

SINGULAR PERTURBATIONS OF TWO CAUCHY PROBLEMS IN A HILBERT SPACE

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1. Introduction. This paper discusses two perturbed Cauchy problems in abstract Hilbert space, one of second and the other of third order, and shows that the solution of each tends uniformly to the solution of the associated degenerate problem. The uniform convergence of the derivatives of these solutions is also discussed. Both problems are first treated on the real line. Results are then extended to abstract Hilbert space. Standard notation is used throughout the paper. The well-known results of spectral theory found in Dunford and Schwartz [2] are employed frequently. Other research to which this paper relates directly includes the work of J. Schmoller [4] and L. Bobisud and J. Calvert [1].

2. The second order problem on the real line. Let the functions $g(t; \lambda)$ and $h(t; \lambda)$ be given functions of the variable t and the real parameter λ . Both functions will be assumed to be defined on the domain $[0, \infty) \times (-\infty, \infty)$. With respect to t and for fixed λ , both functions will be assumed to be differentiable. The function $g(t; \lambda)$ will be assumed to be positive and not equal to zero on any finite interval $[0, T]$.

For $\varepsilon > 0$ we will be concerned with the following perturbed Cauchy problem

$$(1) \quad \begin{aligned} \varepsilon u''_{\varepsilon}(t; \lambda) + g(t; \lambda)u'_{\varepsilon}(t; \lambda) + h(t; \lambda)u_{\varepsilon}(t; \lambda) &= 0, \\ u_{\varepsilon}(0; \lambda) &= x_0, \quad u'_{\varepsilon}(0; \lambda) = x_1. \end{aligned}$$

If we set $\varepsilon = 0$ in the above problem and retain only the first initial condition, we obtain the unperturbed (or degenerate) Cauchy problem

$$(2) \quad \begin{aligned} g(t; \lambda)u'_0(t; \lambda) + h(t; \lambda)u_0(t; \lambda) &= 0, \\ u_0(0; \lambda) &= x_0. \end{aligned}$$

Since both problems (1) and (2) are linear, their solutions u_{ε} and u_0 may be put in the forms

$$u_{\varepsilon}(t; \lambda) = p_{\varepsilon}(t; \lambda)x_0 + q_{\varepsilon}(t; \lambda)x_1,$$

and

$$u_0(t; \lambda) = p_0(t; \lambda)x_0,$$

respectively. Here $p_\varepsilon(t; \lambda)$ and $q_\varepsilon(t; \lambda)$ are the solutions of (1) with the initial data $(x_0, x_1) = (1, 0)$ and $(x_0, x_1) = (0, 1)$, respectively, while $p_0(t; \lambda)$ is the solution of (2) with the initial data $x_0 = 1$.

By known methods it is possible to find a representation of the solutions of each of the above problems and to determine bounds for these solutions and for the functions p_ε , q_ε and p_0 as well as for their first and second derivatives. Having done so, two theorems can be proven (cf. Smith [3] for analogous results).

THEOREM 1. *Let λ be fixed and $T > 0$. Then $\varepsilon \rightarrow 0^+$ implies*

1. $p_\varepsilon(t; \lambda) \rightarrow p_0(t; \lambda)$, and
2. $q_\varepsilon(t; \lambda) \rightarrow 0$,

where convergence is uniform for t in $[0, T]$.

THEOREM 2. *On any interval $[\delta, T]$, $\delta > 0$, $p'_\varepsilon(t; \lambda) \rightarrow p'_0(t; \lambda)$ and $q'_\varepsilon(t; \lambda) \rightarrow 0$ uniformly in t for fixed λ as $\varepsilon \rightarrow 0^+$.*

3. The problem in Hilbert space. We turn next to the associated problem in a Hilbert space H . That is, we consider the abstract problem analogous to (1)

$$(3) \quad \begin{aligned} \varepsilon U''_\varepsilon(t) + g(t; A)U'_\varepsilon(t) + h(t; A)U_\varepsilon(t) &= 0, \\ U_\varepsilon(0) = x_0 \text{ and } U'_\varepsilon(0) = x_1, \quad x_0 \text{ and } x_1 \text{ in } H, \end{aligned}$$

and its associated degenerate problem

$$(4) \quad \begin{aligned} g(t; A)U'_0(t) + h(t; A)U_0(t) &= 0, \\ U_0(0) &= x_0 \end{aligned}$$

which is analogous to (2).

In the above statement of the problem, A is a self-adjoint, possibly unbounded operator whose domain $\mathcal{D}(A)$ is dense in H . The functions $g(t; \lambda)$ and $h(t; \lambda)$ are defined as before with the additional requirement that with respect to λ the functions be integrable Borel functions.

