ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 12, Number 3, Summer 1982

GENERALIZED PURE INJECTIVITY IN THE CONSTRUCTIBLE UNIVERSE

CHARLES MEGIBBEN

In this note we establish for each infinite cardinal m the consistency of the statement "every m-pure injective abelian group is algebraically compact." More precisely, we show that this statement can be proved if we assume Gödel's Axiom of Constructibility. All of our results depend strongly on this axiom and, as is customary, we tag our theorems with the parenthetical notation $(V = L)$ to indicate that they are derived in the $ZFC + V = L$ brand of set theory. It is actually Jensen's combinatorial principal $E(\kappa)$ which is our main tool, but we delay as long as possible introducing it.

Recall that a subgroup H of the abelian group G is said to be *m-pure* provided H is a direct summand of every intermediate subgroup K with $|K/H| < m$ and that an *m-pure injective* group is one which is a direct summand of every group containing it as an m-pure subgroup. Of course $\aleph_0$-purity is equivalent to ordinary purity (i.e., $nG \cap H = nH$ for all positive integers n) and the $\aleph_0$-pure injectives are just the well-known algebraically compact groups. In **[4]** we proved by an *ad hoc* argument that $\aleph_1$-pure injectives are algebraically compact, but were unable to obtain the same conclusion for any larger cardinals. In that paper we also found it convenient to consider certain weaker forms of generalized pure injectivity. In particular we called the group G an *(m, m)-pure injective* if it was a direct summand of any group K in which it appeared as an m-pure subgroup with $|K/G| = m$. That this concept is in fact genuinely weaker than m-pure injectivity was established in **[4]**. Indeed we exhibited there groups which were $(\aleph_1, \aleph_1)$-pure injective but not algebraically compact.

Throughout this note, m and κ denote infinite cardinals. By a *κ-filtration* of the group F we mean a family of subgroups $\{F_\alpha\}_{\alpha<\kappa}$ such that

(1) $\quad |F_\alpha| < \kappa$ and $F_\alpha \subseteqq F_{\alpha+1}$ for all α,

(2) $\quad F_\alpha = \bigcup_{\beta<\alpha} F_\beta$ if α is a limit ordinal, and

(3) $\quad F = \bigcup_{\alpha<\kappa} F_\alpha$.

A subset of the cardinal κ is said to be *stationary* if it has nonempty inter-

Received by the editors on July 7, 1980, and in revised form on March 13, 1981.

section with each subset of κ which is closed and unbounded in the order topology. We shall require the following result from **[1]**.

LEMMA ($V = L$). *Let κ be an uncountable regular cardinal.* If $\{F_\alpha\}_{\alpha<\kappa}$ *is a κ-filtration of F, G is an abelian gruop of cardinality at most κ and* $\{\alpha: \mathrm{Pext}(F_{\alpha+1}/F_\alpha, G) \neq 0\}$ *is stationary in κ, then* $\mathrm{Pext}(F, G) \neq 0$.

The crucial ingredient in our proof that m-pure injectives are algebraically compact is a result which is in all probability known since it can be proved in a rather straightforward fashion from readily available techniques. Since to our knowledge it does not appear in the literature, we shall at the end of this paper sketch a proof of the next theorem.

THEOREM 1. ($V = L$). *Let κ be an uncountable regular cardinal which is not weakly compact. Then there is an abelian group F which is not a direct sum of cyclic groups but does have a κ-filtration $\{F_\alpha\}_{\alpha<\kappa}$ where each F_α is a direct sum of cyclic groups and pure in F. Moreover, we can choose F to be torsion free with $\{\alpha: F_{\alpha+1}/F_\alpha \cong Q\}$ stationary in κ, or torsion with $\{\alpha: F_{\alpha+1}/F_\alpha \cong Q/Z\}$ stationary in κ.*

Our main result will now follow easily from the following theorem.

