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A CLOSED GRAPH THEOREM FOR BANACH BUNDLES

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ABSTRACT. Let (E, p, X) and (F, q, Y) be Banach bundles. A bundle transformation from (E, p, X) into (F, q, Y) is a function $F : E \to F$ for which there is a base map $f_0 : X \to Y$ with $f_0 \circ p = q \circ f$ and for which $f_x \equiv f|_{p^{-1}(x)}$ is a linear transformation from $p^{-1}(x)$ into $q^{-1}(f_0(x))$ for each x in X. It is proved that if $f : E \to F$ is a bundle transformation which preserves relatively compact sets and X is locally compact, then f is continuous if and only if the graph of f is closed. An example is presented in order to show that there is no general open mapping theorem for Banach bundles.

Introduction. Basic theorems such as the closed graph theorem and the open mapping theorem are extremely useful in the study of Banach spaces. The purpose of this note is to investigate the existence of these theorems for Banach bundles. Specifically a closed graph theorem for Banach bundles is proved and an example is presented in order to show that there is no general open mapping theorem for Banach bundles.

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DEFINITION 1. A Banach bundle is a triple (E, p, X) where E and X are topological spaces; p: $E \to X$ is an open surjective map such that each stalk $p^{-1}(x)$ is the under-lying set for a Banach space and

(a) addition is continuous as a map from $E \vee E$ into E where $E \vee E = \{(\zeta_1, \zeta_2) \in E \times E: p(\zeta_1) = p(\zeta_2)\}$ has the subspace topology inherited from $E \times E$;

(b) scalar multiplication is continuous as a mapping: $\mathbf{C} \times E \to E$;

(c) the map: $E \to R$ defined by the norm on each stalk is upper semicontinuous;

(d) if $x \in X$, $0(x)$ is the zero of the Banach space $p^{-1}(x)$, and G is open in E, $0(x) \in G$, then there is a positive number δ and an open set V contained in X such that $x \in V$ and $p^{-1}(V) \cap \{\zeta \in E: \|\zeta\| < \delta\} \subset G$.

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The space X is called the base space. We will assume that all base spaces are completely regular Hausdorff.

In the case that each stalk is also a Hilbert space, (E, p, X) is said to be a Hilbert bundle.

In [4] the map: $E \to R$ given by the norm on each stalk is assumed to be continuous. There is no advantage topologically in assuming that the norm is continuous. The assumption also limits the choice of norms for the bundle. Condition (d) is a very strong topological requirement and this is primarily the reason that in condition (c) the upper semi-continuous requirement suffices.

DEFINITION 2. Let (E, p, X) be a Banach bundle. A selection is a function $\sigma: X \to E$ such that $p \circ \sigma$ is the identity mapping on X. A continuous selection is called a section (or global section). The set of all sections will be denoted by $\Sigma(E)$. A bundle (E, p, X) is said to be full provided for each $\zeta \in E$ there is a section σ such that $\sigma(p(\zeta)) = \zeta$.

In this note all bundles will be assumed to be full. It has been shown by Douady and Dal Soglio-Herault [5] that if the base space is locally paracompact, then the bundle is full.

DEFINITION 3. Let (E, p, X) and (F, q, Y) be Banach bundles. A bundle transformation from (E, p, X) into (F, q, Y) is a function $f: E \to F$ for which there exists a base map $f_0: X \to Y$ making the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f_0]{} & Y \end{array}$$

commutative and for which $f_x \equiv f|_{p^{-1}(x)}$ is a linear transformation from $p^{-1}(x)$ into $q^{-1}(f_0(x))$ for each x in X. If f is continuous, f is said to be a bundle map.

A bundle transformation (map) from (E, p, X) into (F, q, Y) can also be defined as a (continuous) function $f: E \to F$ such that for x in X the map f_x is a linear transformation from $p^{-1}(x)$ into $q^{-1}(y)$ for some y in Y. Note that the map f_0 is given by $f_0(x) = q \circ f \circ 0_E(x)$ for $x \in X$, where 0_E denotes the zero section. Since the zero section is continuous, f_0 is continuous whenever f is continuous.

Also observe that X is homeomorphic to $0_E(X)$ and that $0_E \circ p$ is a retraction of E onto $0_E(X)$.

If f is a bundle transformation from (E, p, X) into (F, q, Y), the symbol $\|f\|$ will signify the $\sup\{\|f(\zeta)\| : \zeta \in E, \|\zeta\| < 1\}$ where $\|f(\zeta)\|$ is the norm of $f(\zeta)$ as an element of the Banach space $q^{-1}(q(f(\zeta)))$. If $\|f\|$ is finite, then f is said to be bounded. Observe that for each x in X, f_x is a linear transformation of $p^{-1}(x)$ into $q^{-1}(f_0(x))$ and thus has the usual norm of a linear transformation, $\|f_x\|$. From the definition of $\|f\|$ it follows easily that $\sup\{\|f_x\| : x \in X\} = \|f\|$.

