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LEBESGUE AND STRONG UNICITY CONSTANTS FOR ZOLOTAREFF POLYNOMIALS

MYRON S. HENRY AND JOHN J. SWETITS

ABSTRACT. For each $f \in C(I)$, let $B_n(f)$ denote the best uniform polynomial approximation of degree less than or equal to n. If $f(x) = x^{n+2} - \sigma_n x^{n+1}$, then the Zolotareff polynomial of degree $n + 2$ is given by $Z_{n+2}(x) = x^{n+2} - \sigma_n x^{n+1} - B_n(f)(x)$, $x \in I$, where $0 \leqq \sigma_n \leqq (n + 2) \tan^2(\pi/2(n + 2))$. Let U_n represent the set of extreme points of Z_{n+2}; then $\mathrm{U} = \{\mathrm{U}_n\}_{n=0}^{\infty}$ is an infinite triangular array of nodes. If $\sigma_n > 0$ for each n, then it is shown that the orders of growth of the Lebesgue and strong unicity constants determined by U are precisely $n + 1$ and n^3, respectively. If $\sigma_n = 0$ for every n, these orders of growth are precisely $\log(n + 2)$ and n, respectively.

1. Introduction. Let $-1 \leqq x_0^n < x_1^n < \cdots < x_n^n < x_{n+1}^n \leqq 1$ be any $n + 2$ points in the interval $I = [-1, 1]$. Then

$$X_n = \{x_i^n\}_{i=0}^{n+1} \tag{1.1}$$

defines a set of nodes contained in I, and

$$X = \{X_n\}_{n=0}^{\infty} \tag{1.2}$$

is an infinite triangular array of nodes [**12**, p. 88]. Let

$$\{\ell_i^{(n)}(x)\}_{i=0}^{n+1} \tag{1.3}$$

be the fundamental Lagrange polynomials determined by (1.1) [**12**, p. 88]. The Lebesgue function of order $n + 1$ determined by X is then

$$\lambda_{n+1}(X, x) = \sum_{i=0}^{n+1} |\ell_i^{(n)}(x)|, \tag{1.4}$$

and the Lebesgue constant Λ_{n+1} of order $n + 1$ determined by X is defined by

$$\Lambda_{n+1}(X) = \max_{-1 \leqq x \leqq 1} \lambda_{n+1}(X, x), \tag{1.5}$$

[**12**, p. 89].

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A classical problem in approximation theory is to estimate the growth of $\Lambda_{n+1}(X)$ as a function of n and X.

Let $C(I)$ denote the space of real valued, continuous functions on the interval I, and let $\Pi_n \subseteqq C(I)$ be the space of polynomials of degree at most n. Denote the uniform norm on $C(I)$ by $\|\cdot\|$. For each $f \in C(I)$ with best approximation $B_n(f)$ from Π_n, there is a smallest constant $M_n(f) > 0$ such that for any $p \in \Pi_n$,

$$\|p - B_n(f)\| \leqq M_n(f)(\|f - p\| - \|f - B_n(f)\|). \tag{1.6}$$

Inequality (1.6) is the well known strong unicity theorem [**2**], and hereafter $M_n(f)$ is defined to be the strong unicity constant. Recently a number of papers [**1, 3, 5, 6, 8, 9, 10**] have examined the growth of the strong unicity constant

$$M_n(f) \tag{1.7}$$

as a function of n and f. Considering the similarity of the questions investigated regarding the growth of Lebesgue and strong unicity constants, it is natural to explore the behaviors of (1.5) and (1.7) on common infinite triangular arrays of nodes.

In this regard, Theorem 1 below gives the strong unicity constant in terms of an appropriate set of nodes. First, some preliminary notation is needed.

For $f \in C(I)$ with best approximation $B_n(f)$, the error function $e_n(f)$ is defined by $e_n(f)(x) = f(x) - B_n(f)(x)$, $x \in I$, and the set of extreme points of the error function is denoted by $E_n(f) = \{x \in I: |e_n(f)(x)| = \|e_n(f)\|\}$.