Recall that the self-adjointness of A assures the existence of the unique spectral family E and permits the representation of A as well as Borel functions of A by means of integrals. That is,

$$(5) \quad A = \int_{-\infty}^{\infty} \lambda dE_\lambda \quad \text{and} \quad f(A) = \int_{-\infty}^{\infty} f(\lambda) dE_\lambda.$$

For x in the domain of A we write

$$(6) \quad Ax = \int_{-\infty}^{\infty} \lambda dE_\lambda x.$$

Proceeding to the problem, we define the three operators

$$(7) \quad \begin{aligned} P_\varepsilon(t) &= p_\varepsilon(t; A) = \int_{-\infty}^{\infty} p_\varepsilon(t; \lambda) dE_\lambda, \\ Q_\varepsilon(t) &= q_\varepsilon(t; A) = \int_{-\infty}^{\infty} q_\varepsilon(t; \lambda) dE_\lambda, \end{aligned}$$

and

$$P_0(t) = p_0(t; A) = \int_{-\infty}^{\infty} p_\varepsilon(t; \lambda) dE_\lambda.$$

We want to show that for elements x_0 and x_1 of an appropriate dense domain determined by the operators of the two problems (3) and (4), the solutions of these two problems are given by

$$(8) \quad U_\varepsilon(t) = P_\varepsilon(t)x_0 + Q_\varepsilon(t)x_1$$

and

$$U_0(t) = P_0(t)x_0,$$

respectively. From the development of Section 2 there can be found a function $\Phi(\lambda)$ determining a self-adjoint operator

$$\Phi(A) = \int_{-\infty}^{\infty} \Phi(\lambda) dE_\lambda.$$

The domain of this operator we will denote with $\Delta(\Phi)$; that is,

$$(9) \quad \begin{aligned} \Delta(\Phi) &= \{x \in H: \int_{-\infty}^{\infty} \Phi^2(\lambda) d(E_\lambda x, x) < \infty\} \\ &= \{x \in H: \|\Phi(A)x\| < \infty\}. \end{aligned}$$

With $\Delta(\Phi)$ thus defined it is possible to show that $x \in \Delta(\Phi)$ is in the domain of each of the operators encountered in (3) and (4). Specifically, $\Delta(\Phi)$ is contained in the domains of $P_\varepsilon(t)$, $Q_\varepsilon(t)$ and $P_0(t)$.

To show that the expressions of (8) are indeed solutions of (3) and (4) it suffices to show that

$$(10) \quad \begin{aligned} \frac{d^k}{dt^k} P_\varepsilon(t)x &= \int_{-\infty}^{\infty} \frac{\partial^k}{\partial t^k} p_\varepsilon(t; \lambda) dE_\lambda x, \quad k = 1, 2, \\ \frac{d^k}{dt^k} Q_\varepsilon(t)x &= \int_{-\infty}^{\infty} \frac{\partial^k}{\partial t^k} q_\varepsilon(t; \lambda) dE_\lambda x, \quad k = 1, 2, \end{aligned}$$

and

$$\frac{d}{dt} P_0(t)x = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} p_\varepsilon(t; \lambda) dE_\lambda x.$$

To this end we prove the following lemma.

LEMMA 1. Let f be a function of t and λ which is defined on $R \times R$, has a continuous derivative with respect to t , and with respect to λ is a Borel function such that

$$\left| \frac{\partial}{\partial t} f(t; \lambda) \right|^2 \leq F(\lambda),$$

where $F(\lambda)$ is integrable on R with respect to $d(E_\lambda x, x)$. Then

$$\frac{d}{dt} \int_{-\infty}^{\infty} f(t; \lambda) dE_\lambda x = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} f(t; \lambda) dE_\lambda x.$$

PROOF. From (5) and by definition of f ,

$$\frac{d}{dt} \int_{-\infty}^{\infty} f(t; \lambda) dE_\lambda x = \frac{d}{dt} f(t; A)x$$

so that

$$(1/\delta) [f(t + \delta; A)x - f(t; A)x] = \int_{-\infty}^{\infty} \frac{f(t + \delta; \lambda) - f(t; \lambda)}{\delta} dE_\lambda x.$$

But by the mean value theorem there exists a number t' , depending on λ , satisfying $|t - t'| \leq \delta$, and such that

$$\frac{f(t + \delta; \lambda) - f(t; \lambda)}{\delta} = \frac{\partial}{\partial t} f(t'; \lambda).$$

Using this we may write

$$(11) \quad \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial t} f(t'; \lambda) - \frac{\partial}{\partial t} f(t; \lambda) \right]^2 d(E_\lambda x, x) \rightarrow 0.$$

as $\delta \rightarrow 0$. (We have used the dominated convergence theorem of Lebesgue together with the fact that $4 \cdot F(\lambda)$ bounds the integrand.) But this implies the conclusion of the lemma, thus completing the proof.