DEFINITION 4. Let (E, p, X) and (F, q, Y) be Banach bundles. A bundle transformation f: $E \to F$ is said to be locally bounded if for each x in X there is a neighborhood N of x such that the set $\{\|f_t\| : t \in N\}$ is bounded. The following results (Theorems 5–10) will be used to prove a closed graph theorem for Banach bundles (Theorem 11).

THEOREM 5. *Let (E, p, X) and (F, q, Y) be Banach bundles with X locally compact and f: $E \to F$ a bundle transformation which preserves relatively compact sets. Then f is locally bounded.*

PROOF. Let $x_0 \in X$. There is a compact neighborhood S of x_0 contained in X. Suppose $\{\|f_x\|: x \in S\}$ is not bounded. Then there is a sequence $\{\zeta_n\}$ contained in $p^{-1}(S)$ such that $\|\zeta_n\| \leqq 1$ and $\|f(\zeta_n)\| \geqq n^2$ for each positive integer n.

CLAIM. The set $G = 0_E(S) \cup \{(1/n)\zeta_n: n = 1, 2, \dots\}$ is a compact subset of E.

Let $\mathfrak{U}$ be an open cover of G. Then for each x in S there is an element U_x in $\mathfrak{U}$ such that $0_E(x) \in U_x$. By (d) of Definition 1, for each x in S, there is a positive number δ_x and an open set V_x contained in X such that $x \in V_x$ and

$$p^{-1}(V_x) \cap \{\zeta \in E: \|\zeta\| < \delta_x\} \subset U_x.$$

Since S is compact there is a finite set $\{x_j: 1 \leqq j \leqq N\}$ such that $S \subset \bigcup\{V_{x_j}: j = 1, 2, \dots, N\}$. Let $\delta = \min\{\delta_{x_j}: j = 1, 2, \dots, N\}$. Then $0_E(S) \subset \{\zeta \in p^{-1}(S): \|\zeta\| < \delta\} \subset \bigcup\{p^{-1}(V_{x_j}) \cap \{\zeta \in E: \|\zeta\| < \delta\}: j = 1, 2, \dots, N\} \subset \bigcup\{U_{x_j}: j = 1, 2, \dots, N\}$. Next, since $\|(1/n)\zeta_n\| = (1/n)\|\zeta_n\| \leqq 1/n$ for each positive integer n, all but a finite number of the elements of the sequence $\{(1/n)\zeta_n\}$ are contained in $\bigcup\{U_{x_j}: j = 1, 2, \dots, N\}$. Thus G is compact.

Now by assumption $f(G)$ is relatively compact and hence bounded. But for each positive integer n, $\|f((1/n)\zeta_n)\| = (1/n)\|f(\zeta_n)\| \geqq (1/n)\, n^2 = n$. This is obviously a contradiction. Thus the set $\{\|f_x\| : x \in S\}$ is bounded and the transformation f is locally bounded.

LEMMA 6. *Let (E, p, X) be a Banach bundle and $\sigma \in \Sigma(E)$. Then the map $T_\sigma: E \to E$ defined by $T_\sigma(\zeta) = \zeta + \sigma(p(\zeta))$ is a homeomorphism of E onto E.*

PROOF. Let $\sigma \in \Sigma(E)$. Denote the selection which maps an x in X to the element $-\sigma(x)$ in $p^{-1}(x)$ by $-\sigma$. By (b) of Definition 1, the selection $-\sigma$ is continuous and hence is a section. It is easily seen that $T_\sigma \circ T_{-\sigma} =$

$T_{-\sigma} \circ T_\sigma = \mathrm{id}_E$, where id_E denotes the identity mapping on E. Thus T_σ is a bijective map. It follows from (a) of Definition 1 that T_σ and $T_{-\sigma}$ are continuous.

THEOREM 7. *Let (E, p, X) and (F, q, Y) be Banach bundles and f: $E \to F$ a locally bounded bundle transformation. Then f is continuous if and only if the restriction $f|_{\sigma(X)}$ is continuous for each section σ in $\Sigma(E)$.*

PROOF. If f is continuous, then it is clear that $f|_{\sigma(X)}$ is continuous for each σ in $\Sigma(E)$.

