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NORMAL FIBRATIONS AND THE EXISTENCE OF TUBULAR NEIGHBORHOODS

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ABSTRACT. To each pair (M, N) of Hilbert cube manifolds for which N is locally flat of codimension n in M, there corresponds a normal Hurewicz fibration over N whose fibers have the homotopy type of S^{n-1}. It is shown that N has a closed tubular neighborhood in M if and only if the normal fibration is fiber homotopically equivalent to some abstract S^{n-1} – bundle over N.

1. Introduction. A closed subspace N of a manifold M is *locally flat* (*with codimension n*) if for each $x_0 \in N$ there is an open neighborhood U of x_0 in N and an open embedding $h: U \times \mathbf{R}^n \to M$ such that $h(x, 0) = x$ for all $x \in U$, where $\mathbf{R}^n$ is n-dimensional Euclidean space. The pair (M, N) is then called a *locally flat pair*.

It is of particular interest to determine, if (M, N) is a locally flat pair, whether N has a tubular neighborhood in M. A *tubular neighborhood* is a neighborhood E of N in M for which there exists a retraction $p: E \to N$ such that (E, p, N) is a locally trivial fiber bundle with 0-section N and fiber F which is either $\mathbf{R}^n$ or the Euclidean n-ball B^n. If E is open in M and $F = \mathbf{R}^n$, E is called an *open tube*; likewise, if E is closed and $F = B^n$, E is called a *closed tube*. The *boundary* ∂E of a closed tube is the combinatorial boundary of E, which is an S^{n-1}-bundle. A locally flat pair of topological (as opposed to differentiable) manifolds need not admit a tubular neighborhood (see, for example, [13]).

The subject of this paper is the tubular neighborhood question for Hilbert cube manifolds. A *Q-manifold* is a separable metric space with a basis consisting of elements homeomorphic to open subsets of the Hilbert cube Q. Equivalently, the basis elements may be required to be homeomorphic to $Q \times [0, 1)$ [1], a fact which will be used repeatedly. Henceforth, (M, N) will always be used to mean a locally flat Q-manifold pair with codimension n.

It is shown in [11] that if (M, N) has codimension 2, then N always has a closed (and thus also an open) tubular neighborhood, a result analogous to the finite dimensional result of Kirby and Siebenmann [10]. (For a

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brief summary of other related finite dimensional results, see **[11]**.) Chapman **[2]** has shown that there exists a pair (M, N) with codimension 3 which does not admit a tubular neighborhood.

The purpose of this paper is to establish a necessary and sufficient condition for the existence of a closed tubular neighborhood of N in M, independent of the codimension. The basic tool is Fadell's notion of a normal fibration **[6]**. In §2 it is shown, for locally flat Q-manifold pairs, that this construction is a Hurewicz fibration whose fibers have the homotopy type of S^{n-1}. §3 consists of technical results concerning neighborhoods of N in M and homotopies of such neighborhoods into the mapping cylinder of the normal fibration. The theorems of §5 depend on these lemmas and Chapman's approximation theorem **[3]**, which is stated in §4. The theorems in §5 can be summarized by the following necessary and sufficient conditions.

THEOREM. *Let (M, N) be a locally flat Q-manifold pair of codimension n. Then N has a closed tubular neighborhood in M if and only if the normal fibration of N in M is fiber homotopically equivalent to some abstract S^{n-1}-bundle over N.*

The normal fibration determines a map $\phi: N \to BG_{n-1}$, the classifying space of Hurewicz fibrations with fibers having the homotopy type of S^{n-1}. The theorem indicates that N admits a tubular neighborhood in M if and only if ϕ lifts to a map $\tilde{\phi}: N \to BH_{n-1}$, the classifying space of locally trivial bundles with fiber S^{n-1}. It will be shown in a subsequent paper **[12]** that the closed tubular neighborhoods of N are actually classified by vertical homotopy classes of lifts of ϕ.

Several of the constructions which follow have been adapted from similar constructions in **[4]**. The author is also grateful to T. A. Chapman for several helpful suggestions made during the course of this work.

2. The normal fibration of a submanifold. Let $\mathscr{E}$ be the space of paths $\{\omega \in M^I: \omega(t) \in N$ if and only it $t = 0\}$ with the compact-open topology. Let $p: \mathscr{E} \to N$ be defined by $p(\omega) = \omega(0)$. $(\mathscr{E}, p, N)$ is called the *normal fibration* of N in M. This terminology is justified by the following lemma.