Bounds integrable with respect to $d(E_\lambda x, x)$ for the functions $(\partial^k/\partial t^k)p_\varepsilon(t; \lambda)$, $(\partial^k/\partial t^k)q_\varepsilon(t; \lambda)$, $k = 1, 2$, and for $(\partial/\partial t)p_0(t; \lambda)$ can be found, thus allowing us to conclude from Lemma 1 that

$$\frac{d}{dt} P_\varepsilon(t)x = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} p_\varepsilon(t; \lambda) dE_\lambda x.$$

Proof in the case of the second derivative of $P_\varepsilon(t)x$ follows also from application of Lemma 1. The remaining cases involving $Q_\varepsilon(t)$ and $P_0(t)$ are disposed of similarly.

Continuing, we have from (8)

$$\begin{aligned} \varepsilon U_\varepsilon''(t) + g(t; A)U_\varepsilon'(t) + h(t; A)U_\varepsilon(t) \\ = \varepsilon P_\varepsilon''(t)x_0 + g(t; A)P_\varepsilon'(t)x_0 + h(t; A)P_\varepsilon(t)x_0 \\ + \varepsilon Q_\varepsilon''(t)x_1 + g(t; A)Q_\varepsilon'(t)x_1 + h(t; A)Q_\varepsilon(t)x_1. \end{aligned}$$

The sum of the first three terms of the right member equals

$$\int_{-\infty}^{\infty} \left[\varepsilon p''_e(t; \lambda) + g(t; \lambda) p'_e(t; \lambda) + h(t; \lambda) p_e(t; \lambda) \right] dE_\lambda x_0$$

which in turn, equals zero. Similarly, the remaining terms equal zero so that $U_\varepsilon(t)$ is a solution of (3) on the interval $[0, T]$.

In the same manner $U_0(t)$ is shown to be a solution of (4). Thus follows another lemma.

LEMMA 2. $U_\varepsilon(t)$ and $U_0(t)$ are solutions of the problems (3) and (4), respectively, on $[0, T]$.

This lemma is complemented by the next one.

LEMMA 3. The solutions of (3) and (4) are unique.

PROOF. It suffices to show that $P_\varepsilon(t)x_0$ and $Q_\varepsilon(t)x_1$ are unique solutions of (3) with $x_1 = 0$ and $x_0 = 0$, respectively, and that $P_0(t)x_0$ is the unique solution of (4).

Consider the case involving $P_\varepsilon(t)x_0$. We want to show that if $\bar{P}_\varepsilon(t)x_0$ is a second solution of (3), $x_1 = 0$, then $R_\varepsilon(t) \equiv P_\varepsilon(t)x_0 - \bar{P}_\varepsilon(t)x_0 \equiv 0$. If $x_1 = x_0 = 0$,

$$\varepsilon R''_\varepsilon(t) + g(t; A)R'_\varepsilon(t) + h(t; A)R_\varepsilon(t) = 0.$$

Now let n be an integer and define the bounded operators

$$g_n(t; A) \equiv \int_{-n}^n g(t; \lambda) dE_\lambda = (E_n - E_{-n})g(t; A),$$

$$h_n(t; A) \equiv \int_{-n}^n h(t; \lambda) dE_\lambda = (E_n - E_{-n})h(t; A),$$

and

$$G_{\varepsilon, n}(t, s; A) \equiv \frac{1}{\varepsilon} \int_0^{t-s} \exp\left[-\frac{1}{\varepsilon} \int_0^\tau g_n(\sigma + s; A) d\sigma\right] d\tau, \quad 0 \leq s \leq t.$$

Also let $R_{\varepsilon, n}(t)$ be defined by

$$R_{\varepsilon, n}(t) \equiv (E_n - E_{-n})R_\varepsilon(t).$$

Then

$$\begin{aligned} & \varepsilon R''_{\varepsilon, n}(t) + g_n(t; A)R'_{\varepsilon, n}(t) + h_n(t; A)R_{\varepsilon, n}(t) \\ (12) \quad & = (E_n - E_{-n})[\varepsilon R''_\varepsilon(t) + g(t; A)R'_\varepsilon(t) + h(t; A)R_\varepsilon(t)] \\ & = 0. \end{aligned}$$

We have employed here the fact that the operators included in the above expressions are permutable and that $(E_n - E_{-n})$ is idempotent.

