5.23) in view of (5.28). If these hold, then $x(t)$ is a solution of (5.19) which implies y is a solution of (4.13). To get (5.24) and (5.25) we apply (5.11) to $\hat{x}(s)$ in (5.28). From (5.27) and (5.22) we see that $s\phi_j(s) = \phi_{j+1}(s) + \zeta^{(j)} \to \zeta^{(j)}$ as $\mathrm{Re}(s) \to +\infty$ for $j = 1, \ldots, q-1$. But $s\phi_j(s) \to y^{(j-1)}(0)$ in view of (5.11) and the form of x in (4.20), so we get relations (5.24). Also

$$s\phi_q(s) = s^q\hat{y}(s) - \sum_{j=1}^{q-1} s^{q-j}\zeta^{(j)} \to y^{(q-1)}(0)$$

as $\mathrm{Re}(s) \to +\infty$ and this is (5.25).

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