THEOREM 1. (BARTELT AND SCHMIDT [**1**].) *If $f \in C(I) - \Pi_n$, then*

$$M_n(f) = \max_{p \in \Pi_n} \{\|p\|: \operatorname{sgn} e_n(f)(x)\, p(x) \leqq 1 \textit{ for } x \in E_n(f)\}. \tag{1.8}$$

Theorem 1 demonstrates that $M_n(f)$ is also defined in terms of a set of nodes, namely the set of extreme points of $e_n(f)$. If $E_n(f)$ consists of precisely $n + 2$ points, Theorem 1 can be sharpened [**6**]. In particular, if $E_n(f)$ consists of

$$-1 \leqq x_0 < x_1 < \cdots < x_n < x_{n+1} \leqq 1, \tag{1.9}$$

and if $q_{in} \in \Pi_n$ satisfies

$$\begin{aligned} q_{in}(x_j) = \operatorname{sgn} e_n(f)(x_j),\ & j = 0, 1, \ldots, n + 1, \\ & i = 0, \ldots, n + 1, \\ & i \neq j, \end{aligned} \tag{1.10}$$

then

(1.11) $$M_n(f) = \max_{0 \leqq i \leqq n+1} \|q_{in}\|.$$

Now let $\{f_n\}_{n=0}^{\infty} \subseteqq C(I)$, and assume that $E_n(f_n)$ contains precisely $n + 2$ points $\{x_i^n\}_{i=0}^{n+1}$. Then

(1.12) $$E(F) = \{E_n(f_n)\}_{n=0}^{\infty}$$

is an infinite triangular array of nodes and consequently determines a Lebesgue constant $\Lambda_{n+1}[E(F)]$ in the manner outlined by (1.2)–(1.5). Similarly, for $n = 0, 1, \ldots,$ a strong unicity constant $M_n(f_n)$ is determined by (1.9) through (1.11).

The example that follows demonstrates the above theory. Let $G_n(x) = x^{n+1}$, $x \in I$. Then it is well known that

(1.13) $$e_n(G_n)(x) = (1/2^n)C_{n+1}(x),\ x \in I,$$

where C_{n+1} is the Chebyshev polymomial of degree $n+1$. Thus the set of extreme points $E_n(G_n)$ of $e_n(G_n)$ consists of the extreme points of C_{n+1}. Consequently, if $G = \{E_n(G_n)\}_{n=0}^{\infty}$, then **[4]**

(1.14) $$\Lambda_{n+1}(G) = O(\log(n + 1)).$$

On the other hand, it can be shown **[3, 5, 8]** that

(1.15) $$M_n(G_n) = 2n + 1.$$

Thus if G is the infinite triangular array of nodes whose n-th row consists of the $n + 2$ extreme points of C_{n+1}, then the growth of $\Lambda_{n+1}(G)$ and $M_n(G_n)$ are known precisely.

The principal objective of the present paper is to calculate the companions to (1.14) and (1.15) for the natural but somewhat more complex extensions of the Chebyshev polynomials, namely the Zolotareff polynomials **[11**, p. 41**]**.

2. Zolotareff polynomials.

DEFINITION 1. Let $\{\delta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ be sequences of positive real numbers, and let α and β be positive real numbers not depending on n. If there exists a natural number N such that for all $n \geqq N$,

(2.1) $$\alpha\gamma_n \leqq \delta_n \leqq \beta\gamma_n,$$

then δ_n is said to be of precise order γ_n.

We note from Definition 1, (1.14), and (1.15), that $\Lambda_{n+1}(G)$ is of precise order $\log(n + 1)$, and $M_n(G_n)$ is of precise order n. The goal of the present section is to establish the precise orders of the Lebesgue and strong unicity constants determined by infinite triangular arrays whose rows consist of the extreme points of certain Zolotareff polynomials.

Let

$$f_n(x) = x^{n+2} - \sigma_n x^{n+1},\ x \in I, \tag{2.2}$$

where

$$0 \leqq \sigma_n \leqq (n+2)\tan^2(\pi/2(n+2)). \tag{2.3}$$

Then it is known [11, p. 41] that the error function $e_n(f_n)$ satisfies

$$e_n(f_n)(x) = (1/2^{n+1})(1 + \sigma_n/(n+2))^{n+2}\, C_{n+2}\left(\frac{x - \sigma_n/(n+2)}{1 + \sigma_n/(n+2)}\right),\ x \in I, \tag{2.4}$$

where C_{n+2} is the Chebyshev polynomial of degree $n + 2$. Hereafter the right side of (2.4) is designated by Z_{n+2}, the $(n + 2)$nd degree Zolotareff polynomial. If $\sigma_n > 0$, there are precisely $n + 2$ extreme points