Suppose $f|_{\sigma(X)}$ is continuous for each σ in $\Sigma(E)$. Let $\zeta_0 \in E$, $x_0 = p(\zeta_0)$ and H be an open subset of F containing $f(\zeta_0)$. Since all bundles are assumed to be full, there is an element τ in $\Sigma(F)$ such that $\tau \circ q(f(\zeta_0)) = f(\zeta_0)$. Then by Lemma 6 and (d) of Definition 1 there is a positive number δ and an open set W contained in Y such that $q(f(\zeta_0)) \in W$ and $q^{-1}(W) \cap \{\eta \in F\colon \|\eta - \tau \circ q(\eta)\| < \delta\} \subset H$. There is a section σ in $\Sigma(E)$ such that $\sigma(x_0) = \zeta_0$. Then $(f \circ \sigma - \tau \circ f_0)(x_0) = f \circ \sigma(x_0) - \tau \circ f_0(x_0) = f(\zeta_0) - \tau \circ q(f(\zeta_0)) = f(\zeta_0) - f(\zeta_0) = 0$. By (c) of Definition 1 the map given by the norm on each stalk of F is upper semi-continuous and hence the set $\{\eta \in F\colon \|\eta\| < \delta/2\}$ is open in F. Observe that $f_0 = q \circ f \circ 0_E$ and thus f_0 is continuous. Then

$$\begin{aligned} V_1 &= (f \circ \sigma - \tau \circ f_0)^{-1}(\{\eta \in F\colon \|\eta\| < \delta/2\}) \\ &= \{x \in X\colon \|f \circ \sigma(x) - \tau \circ f_0(x)\| < \delta/2\} \end{aligned}$$

is a neighborhood of x_0.

Since f is locally bounded, there is a neighborhood V_2 of x_0 and a positive constant M such that $\|f_x\| < M$ for all x in V_2. Let $V = V_1 \cap V_2 \cap f_0^{-1}(W)$. Observe that x_0 is in V.

CLAIM. $f(p^{-1}(V) \cap \{\zeta \in E\colon \|\zeta - \sigma \circ p(\zeta)\| < \delta/2M\}) \subset q^{-1}(W) \cap \{\eta \in F\colon \|\eta - \tau \circ q(\eta)\| < \delta\}$ which is contained in H.

To see this let $\zeta \in p^{-1}(V)$ such that $\|\zeta - \sigma \circ p(\zeta)\| < \delta/2M$. Note $q(f(\zeta)) = f_0(p(\zeta)) \in f_0(V) \subset W$ and thus $f(\zeta) \in q^{-1}(W)$. Now

$$\begin{aligned} &\|f(\zeta) - \tau \circ q(f(\zeta))\| \\ &\quad = \|f(\zeta) - f \circ \sigma(p(\zeta)) + f \circ \sigma(p(\zeta)) - \tau \circ q(f(\zeta))\| \\ &\quad \leqq \|f(\zeta) - f \circ \sigma(p(\zeta))\| + \|f \circ \sigma(p(\zeta)) - \tau \circ f_0(p(\zeta))\| \\ &\quad < M\,\|\zeta - \sigma(p(\zeta))\| + \delta/2 < \delta/2 + \delta/2 = \delta. \end{aligned}$$

Thus f is continuous.

It is well known that a linear transformation from one Banach space into another is continuous if and only if it is bounded. This is not true for bundle transformations (see Examples B and C). However the following statement is true for bundle transformations.

THEOREM 8. *Let (E, p, X) and (F, q, Y) be Banach bundles with X compact and $f\colon E \to F$ a bundle transformation. Then f is continuous if and only if f is bounded and $f|_{\sigma(X)}$ is continuous for each σ in $\Sigma(E)$.*

PROOF. Assume f is continuous. Then it is obvious that $f|_{\sigma(X)}$ is continuous for each σ in $\Sigma(E)$. Also since f is continuous, f preserves relatively compact sets. Then we observe that Theorem 5 is true with "compact" in place of "locally compact" and "bounded" in place of "locally bounded".

The converse follows from Theorem 7.

The next theorem is found in [9].

THEOREM 9. *Let X be a topological space, Y a Hausdorff space, and $f\colon X \to Y$ a continuous function. Then the graph of f is closed.*

In [1], Theorem 2.7, the following topological version of a closed graph theorem is found.

THEOREM 10. *Let X be a Hausdorff space and Y a compact Hausdorff space. Then a function $f\colon X \to Y$ is continuous if and only if the graph of f is closed.*

Finally we are able to prove a closed graph theorem for Banach bundles. We will use $G(f)$ to signify the graph of a function f.

THEOREM 11. *Let (E, p, X) and (F, q, Y) be Banach bundles with X locally compact, E and F Hausdorff, and $f\colon E \to F$ a bundle transformation which preserves relatively compact sets. Then f is continuous if and only if the graph of f is closed.*

PROOF. Suppose that f is continuous. Since F is Hausdorff, Theorem 9 implies that the graph of f is closed.

Assume $G(f)$ is closed. By Theorem 5, the bundle transformation f is locally bounded. Thus by Theorem 7, it is sufficient to prove that $f|_{\sigma(X)}$ is continuous for each $\sigma \in \Sigma(E)$.