From the definition of $R_{\epsilon, n}(t)$ and from (12) it follows that $R_{\epsilon, n}(t)$ satisfies the integral equation

$$R_{\epsilon, n}(t) = - \int_0^t h_n(s; A)G_{\epsilon, n}(t, s; A)R_{\epsilon, n}(s)ds.$$

Taking the norms of both sides leads to the inequality

$$\|R_{\epsilon, n}(t)\| \leq \int_0^t \|h_n(s; A)\| \|G_{\epsilon, n}(t, s; A)\| \|R_{\epsilon, n}(s)\| ds.$$

But the application of Gronwall's lemma shows that $\|R_{\epsilon, n}(t)\| \equiv 0$. Since $R_{\epsilon}(t) = \lim_{n \rightarrow \infty} R_{\epsilon, n}(t)$, it follows that $R_{\epsilon}(t) \equiv 0$, which was to be shown. The uniqueness of the remaining solutions $Q_{\epsilon}(t)x_1$ and $P_0(t)x_0$ is shown analogously.

We prove a final theorem.

THEOREM 3. For x_0 and x_1 in $\Delta(\Phi)$ $\|U_{\epsilon}(t) - U_0(t)\| \rightarrow 0$ uniformly as $\epsilon \rightarrow 0^+$ on $[0, T]$ and $\|U'_{\epsilon}(t) - U'_0(t)\| \rightarrow 0$ uniformly as $\epsilon \rightarrow 0^+$ on $[\delta, T]$, $0 < \delta \leq T$.

PROOF. From the definition of $U_{\epsilon}(t)$ and $U_0(t)$ (cf. (8)) it is clear that

$$\|U_{\epsilon}(t) - U_0(t)\| \leq \|P_{\epsilon}(t)x_0 - P_0(t)x_0\| + \|Q_{\epsilon}(t)x_1\|$$

and

$$\|U'_{\epsilon}(t) - U'_0(t)\| \leq \|P'_{\epsilon}(t)x_0 - P'_0(t)x_0\| + \|Q'_{\epsilon}(t)x_1\|.$$

Thus it suffices to show that each of the four quantities

$$\|P_{\epsilon}(t)x_0 - P_0(t)x_0\|^2 = \int_{-\infty}^{\infty} |p_{\epsilon}(t; \lambda) - p_0(t; \lambda)|^2 d(E_{\lambda}x_0, x_0),$$

$$\|Q_{\epsilon}(t)x_1\|^2 = \int_{-\infty}^{\infty} |q_{\epsilon}(t; \lambda)|^2 d(E_{\lambda}x_1, x_1),$$

$$\|P'_{\epsilon}(t)x_0 - P'_0(t)x_0\|^2 = \int_{-\infty}^{\infty} |p'_{\epsilon}(t; \lambda) - p'_0(t; \lambda)|^2 d(E_{\lambda}x_0, x_0)$$

and

$$\|Q'_{\epsilon}(t)x_1\|^2 = \int_{-\infty}^{\infty} |q'_{\epsilon}(t; \lambda)|^2 d(E_{\lambda}x_1, x_1)$$

tends to zero uniformly with ϵ .

From Section 1 it can be shown that $|p_{\epsilon}(t; \lambda) - p_0(t; \lambda)|^2$ has a bound dependent on T and λ . Also, from Theorem 1 $|p_{\epsilon}(t; \lambda) - p_0(t; \lambda)| \rightarrow 0$ uniformly with ϵ . It follows from the Lebesgue dominated convergence theorem that $\|P_{\epsilon}(t)x_0 - P_0(t)x_0\| \rightarrow 0$ uniformly with ϵ .

The uniform convergence of the remaining three quantities follows analogously, concluding the proof of the theorem.

4. The third order problem. Consider the third order Cauchy problem

$$(13) \quad \begin{aligned} \varepsilon u_\varepsilon'''(t; \lambda) + a u_\varepsilon'(t; \lambda) + h(t; \lambda) u_\varepsilon(t; \lambda) &= 0, \\ u_\varepsilon(0; \lambda) = x_0, u_\varepsilon'(0; \lambda) = x_1, u_\varepsilon''(0; \lambda) &= x_2 \end{aligned}$$

where $h(t; \lambda)$ is a function of the real variable t and the real parameter λ . The function h will be assumed defined on $[0, \infty) \times (-\infty, \infty)$ and, for fixed λ , differentiable with respect to t . The coefficient a is a positive real constant and ε a small parameter > 0 .

The degenerate problem has the form

$$(14) \quad \begin{aligned} a u_0'(t; \lambda) + h(t; \lambda) u_0(t; \lambda) &= 0, \\ u_0(0; \lambda) &= x_0. \end{aligned}$$

Employing methods analogous to those shown above, the following results are available.

LEMMA 4. $U_\varepsilon(t)$ and $U_0(t)$ are solutions of (13) and (14), respectively. These solutions are unique.

THEOREM 5. The solution $U_\varepsilon(t)$ of the perturbed Cauchy problem (13) tends uniformly as $\varepsilon \rightarrow 0^+$ to the solution $U_0(t)$ of the degenerate problem (14).

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