LEMMA 2.1. *$(\mathscr{E}, p, N)$ is a Hurewicz fibration.*

PROOF. We consider first the special case $N = Q \times \{0\} \subset Q \times \mathbf{R}^n$. We show that p has the path lifting property (see, for example, **[8**, p. 82]). It is clear that this property is equivalent to the following condition: Let $Z = \{(\omega, \sigma) \in \mathscr{E} \times N^I: \omega(0) = \sigma(0)\}$; then there is a map $q: Z \to (Q \times \mathbf{R}^n)^{I \times I}$ such that

(1) $q(\omega, \sigma)(s, t) \in Q \times \{0\}$ if and only if $t = 0$,

(2) $q(\omega, \sigma)(s, 0) = \sigma(s)$, $0 \leqq s \leqq 1$,

and

(3) $q(\omega, \sigma)(0, t) = \omega(t)$, $0 \leqq t \leqq 1$.

Such a map is constructed as follows. Let $\lambda(\omega, \sigma, s): [0, s] \to Q \times \mathbf{R}^n$ be the linear map such that $\lambda(\omega, \sigma, s)(0) = \sigma(s)$ and $\lambda(\omega, \sigma, s)(s) = \omega(s)$. Then define q by

$$q(\omega, \sigma)(0, t) = \omega(t), \qquad 0 \leqq t \leqq 1 \quad \text{and}$$

$$q(\omega, \sigma)(s, t) = \begin{cases} \lambda(\omega, \sigma, s)(t), & 0 \leqq t \leqq s, \\ \omega(t), & s \leqq t \leqq 1. \end{cases} \quad 0 < s \leqq 1,$$

This suffices to prove the special case.

In order to prove the general case, let α be an open covering of N so that, for each $U \in \alpha$, there is an open set V in N with $\mathrm{cl}(U) \subset V$ and there is an open embedding $\phi: Q \times [0, 1) \times \mathbf{R}^n \to M$ with $\phi(Q \times [0, 1) \times \{0\}) = V$ and $\phi(Q \times [0, 1/2] \times \{0\}) = \mathrm{cl}(U)$. The existence of such an open embedding is an immediate consequence of the locally flat structure. The technique used in [9, Lemma 5.1] provides a fiber homotopy equivalence

$$\psi: p^{-1}\mathrm{cl}(U) \to \{\omega \in p^{-1}\mathrm{cl}(U): \omega([0, 1]) \subset \phi(Q \times [0, 3/4] \times \mathbf{R}^n)\}.$$

Thus it follows routinely from the special case that p is a Hurewicz fibration over each $U \in \alpha$. Therefore, by [5], p is a Hurewicz fibration.

REMARKS. The definition is due to Fadell [6]. The above proof of Lemma 2.1 also works for finite dimensional manifolds. A different proof for the finite dimensional case is given in [6, Proposition 4.1].

LEMMA 2.2 *The fibers of $\mathcal{E}$ have the homotopy type of S^{n-1}.*

PROOF. Let $b_0 \in N$ and let $F = p^{-1}(b_0)$. There is a neighborhood U of b_0 in M so that the pair $(U, U \cap N)$ is homeomorphic to $(Q \times [0, 1) \times \mathbf{R}^n, Q \times [0, 1) \times \{0\})$. For simplicity we identify these pairs. Let F' be the subspace of all paths in F whose images are in U. Again by the technique of [9, Lemma 5.1], F is homotopically equivalent to F'.

Define $u: F' \to U - N$ by $u(\omega) = \omega(1)$ and $v: U - N \to F'$ by letting $v(x)$ be the linear path from b_0 to x. Then $uv = \mathrm{id}$. Moreover, $vu \cong \mathrm{id}$ by a kind of Alexander trick. To be specific, a homotopy λ_t is defined by letting $\lambda_1 = vu$ and, for $t < 1$, defining

$$\lambda_t(\omega)(s) = ((1 - t)q_{s/(1-t)} + tq_0, (1 - t)x_{s/(1-t)}) \text{ for } 0 \leqq s \leqq 1 - t,$$

$$\lambda_t(\omega)(s) = vu(\omega)(s) \text{ for } 1 - t \leqq s \leqq 1,$$

where $(q_s, x_s) = \omega(s)$. Thus F' is homotopically equivalent to $U - N = Q \times [0, 1) \times (\mathbf{R}^n - \{0\})$. Since $Q \times [0, 1)$ is contractible, F', and therefore F, is homotopically equivalent to $\mathbf{R}^n - \{0\}$, that is, to S^{n-1}.