Let $\sigma \in \Sigma(E)$ and $x_0 \in X$. Then there is a compact neighborhood S of x_0 contained in X. The set $\sigma(S)$ is a continuous image of S and hence is compact. Thus $f|_{\sigma(S)}\colon \sigma(S) \to \mathrm{cl}(f(\sigma(S)))$ is a map from a Hausdorff space into a compact, Hausdorff space. Observe that $G(f|_{\sigma(S)}) = G(f) \cap (\sigma(S) \times F)$ which is closed in $E \times F$. Thus by Theorem 10, $f|_{\sigma(S)}\colon \sigma(S) \to \mathrm{cl}(f(\sigma(S)))$ is continuous and hence f is continuous.

This note is concluded with several examples.

EXAMPLE A. Let $X = [-1, 1]$, $E = [(X - \{0\}) \times \mathbf{C}] \cup \{(0, 0)\}$ and let the map $p\colon E \to X$ be given by $p(x, \alpha) = x$ for $(x, \alpha) \in E$. Give $X \times \mathbf{C}$ the product topology and let E have the relative topology as a subset of

$X \times \mathbf{C}$. For $(x, \alpha) \in E$ define the norm of (x, α) by $\|(x, \alpha)\| = |\alpha|$, where $|\alpha|$ is the absolute value of the complex number α. Then (E, p, X) is a Hilbert bundle (also a C^*-bundle). Define the function $f: E \to E$ by

$$f(x, z) = \begin{cases} (x, z/x) \text{ if } (x, z) \neq (0, 0) \\ (0, 0) \text{ if } (x, z) = (0, 0) \end{cases}$$

where $(x, z) \in E$. The function f is a bundle transformation that is locally bounded at each point of X except the origin. Observe that the set $\{(x, x) \in E: x \in X\}$ is compact, but that the image of this set under f (namely $\{(x, 1) \in E: x \neq 0\} \cup \{(0, 0)\}$ is not relatively compact.

EXAMPLE B. Let X be the space of real numbers and give $E = X \times \mathbf{C}$ the product topology. Let p be the projection of E onto X. Then (E, p, X) is a Hilbert bundle. (The construction of the bundle is similar to that of Example A.) Define $f: E \to E$ by $f(x, z) = (x, xz)$ for $(x, z) \in E$. The bundle transformation f is continuous but is not bounded.

EXAMPLE C. Let $E = X \times \mathbf{C}$ where X is the closed interval $[-1, 1]$ with its usual topology. Give $X \times \mathbf{C}$ the product topology and let p be the projection of E onto X. Then (E, p, X) is a Hilbert bundle. (The construction of this bundle is also similar to that of Example A.) Define $f: E \to E$ by

$$f(x, z) = \begin{cases} (x, xz) & \text{if } x \neq 0 \\ (0, z) & \text{if } x = 0 \end{cases}$$

where $(x, z) \in E$. This bundle transformation is bounded but not continuous.

EXAMPLE D. Let $X = [-1, 1]$, $E = X \times \mathbf{C}$, and let $p: E \to X$ be the projection. Next let R be the equivalence relation on E whose cosets are the fibers $E_x = \{x\} \times \mathbf{C}$ for $x \neq 0$ and are singleton otherwise. Now set $F = E/R$ and let $q: F \to X$ be the induced map given by $q(R(x, z)) = x$. Then (F, q, X) is a Banach bundle for which the base space X is compact Hausdorff and the bundle space F is not Hausdorff. Indeed F has only one non-zero fiber, namely, the one above 0 and no pair of points on this fiber can be separated by disjoint neighborhoods. This bundle has an upper semicontinuous norm which is not continuous. The function $f: E \to F$ given by $f(x, z) = R(x, z)$ is an open bundle map. However $G(f)$ contains the sequence $((1/n, 0), R(1/n, 0))$ but does not contain the point $((0, 0), R(0, 1))$ which is a cluster point of the sequence. Thus the graph of f is not closed.

It was hoped that a general open mapping theorem could be proved for

Banach bundles. However the following example shows that such a theorem is not possible without severe restrictions on the bundle map.

EXAMPLE E. Let (E, p, X) be the Hilbert bundle given in Example A. Let the map $f: E \to E$ be given by $f(x, z) = (x, xz)$ for $(x, z) \in E$. Then f is a continuous, bijective bundle transformation over a compact base space. However

$$f([-1, 1] \times \{z \in \mathbf{C}: |z| < 1\} \cap E) \\ = \{(z, w) \in E: |w| < |x|, x \neq 0\} \cup \{(0, 0)\}$$

which does not contain a neighborhood of the origin. Thus the map f is not open.

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