THEOREM 2 ($V = L$). *Let κ be an uncountable regular cardinal which is not weakly compact. Then a (κ, κ)-pure injective group of cardinality at most κ is algebraically compact.*

PROOF. Suppose G is (κ, κ)-pure injective and $|G| \leqq \kappa$. To show that G is algebracially compact we need only verify that $\mathrm{Ext}(Q, G) = 0$ and $\mathrm{Pext}(Q/Z, G) = 0$ (see Proposition 38.5 in **[2]**). To establish the second of these conditions, consider a torsion group F as in Theorem 1. Since every subgroup of F of cardinality less than κ is a direct sum of cyclic groups and therefore pure projective, G will be κ-pure in any group K in which it is pure and $K/G \cong F$. Thus $\mathrm{Pext}(F, G) = 0$ since G is (κ, κ)-pure injective. If, however, $\mathrm{Pext}(Q/Z, G) \neq 0$ we would then have a contradiction of Eklof's lemma above. That $\mathrm{Ext}(Q, G) = 0$ follows similarly by taking F to be a torsion free group satisfying the conditions of Theorem 1.

THEOREM 3 ($V = L$). *An m-pure injective abelian group is algebraically compact.*

PROOF. Suppose G is m-pure injective and let κ be the smallest uncountable regular cardinal which is not weakly compact and such that $\kappa \geqq m$ and $\kappa \geqq |G|$. Since κ-purity implies m-purity, G is a (κ, κ)-pure injective of cardinality not exceeding κ and therefore algebraically compact by Theorem 2.

Theorem 2 sheds light on other questions left unresolved in **[4]**. Since the theorem can be reformulated as saying that a (κ, κ)-pure injective which is not algebraically compact has cardinality at least κ^+, it is now clear why the non-algebraically compact $(\aleph_1, \aleph_1)$-pure injectives that we constructed there all had cardinality at least $2^{\aleph_1}$. Also it explains why we were unable to prove that an $(\aleph_1, \aleph_1)$-pure injective group G was necessarily $(\aleph_1, \aleph_2)$-pure injective, that is, a direct summand of any K with G $\aleph_1$-pure in K and $|K/G| \leqq \aleph_2$. Indeed in $ZFC + V = L$ this is false since we can construct a non-algebraically compact G which is $(\aleph_1, \aleph_1)$-pure injective and such that $|G| = 2^{\aleph_1}$. In $ZFC + V = L$ we have $2^{\aleph_1} = \aleph_2$ and hence this group cannot be $(\aleph_2, \aleph_2)$-pure injective and consequently not $(\aleph_1, \aleph_2)$-pure injective.

Finally, we sketch a proof of our Theorem 1 above. In $ZFC + V = L$ the combinatorial principle $E(\kappa)$ holds for such cardinals (see statement of Theorem 6.1 in [3]). Since the limit ordinals in a closed and unbounded set form once again such a set, this implies that there will be a stationary subset E of κ such that for each limit $\sigma < \kappa$, $E \cap \sigma$ is not stationary in σ. Moreover, from a careful scrutiny of the proof in [3], one sees that all the ordinals in E are limits cofinal with ω. For each $\alpha \in E$, let $\bar{\alpha} = \{\alpha_i\}_{i<\omega}$ be a strictly increasing sequence of ordinals with sup $\alpha_i = \alpha$ and $\alpha_i \notin E$. To construct a torsion-free F with the desired properties, we set $H_\alpha = Z$ for each $\alpha \in \kappa \backslash E$ and work within the direct product P of all such H_α's. For each $\alpha \in E$ we select a sequence $\{z_{\alpha, i}\}_{i<\omega}$ in P such that the support of each $z_{\alpha, i}$ lies in $\bar{\alpha}$ and $B_\alpha/A_\alpha \cong Q$ where A_α is the direct sum of the H_β's with $\beta \in \bar{\alpha}$ and B_α is generated by A_α and the $z_{\alpha, i}$'s. We then construct inductively an ascending sequence of subgroups F_α in P by taking $F_{\alpha+1} = F_\alpha \oplus H_\alpha$ if $\alpha \notin E$, $F_{\alpha+1} = F_\alpha + B_\alpha$ if $\alpha \in E$ and $F_\alpha = \bigcup_{\beta<\alpha} F_\beta$ whenever α is a limit ordinal. One can then prove inductively that the F_α's are pure in P and that $F_{\alpha+1}/F_\alpha \cong Q$ for all $\alpha \in E$. Now take F to be the union of the F_α's and observe that F is generated by the direct sum of the H_α's and all the $z_{\alpha, i}$'s. It is easy to show that each F_β with $\beta \notin E$ is a direct summand of F. Indeed whenever $(\alpha, i) \in E \times \omega$. We can write $z_{\alpha, i} = x_{\alpha, i} + y_{\alpha, i}$ where $x_{\alpha, i} \in F_\beta$ and the support of $y_{\alpha, i}$ involves only ordinals $\gamma \geqq \beta$. Then $F = F_\beta \oplus K_\beta$ where K_β is the subgroup of F generated by the $y_{\alpha, i}$'s and the H_γ's with $\gamma \geqq \beta$.