3. Some preliminary lemmas.

LEMMA 3.1. *There exists a neighborhood U of N in M and a pseudoisotopy $H_t: M \to M$ such that*
(1) *H_t is a homeomorphism for* $0 < t \leqq 1$,
(2) $H_t|N = \text{id}$ (*the identity map*), $0 \leqq t \leqq 1$,
(3) $H_1 = \text{id}$, *and*
(4) $H_0(U) \subset N$.

PROOF. First consider the case for which N is compact. By the local flatness, there is a finite sequence of open embeddings $\phi_i: Q \times [0, 1) \times \mathbf{R}^n \to M$, $i = 1, \ldots, k$ such that $\phi_i(Q \times [0, 1) \times \mathbf{R}^n) \cap N = \phi_i(Q \times [0, 1) \times \{0\})$ and $\{\phi_i(Q \times [0, 1/4) \times \{0\}): i = 1, \ldots, k\}$ is an open cover of N. For each i, the $\mathbf{R}^n$ norm induces a pseudoisotopy $h^i_t: M \to M$ which satisfies properties (1)–(3), which is supported on $\phi_i(Q \times [0, 3/4] \times 2B^n)$, and for which $h^i_0\phi_i(Q \times [0, 1/2] \times B^n) \subset N$. Choose positive numbers $r_1 = 1, r_2, \ldots, r_k$ such that

$$h^{i-1}_t \cdots h^1_t\phi_i(Q \times [0, 1/4] \times r_iB^n) \subset \phi_i(Q \times [0, 1/2) \times \text{int}(B^n).$$

Let U be the union of the sets $\phi_i(Q \times [0, 1/4) \times r_i \,\text{int}(B^n)$ and let $H_t = h^k_t h^{k-1}_t \cdots h^1_t$. These clearly satisfy the required properties.

In order to prove the general case, in which N is locally compact, represent N as the countable union of compact subsets N_i such that, for each i, N_i is contained in the interior of N_{i+1}. It follows easily from the compact case that there is a neighborhood U_1 of the union of the sets $N_{2i} - \text{int}(N_{2i-1})$ and a pseudoisotopy h_t such that $h_0(U_1) \subset N$ and that h_t satisfies properties (1)–(3). It is also clear that a second application of the compact case suffices to produce a U_2 and a g_t so that $U = U_1 \cup U_2$ and $H_t = g_th_t$ satisfy the conclusions of the lemma.

COROLLARY 3.2. *Let $\mathscr{E}$ be the normal fibration of N in M and let $\mathscr{E}_0$ be the space of all constant paths in N. Then there exists a neighborhood U of N in M and a map $\sigma: U \to \mathscr{E} \cup \mathscr{E}_0$ such that $\sigma(U - N) \subset \mathscr{E}$.*

PROOF. Let U be the neighborhood from Lemma 3.1, let $\sigma(x)$ be the constant path at x if $x \in N$, and let $\sigma(x)$ be the path defined by $\sigma(x)(t) = H_t(x)$ if $x \in U - N$.

LEMMA 3.3. *If E is a tubular neighborhood of N with retraction p, then there is a pseudoisotopy $H_t: E \to E$ satisfying properties* (1)–(3) *of Lemma* 3.1 *and such that $H_0 = p$ and $pH_t = p$*, $0 \leqq t \leqq 1$.

PROOF. The proof is the same as that for Lemma 3.1. The final conclusion is possible because the individual pseudoisotopies can be defined over bundle charts and change only the second coordinate.

We now turn our attention to the *mapping cylinder* of the normal fibration of N in M. Let $M_p = N \cup (\mathscr{E} \times (0, 1])$, where $p: \mathscr{E} \to N$ is the normal fibration. Instead of the quotient topology usually used for mapping cylinders, let the topology for M_p be determined by a subbasis consisting of (a) open sets in $\mathscr{E} \times (0, 1]$ and (b) sets of the form $G \cup (p^{-1}(G) \times (0, \varepsilon))$ where G is open in N and $\varepsilon > 0$.