$$-1 = x_0^n < x_1^n < \cdots < x_n^n < x_{n+1}^n \leqq 1 \tag{2.5}$$

of (2.4) in this interval I, [11]. Furthermore, if

$$-1 = t_0^{n+2} < t_1^{n+2} < \cdots < t_{n+1}^{n+2} < t_{n+2}^{n+2} = 1 \tag{2.6}$$

are the $n + 3$ extreme points of C_{n+2}, then for $i = 0, \ldots, n + 1$,

$$t_i^{n+2} = \frac{x_i^n - \sigma_n/(n+2)}{1 + \sigma_n/(n+2)}. \tag{2.7}$$

For σ_n in (2.3) positive, let

$$E_n(f_n) = \{x_i^n\}_{i=0}^{n+1}, \tag{2.8}$$

and let

$$\mathbf{U} = \{E_n(f_n)\}_{n=0}^{\infty}. \tag{2.9}$$

Then $\mathbf{U}$ is an infinite triangular array of nodes of the type given by (1.12), and consequently $\mathbf{U}$ determines a Lebesgue and strong unicity constant. The first theorem of the present section establishes the growth rate of the strong unicity constant determined by $\mathbf{U}$.

THEOREM 2. *Let $\{f_n\}_{n=0}^{\infty}$ be as in* (2.2) *with error function* (2.4). *Assume σ_n in* (2.3) *is positive. Then the precise order of $M_n(f_n)$ is n^3.*

PROOF. For simplicity, the superscripts in (2.5), (2.6), (2.7), and (2.8) are henceforth assumed but are not explicitly displayed. Define $Q_{n+1} \in \Pi_{n+1}$ by

$$Q_{n+1}(x_i) = \operatorname{sgn} e_n(f_n)(x_i),\ i = 0, 1, \ldots, n + 1. \tag{2.10}$$

From (2.4) and (2.7) it follows that

$$Q_{n+1}(x) = C_{n+2}\left(\frac{x-\sigma_n/(n+2)}{1+\sigma_n/(n+2)}\right) - 2^{n+1}\left(\frac{1}{1+\sigma_n/(n+2)}\right)^{n+2} \prod_{j=0}^{n+1}(x - x_j). \tag{2.11}$$

Again appealing to (2.7) we have for $y = (x - \sigma_n/(n+2)/(1 + \sigma_n/(n+2))$ that

$$\begin{aligned} 2^{n+1}\left(\frac{1}{1+\sigma_n/(n+2)}\right)^{n+2} \prod_{j=0}^{n+1}(x - x_j) &= 2^{n+1}\prod_{j=0}^{n+1}(y - t_j) \\ &= (y+1)C'_{n+2}(y)/(n+2) \end{aligned} \tag{2.12}$$

Since $\max|C'_{n+2}(y)| \leqq \max_{-1\leqq y\leqq 1} |C'_{n+2}(y)| = (n+2)^2$, $-1 \leqq y \leqq (1 - \sigma_n/(n+2)/(1 + \sigma_n/(n+2))$, (2.11) and (2.12) imply that

$$\|Q_{n+1}\| \leqq 1 + 2(n+2). \tag{2.13}$$

For each $i = 0, 1, \ldots, n+1$ define q_{in} as in (1.10). Then

$$q_{in}(x) = Q_{n+1}(x) - a_{n+1} \prod_{\substack{j=0\\ j\neq i}}^{n+1} (x - x_j), \tag{2.14}$$

where a_{n+1} is the coefficient of x^{n+1} in Q_{n+1}. From (2.11) and (2.12) a direct calculation shows that

$$|a_{n+1}| = \left(\frac{2}{1+\sigma_n/(n+2)}\right)^{n+1}. \tag{2.15}$$

Utilizing (2.7), (2.14), and (2.15) results in

$$\begin{aligned} |q_{in}(x)| &\leqq |Q_{n+1}(x)| + 2^{n+1}\prod_{j=0}^{n+1} |y - t_j|/|y - t_i| \\ &= |Q_{n+1}(x)| + |(y+1)C'_{n+2}(y)/(n+2)(y - t_i)|, \end{aligned}$$