It remains to verify that F is not a free group, but that all the F_α's are free. Assume to the contrary that F is free say, $F = \bigoplus_{\alpha<\kappa} C_\alpha$ where each C_α is an infinite cyclic group. Under this assumption one readily checks that the set S of those α's with F_α a direct sum of some of the C_γ's forms a closed and unbounded subset of κ. But then the set of those α with $F_{\alpha+1}/F_\alpha$ free would contain S, contrary to the fact that $E = \{\alpha\colon F_{\alpha+1}/F_\alpha \cong Q\}$ is stationary in κ. To prove that each F_α is free we proceed by induction. Assume then that all the F_β's with $\beta < \alpha$ are free. If $\alpha = \beta + 1$ for $\beta \notin E$, then $F_\alpha = F_{\beta+1} = F_\beta \oplus H_\beta$ is free. If $\alpha = \beta + 1$ for some $\beta \in E$, then we have $F_\alpha = F_\beta + B_\beta$ with B_β countable. By a standard argument, we have a direct decomposition $F_\alpha = A \oplus C$ where A is a direct summand of F_β and C is countable. Since the group P is $\aleph_1$-free, the group C is free and consequently F_α is free in this case also. Finally, suppose α is a limit ordinal. Since $E \cap \alpha$ is not stationary in α, there is a function $\mathfrak{f}: \alpha \to \alpha$ whose range is closed and unbounded in α and disjoint from E. Then F_α is the union of a smooth chain of subgroups $\{L_\beta\}_{\beta<\alpha}$ where $L_\beta = F_{\mathfrak{f}(\beta)}$ and it suffices to argue that $L_{\beta+1}/L_\beta$ is free for all β in order to conclude that F_α itself is free. But since $\mathfrak{f}(\beta) \notin E$, we have the direct decomposition $F = L_\beta \oplus K_{\mathfrak{f}(\beta)}$. From this it follows $L_{\beta+1}/L_\beta$ is free since it is isomorphic to $L_{\beta+1} \cap K_{\mathfrak{f}(\beta)}$, a subgroup of the free group $L_{\beta+1} = F_{\mathfrak{f}(\beta+1)}$.

The proof for a torsion group F is essentially the same except that each H_α is taken to be the countable group $\oplus_{n=1}^{\infty} Z/nZ$ and, of course, the $z_{\alpha,i}$'s are chosen so that $B_\alpha/A_\alpha \cong Q/Z$.

REFERENCES

1. P. Eklof, *Homological algebra and set theory*, Trans. Amer. Math. Soc. **227** (1977), 207–225

2. L. Fuchs, *Infinite Ablian Groups*, *Vol I*, Academic Press, New York and London, 1970.

3. R. Jensen, *The fine structure of the constructible hierarchy*, Ann. Math. Logic **4** (1972), 229–308.

4. C. Megibben, *Generalized Pure Injectivity*, Symposia Math **13** (1974), 257–271.

VANDERBILT UNIVERSITY, NASHVILLE, TN 37235