Normalize the metric on M so that $d(x, y) \leqq 1$, and let σ be the map from Corollary 3.2. Now define a function f: $U \to M_p$ by $f(x) = x$ if $x \in N$ and $f(x) = (\sigma(x), d(x, N))$ if $x \in U - N$. The topology of M_p has been chosen so that f is continuous. Our goal is to define a map $g: M_p \to M$ satisfying the homotopy conditions to be stated in Lemma 3.5. In order to satisfy continuity, we first reparameterize the paths in $\mathscr{E}$ by using the following lemma.

LEMMA 3.4. *There is a map* ψ: $\mathscr{E} \to \mathscr{E}$ *such that*

(1) $\psi(\omega)([0, 1]) = \omega([0, 1])$ *for each* $\omega \in \mathscr{E}$,

(2) *for each natural number* k *and each* $\omega \in \mathscr{E}$, if $t < 2^{-k}$, *then* $d(\psi(\omega)(t), \omega(0)) < 2^{-k}$, *and*

(3) ψ *is fiber homotopic to* $\mathrm{id}\mathscr{E}$.

PROOF. There exists a decreasing sequence of maps t_k: $\mathscr{E} \to (0, 1)$ so that $d(\omega(t), \omega(0)) < 2^{-k}$ whenever $t \in [0, t_k(\omega)]$. Each t_k can be defined inductively by piecing together maps defined over the elements of a sufficiently fine cover of N. Then ψ is a reparamenterization so that $\psi(\omega)(2^{-k}) = \omega(t_k)$.

We can now define a continuous functing g: $M_p \to M$ by $g(x) = x$ if $x \in N$ and $g(\omega, t) = \psi(\omega)(t)$ if $x \in \mathscr{E} \times (0, 1)$.

LEMMA 3.5. (1) *gf is homotopic to* id_U *by a homotopy which is the identity on M and which takes* $U - N$ *into* $M - N$ *at all levels.*

(2) *There is a neighborhood* $V \subset g^{-1}(U)$ *of N in* M_p *such that* $fg|V$ *is homotopic to* id_V *by a homotopy which is the identity on N and which takes* $V - N$ *into* $M_p - N$ *at all levels.*

PROOF. Condition (1) is easy to establish. Since, for each $x \in U - N$, $gf(x) = \psi\sigma(x)(d(x, N)) = \sigma(x)(t_x)$ for some $t_x \in (0, 1)$, simply define a homotopy in terms of $\sigma(x) \mid [t_x, 1]$.

In order to establish condition (2), note first that $fg(\omega, t) = (\sigma(\psi(\omega)(t)), d(\psi(\omega)(t), N))$ for each $(\omega, t) \in g^{-1}(U) - N$. Consider the map η_1: $g^{-1}(U) - N \to \mathscr{E}$ defined by $\eta_1(\omega, t) = \sigma(\psi(\omega)(t))$. It is easy to define a homotopy from fg to id if we can show that η_1 is homotopic to the projection map taking (ω, t) to ω. Note that $\eta_1(\omega, t)(1) = \omega(\tau)$ for some τ which depends continuously on (ω, t). Let $\eta_2(\omega, t)$ be a reparameterization of $\omega| [0, \tau]$. Clearly η_2 is homotopic to projection, so the problem reduces to finding a neighborhood V for which η_1 is homotopic to η_2.

We consider first the special case in which $(M, N) = (Q \times [0, 1) \times \mathbf{R}^n,$

$Q \times [0, 1) \times \{0\}$). Let $\eta_3(\omega, t)$ be the path from $\omega(0)$ to $\omega(\tau)$ whose $Q \times [0, 1)$ coordinates are determined by η_2 and whose $\mathbf{R}^n$ coordinates are determined by η_1. Then η_1 is clearly homotopic to η_3 by a homotopy which moves only the $Q \times [0, 1)$ coordinates. Since $\eta_3(\omega, t)$ and $\eta_2(\omega, t)$ are paths which begin and end at the same points, η_3 and η_2 are each homotopic to the map taking (ω, t) to the linear path from $\omega(0)$ to $\omega(\tau)$ by the Alexander trick used in Lemma 2.2. Thus we have shown for the special case that η_1 is homotopic to η_2 over all of $M_p - N$. Moreover, the homotopy clearly extends to N by means of the identity homotopy.