where again

$$y = \frac{x - \sigma_n/(n+2)}{1 + \sigma_n/(n+2)}. \tag{2.16}$$

Thus $|q_{in}(x)| \leqq |Q_{n+1}(x)| + (|y+1|/(n+2))|C''_{n+2}(\varepsilon)|$, where ε is between y and t_i, $i = 0, 1, \ldots, n$. Since $\|C''_{n+2}\| = O[(n+2)^4]$, this inequality and (2.13) imply that there is a β independent of n such that

$$|q_{in}(x)| \leqq \beta n^3,\ i = 0, 1, \ldots, n+1. \tag{2.17}$$

For $i = n+1$, (2.14) yields

$$|q_{n+1,n}(x)| \geqq |a_{n+1}| \prod_{j=0}^{n} |x - x_j| - |Q_{n+1}(x)|.$$

Therefore, if $x = x_{n+1}$,

$$|q_{n+1,n}(x_{n+1})| \geqq |a_{n+1}| \prod_{j=0}^{n} |x_{n+1} - x_j| - 1.$$

Utilizing (2.7), (2.15), and (2.16) in this inequality results in

$$
\begin{aligned}
|q_{n+1,n}(x_{n+1})| &\geqq 2^{n+1} \prod_{j=0}^{n} |t_{n+1} - t_j| - 1 \\
&= \left| \frac{C''_{n+2}(t_{n+1})}{n+2} \right| - 1 \qquad (2.18)\\
&= \left| \frac{C''_{n+2}(\cos(\pi/(n+2)))}{n+2} \right| - 1 \geqq \alpha n^3,
\end{aligned}
$$

where α does not depend on n and $\alpha \geqq 0$. Inequalities (2.17), (2.18), and equality (1.11) imply the conclusion of Theorem 2.

In the next theorem the precise order of $\Lambda_{n+1}(\mathrm{U})$ is established. First, a lemma that will facilitate the proof of Theorem 3 below is given.

LEMMA 1. *For $f \in C(I)$, suppose that $E_n(f)$ contains exactly $n + 2$ points $\{x_i\}_{i=0}^{n+1}$. Let Λ_{n+1}^j denote the Lebesgue constant determined by the infinite triangular array of nodes whose n-th row consists of the points $E_n^j(f) = E_n(f) - \{x_j\}$, $j = 0, 1, \ldots, n + 1$. Then*

$$M_n(f) = \max_{0 \leqq j \leqq n+1} \Lambda_{n+1}^j. \tag{2.19}$$

This lemma provides an interesting connection between the strong unicity constant and certain Lebesgue constants. The proof is given in [8].

THEOREM 3. *Let* U *be the infinite triangular array of nodes given by* (2.9). *Assume δ_n satisfying* (2.3) *is positive. Then the Lebesgue constant $\Lambda_{n+1}(\mathrm{U})$ is of precise order $n + 1$.*

PROOF. From (1.4), (2.7), (2.8), and (2.16) we have

$$\lambda_{n+1}(\mathrm{U}, x) = \sum_{i=0}^{n+1} \prod_{\substack{j=0\\ j\neq i}}^{n+1} \left| \frac{x - x_j}{x_i - x_j} \right| = \sum_{i=0}^{n+1} \prod_{\substack{j=0\\ j\neq i}}^{n+1} \left| \frac{y - t_j}{t_i - t_j} \right|.$$

Thus

$$\Lambda_{n+1}(\mathrm{U}) \leqq \max_{-1 \leqq y \leqq 1} \sum_{i=0}^{n+1} \prod_{\substack{j=0\\ j\neq i}}^{n+1} \left| \frac{y - t_j}{t_i - t_j} \right|. \tag{2.20}$$

Let $G_{n+1}(y) = y^{n+2}$, $y \in I$, and let $\Lambda_{n+2}(T)$ be the Lebesgue constant determined by the infinite triangular array T whose n-th row consists of the $n + 2$ points $\{t_0, t_1, \ldots, t_{n+1}\}$. Thus the n-th row of T consists of the first $n + 2$ extreme points of C_{n+2}. From (1.15) $M_{n+1}(G_{n+1}) = 2n + 3$. From Lemma 1 $\Lambda_{n+2}(T) \leqq 2n + 3$, and consequently (2.20) implies that

$$\Lambda_{n+1}(\mathrm{U}) \leqq 2n + 3. \tag{2.21}$$

Now define $Q_{n+1} \in \Pi_{n+1}$ by (2.10). From (1.4), (1.5), and (2.10) it is clear that