We can now do the construction for the general case by working over a sequence of product neighborhoods. Choose (G_i, C_i) so that $\{G_i\}$ is a locally finite sequence of open sets in M, each triple $(G_i, G_i \cap N, C_i)$ is homeomorphic to $(Q \times [0, 1) \times \mathbf{R}^n, Q \times [0, 1) \times \{0\}, Q \times [0, 1/4] \times \{0\})$, and the collection of $\text{int}(C_i)$ cover N. We now proceed inductively.

Identify G_1 with $Q \times [0, 1) \times \mathbf{R}^n$. By Lemma 3.4 it is possible to find a neighborhood V_1 of C_1 in G_1 and a number t_1 sufficiently small that, whenever $(\omega, t) \in p^{-1}(V_1) \times (0, t_1)$, $\eta_1(\omega, t)$ and $\eta_2(\omega, t)$ both have images in $Q \times [0, 1/2) \times \text{int}(B^n)$. The special case provides a homotopy between η_2 and η_1 so long as the maps are restricted so that all paths remain in G_1. It is then routine to phase out the motions of this homotopy and construct a map $h_1: g^{-1}(U) - N \to \mathscr{E}$ so that $h_1(\omega, t)$ is a path which ends at the same point as $\eta_1(\omega, t)$ and $\eta_2(\omega, t)$ and so that the following conditions are satisfied:

(a) $h_1 = \eta_1$ on $p^{-1}(V_1) \times (0, t_1)$,
(b) $h_1 = \eta_2$ off $p^{-1}(G_1 \cap N) \times (0, 1]$,
(c) $\eta_2 \cong h_1$ rel $(g^{-1}(U) - N) - (p^{-1}(G_1 \cap N) \times (0, 1])$.

All that is required for the induction step is some care regarding the overlap of the G_i. One more step is sufficient to illustrate how this is achieved. Indentify G_2 with $Q \times [0, 1) \times \mathbf{R}^n$. Choose V_2 and t_2 so that if $(\omega, t) \in p^{-1}(V_2) \times (0, t_2)$, then $\eta_1(\omega, t)$ and $h_1(\omega, t)$ both have images in $Q \times [0, 1/2) \times \text{int}(B^n)$. If $V_1 \cap G_2$ is nonempty, choose a neighborhood V_1' of C_1 in N so that $\text{cl}(V_1') \subset V_1$. Now the techniques of the special case can be used over an open set of $g^{-1}(U)$ for which all paths involved remain in G_2. First, as before, move the path $h_1(\omega, t)$ to a path ω' which beigns and ends at the same points as does $\eta_1(\omega, t)$. Note that $h_1(\omega, t)$ is not moved if $(\omega, t) \in p^{-1}(V_1) \times (0, t_1)$.

By a Tietze extension argument there is a choice of s, $0 \leqq s \leqq 1$, depending continuously on (ω, t), so that $s = 0$ if $(\omega, t) \in p^{-1}(V_1') \times (0, t_1/2)$, and $s = 1$ if $(\omega, t) \notin p^{-1}(V_1) \times (0, t_1)$. Now move ω' to $\lambda_s(\omega')$, where λ_t is a homotopy defined using the Alexander trick of Lemma 2.2. Since $\lambda_s(\omega') = \lambda_s\eta_1(\omega, t)$ in every case, $\lambda_s(\omega')$ is moved to $\eta_1(\omega, t)$ using the same kind of homotopy. Note that $h_1(\omega, t) = \eta_1(\omega, t)$ has not been moved at all if $(\omega, t) \in p^{-1}(V_1') \times (0, t_1/2)$. Thus it is possible to phase out these movements and construct a map $h_2: g^{-1}(U) - N \to \mathscr{E}$ such that

(a) $h_2 = \eta_1$ on $p^{-1}(V_2) \times (0, t_2) \cup p^{-1}(V_1') \times (0, t_1/2)$,
(b) $h_2 = h_1$ off $p^{-1}(G_2 \cap N) \times (0, 1]$, and
(c) $h_1 \cong h_2$ rel $(g^{-1}(U) - N) - p^{-1}(G_2 \cap N \times (0, 1])$.