$$\|Q_{n+1}\| \leqq \Lambda_{n+1}(\mathbf{U}). \tag{2.22}$$

On the other hand, if y is given by (2.16), then for $x \in I$, $-1 \leqq y \leqq (1 - \sigma_n/(n + 2))/(1 + \sigma_n/(n + 2))$. From (2.3), (2.5), and (2.7) it follows that $\cos(\pi/(n + 2)) \leqq (1 - \sigma_n/(n + 2))/(1 + \sigma_n/(n + 2))$. Therefore $z_{n+1} = \cos(3\pi/(2n + 4)) \leqq (1 - \sigma_n/(n + 2))/(1 + \sigma_n/(n + 2))$. We note that $C_{n+2}(z_{n+1}) = 0$. Therefore (2.11), (2.12) and (2.16) imply that

$$\begin{aligned}
\|Q_{n+1}\| &\geqq |Q_{n+1}[(1 + \sigma_n/(n + 2))z_{n+1} + \sigma_n/(n + 2)] \\
&= |z_{n+1} + 1|\,|C'_{n+2}(z_{n+1})/(n + 2)| \\
&= |\cos(3\pi/2(n + 2)) + 1|\left|\frac{1}{\sin(3\pi/2(n + 2))}\right|.
\end{aligned}$$

This inequality implies that there exists a positive number α not depending on n such that $\|Q_{n+1}\| \geqq \alpha(n + 1)$. Now (2.22) implies that

$$\Lambda_{n+1}(\mathbf{U}) \geqq \alpha(n + 1). \tag{2.23}$$

Inequalities (2.21) and (2.23) are equivalent to the conclusion of Theorem 3.

Theorems 2 and 3 state that the strong unicity and Lebesgue constant determined by the infinite triangular array of nodes whose n-th row consists of the extreme points of the Zolotareff polynomial of degree $n + 2$ (with $\sigma_n > 0$ satisfying (2.3)) are of precise order n^3 and $n + 1$, respectively.

In light of the corresponding results for the strong unicity and Lebesgue constants determined by the infinite triangular array of nodes whose n-th row consists of the extreme points of C_{n+1} (precise order n and $\log(n + 1)$, respectively), the conclusions of Theorems 2 and 3 are perhaps somewhat surprising. If in (2.3) $\sigma_n = 0$ for all n, the conclusions of Theorems 2 and 3 are significantly modified. Theorem 4 and 5 below address these modifications.

3. The zero case. Let σ_n in (2.3) be zero for every n, and let $\mathbf{U}_0$ be the infinite triangular array of nodes whose n-th row consists of the extreme points of the corresponding Zolotareff polynomial of degree $n + 2$. From (2.2) and (2.4) we note for $\sigma_n = 0$ that $Z_{n+2}(x) = (1/2^{n+1})C_{n+2}(x)$, $x \in I$. Consequently the extreme points of Z_{n+2} are merely the $n + 3$ extreme points of C_{n+2}, and the n-th row of $\mathbf{U}_0$ now consists of the $n + 3$ extreme points of C_{n+2}.

Based on these observations and (1.14), it is immediate that

$$\Lambda_{n+1}(\mathbf{U}_0) = \Lambda_{n+2}(G) = O(\log(n + 2)). \tag{3.1}$$

The above analysis constitutes the proof of the next theorem.

THEOREM 4. *Let σ_n in (2.3) equal zero for each n, and let $\mathbf{U}_0$ be the infinite* *triangular array of nodes whose n-th row consists of the extreme points of* Z_{n+2}. *Then* $\Lambda_{n+1}(\mathrm{U}_0)$ *is of precise order* $\log(n + 2)$.

Let

$$\hat{f}_n(x) = x^{n+2}. \tag{3.2}$$

To establish the precise order of $M_n(\hat{f}_n)$ is somewhat more complex, primarily because the cardinality of $E_n(\hat{f}_n)$ is now $n + 3$, and consequently (1.11) cannot be directly employed to estimate $M_n(\hat{f}_n)$.

THEOREM 5. *If* $\hat{f}_n$ *is given by* (3.2), *then* $M_n(\hat{f}_n)$ *is of precise order* n.