The general induction step now follows in like manner. Since the G_i are locally finite, all but finitely many of the h_i agree in a neighborhood of any fixed point of $g^{-1}(U) - N$. Thus $h = \lim_{i\to\infty} h_i$ is a well defined map which is homotopic to η_2. The homotopy clearly extends to N by the identity. Moreover, the local finiteness means that the neighborhoods $p^{-1}(V_i) \times (0, t_i)$ are cut back only finitely many times. The union of these modified neighborhoods is a neighborhood V of N in M_p and $h = \eta_1$ on V.

4. The approximation theorem. The last tool we need is Champan's approximation theorem [3] for fiber bundles over a Q-manifold. This theorem is stated below. First we give some definitions.

Let X and Y be locally compact spaces and let $f: X \to Y$ be a *proper map*, i.e., if C is a compact subset of Y, then $f^{-1}(C)$ is compact. If α is an open cover of Y, then f is said to be an *α-equivalence* if there is a proper map $g: Y \to X$ such that (1) fg is *α-homotopic* to id, i.e., if there is a proper homotopy $fg \cong$ id so that the track of each point of Y under the homotopy lies in some element of α, and (2) gf is $f^{-1}(\alpha)$-homotopic to id, where $f^{-1}(\alpha) = \{f^{-1}(U): U \in \alpha\}$. If $A \subset Y$, f is said to have a property *over A* if it has the property when restricted to the inverse image of A.

The following theorem is a relative version of the theorem given in [3]. Its proof, however, is implicit on the proof given in [3].

APPROXIMATION THEOREM. *Let B be a Q-manifold, let $C \subset U \subset B$, where C is closed and U is open, and let α be an open cover of B. Then there is an open cover β of B so that if $p: E \to B$ is an S^n-bundle, then any proper map f from a Q-manifold M to E which is a $p^{-1}(\beta)$-equivalence over $p^{-1}(U)$ is $p^{-1}(\alpha)$-homotopic to a map $g: M \to E$ which is a homeomorphism over $p^{-1}(C)$.*

As is stated in [3], the conclusion also holds whenever the fiber of the bundle, in this case S^n, is a compact ANR for which π_1 of each component is free or free abelian.

5. Tubular neighborhoods. We continue to use (M, N) to mean a locally flat pair of Q-manifolds with codimension n.

THEOREM 5.1. *Let $q: E \to N$ be a tubular neighborhood of N in M. Then $E - N$ (and thus, in case E is a closed tube, ∂E) is fiber homotopically equivalent to the normal fibration $p: \mathscr{E} \to N$.*

PROOF. Let $\sigma: E - N \to \mathscr{E}$ be defined by $\sigma(x)(t) = H_t(x)$, where H_t is the pseudoisotopy of Lemma 3.3. Note that σ is a fiber preserving map. It is easy to imitate the technique of Lemma 2.2 and show that σ is a homotopy equivalence on a singular fiber. Therefore [7], it is a fiber homotopy equivalence.

We conclude by showing, conversely, that the existence of any S^{n-1}-bundle over N which is fiber homotopically equivalent to the normal fibration of N in M implies the existence of a tubular neighborhood of N in M.

THEOREM 5.2. *Let $q: E \to N$ be an S^{n-1}-bundle. If E is fiber homotopically equivalent to the space of the normal fibration of N in M, then N has a closed tubular neighborhood in M whose boundary is homeomorphic to E.*

PROOF. Let $\tilde{q}: \tilde{E} \to N$ be the disk bundle associated with q, i.e. $\tilde{E}$ is the mapping cylinder of q (in the usual sense), which can be represented as $E \times (0, 1] \cup N$, and $\tilde{q}$ is naturally induced by q. Note that each fiber $\tilde{q}^{-1}(x)$ is just the cone over $q^{-1}(x)$, that is B^n. Moreover, N is a 0-section of $\tilde{E}$.

The strategy of the proof is to construct a homeomorphism between a closed neighborhood of N in M and the subbundle $E \times (0, 1/2] \cup N$ of $\tilde{E}$ by means of the Approximation Theorem.

As a first step in the construction, let $u: \mathscr{E} \to E$ (where $\mathscr{E}$ is the total space of the normal fibration) be a fiber homotopy equivalence with fiber homotopy inverse $v: E \to \mathscr{E}$. Let U be the neighborhood of N in M and f and g be the maps constructed in Section 3 above. Define $\tilde{f}: U \to \tilde{E}$ by $\tilde{f} = \text{id}$ on N and $\tilde{f} = (u \times \text{id})f$ on $U - N$. In like manner, define $\tilde{g}: \tilde{E} \to M$ from g and v. A routine check verifies that these functions are continuous. It also follows immediately from Lemma 3.5 that

(1) $\tilde{g}\tilde{f}$ is homotopic to id_U by a homotopy which is the identity on N and which takes $U - N$ into $M - N$ at each level, and;

(2) there is a neighborhood $V \subset \tilde{g}^{-1}(U)$ of N in $\tilde{E}$ such that $\tilde{f}\tilde{g}|V$ is homotopic to id_V by a homotopy which is the identity on N and which takes $V - N$ into $\tilde{E} - N$ at each level.

We must now modify these homotopies in order to satisfy the hypothesis of the Approximation Theorem. First consider the case for which N is compact.

It follows easily from well known properties of ANRs that there is some $r_0 \in (0, 1)$ so that $\tilde{f}$ is a proper homotopy equivalence over $E \times (0, r_0] \cup N$. Furthermore, choose inductively a decreasing sequence of positive numbers r_i with limit 0 so that f is an α-equivalence over $E \times (0, r_1)$, where α is the cover consisting of $E \times (r_{i+2}, r_i)$, $i = 1, 2, \ldots$. The existence of such a sequence follows from elementary continuity and compactness arguments.

Now consider the S^{n-1}-bundle $q \times \text{id}: E \times (0, 1] \to N \times (0, 1]$. Let $A_j = E \times [2^{-(j+1)}, 2^{-j}]$, let $B_j = (q \times \text{id})(A_j)$ and let β be any open cover of $N \times (0, 1]$. Since N is assumed to be compact, there exists a sequence of positive numbers ε_j so that the cover consisting of all elements of the form

$$\{(x, t): \mathrm{d}(x, x_0) < \varepsilon_j, |t - t_0| < \varepsilon_j\}, (x_0, t_0) \in B_j$$

is a refinement of β. Choose i_1 sufficiently large that $\tilde{f}$ is a $q^{-1}(\varepsilon_1)$-equivalence over $E \times (0, r_{i_1}]$. Proceed inductively, for $j > 1$, to choose $i_j > i_{j-1}$ sufficiently large that $\tilde{f}$ is a $q^{-1}(\varepsilon_j)$-equivalence over $E \times (0, r_{i_j}]$ and $i_j - i_{j-1} > 3/\varepsilon_{j-1}$.

There is then clearly a homeomorphism η: $(0, 1] \to (0, 1]$ such that $\eta(r_{i_j}) = 2^{-j}$ and such that $(\mathrm{id} \times \eta)\tilde{f}$ is a $(q \times \mathrm{id})^{-1}(\beta)$-equivalence over each A_j. Thus by application of the Approximation Theorem we obtain a homotopy from $(\mathrm{id} \times \eta)\tilde{f}$ to f_1 such that f_1 is a homeomorphism over each A_{2j-1}. Moreover, we can require that f_1 remain a proper homotopy equivalence with cover control over each A_{2j}. Thus a second application of the Approximation Theorem and a routine application of Z-set unknotting (see [1]) provides us with a homotopy between f_1 and a map f_2 which is a homeomorphism over all of $E \times (0, 1/2]$. Since the Approximation Theorem allows us to require that f_2 be $(q \times \mathrm{id})^{-1}(\alpha)$-homotopic to $(\mathrm{id} \times \eta)\tilde{f}$ for some cover α of $N \times (0, 1/2]$ whose mesh approaches 0 near $N \times \{0\}$, f_2 extends continuously by the identity on N. Then $f_2^{-1}(E \times (0, 1/2] \cup N)$ is the required tubular neighborhood of N in M.

The proof for N locally compact is similar. Since uniform continuity can no longer be used, the numbers r_i must be replaced by maps r_i: $N \to (0, 1)$, which are defined over a sequence of compact subsets of N. The modification of the details of the proof are routine and are left to the reader.

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