PROOF. Since $E_n(\hat{f}_n)$ contains $n + 3$ points of alternation [2], it is clear that $E_n(\hat{f}_n) = E_{n+1}(\hat{f}_n)$. Thus (1.13), (1.15), and (3.2) imply that $M_{n+1}(G_{n+1}) = M_{n+1}(\hat{f}_n) = 2n + 3$. Theorem 1 now implies that $M_n(\hat{f}_n) \leqq 2n + 3$.

To complete the proof we must show that there exists an N and an $\alpha > 0$ not depending on n such that $\alpha n \leqq M_n(\hat{f}_n)$ for all $n \geqq N$. Let the extreme points $E_n(\hat{f}_n)$ of $e_n(\hat{f}_n)$ be labeled

$$-1 = x_0 < x_1 < \cdots < x_{n+1} < x_{n+2} = 1. \tag{3.3}$$

Define $q_n \in \Pi_n$ by

$$q_n(x_i) = \operatorname{sgn} e_n(\hat{f}_n)(x_i), \tag{3.4}$$

$i = 1, 2, \ldots, n + 1$. Since $\operatorname{sgn} e_n(\hat{f}_n)(x_i) = -\operatorname{sgn} e_n(\hat{f}_n)(x_{i+1})$, $i = 0, \ldots, n + 1$ it follows from (3.4) that

$$\operatorname{sgn} e_n(\hat{f}_n)(x_i) q_n(x_i) \leqq 1,\ i = 0, \ldots, n + 2. \tag{3.5}$$

Expressions (3.4), (3.5), and (1.8) now imply that

$$\|q_n\| \leqq M_n(\hat{f}_n). \tag{3.6}$$

Define $Q_{n+1} \in \Pi_{n+1}$ by

$$Q_{n+1}(x_i) = \operatorname{sgn} e_n(\hat{f}_n)(x_i), \tag{3.7}$$

$i = 0, 1, \ldots, n + 1$. If a_{n+1} is the coefficient of x^{n+1} in Q_{n+1}, it follows from (3.7) and the theorem of de la Vallée Poussin [2, 9] that

$$|a_{n+1}| \geqq 2^n. \tag{3.8}$$

Now (3.4) and (3.7) imply that

$$q_n(x) = Q_{n+1}(x) - a_{n+1} \prod_{j=1}^{n+1} (x - x_j). \tag{3.9}$$

Therefore (3.6), (3.7), (3.8), and (3.9) imply that

$$\begin{aligned} M_n(\hat{f}_n) &\geqq \|q_n\| \geqq |q_n(-1)| \\ &\geqq 2^n \prod_{j=1}^{n+1} |1 + x_j| - 1 \\ &= \frac{|C'_{n+2}(-1)|}{2(n+2)} - 1 \\ &= (n+2)/2 - 1 = n/2. \end{aligned}$$

REMARK. When compared with (1.14) and (1.15), Theorems 4 and 5 portray an expected phenomena. These two theorems tend to reinforce an earlier assertion that the results of Theorems 2 and 3 are somewhat surprising.

4. Conclusions and observations. In the preceeding sections, the precise orders of the Lebesgue and strong unicity constrants determined by the infinite triangular array of nodes whose n-th row consists of the extreme points of certain Zolotareff polynomials are calculated. In particular, the Zolotareff polynomial of degree $n + 2$ is

$$Z_{n+2}(x) = x^{n+2} - \sigma_n x^{n+1} - B_n(f)(x),\ x \in I,$$

where $0 \leqq \sigma_n \leqq (n + 2)\tan^2(\pi/2(n + 2))$. If σ_n is positive for every n, then the Lebesgue constant alluded to above is of precise order $n + 1$, and the strong unicity constant is of precise order n^3. If $\sigma_n = 0$ for all n, then these precise orders are $\log(n + 2)$ and n, respectively.

In a related paper [7], the authors have established the precise orders of the Lebesgue and strong unicity constants determined by (1.12) for a class of non-polynomial functions. In particular, if $f_n = f \in C^\infty(I)$ for $n = 0, 1, 2, \ldots$, and if f satisfies a certain derivative condition [7], then the Lebesgue constant determined by (1.2) is of precise order $\log(n + 1)$, and the strong unicity constant is of precise order n.

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Department of Mathematics, Central Michigan University, Mount Pleasant, MI 48859

Department of Mathematical Sciences, Old Dominion University, Norfolk, VA 23508