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THE DUNFORD-PETTIS PROPERTY OF SOME SPACES OF AFFINE VECTOR-VALUED FUNCTIONS

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ABSTRACT. Let K be a Choquet simplex, E be a Banach space, let $C(K, E)$ denote the Banach space of all continuous E-valued functions defined on K with supnorm, and let $A(K, E)$ be the subspace of $C(K, E)$ consisting of affine functions. We show that $A(K, E)$ has the Dunford-Pettis property whenever $C(K, E)$ has the same property. We also exhibit a compact convex set C that is neither a Choquet simplex, nor a dual unit ball of a Banach space with the Dunford-Pettis property such that $A(C, \mathbf{R})$ has the Dunford-Pettis property.

Introduction. Let K be a compact convex subset of a locally convex Hausdorff space, and let E be a real or complex Banach space. In this paper we investigate the *Dunford-Pettis property* of $A(K, E)$, the space of all continuous and affine E-valued functions defined on a Choquet simplex K. This study is motivated by the fact that when K is a Choquet simplex, it is well known **[10]** that $A(K, \mathbf{R})^*$ is linearly isometric to an L^1-space, thus $A(K, \mathbf{R})$ has the Dunford-Pettis property. This raises the following interesting problem.

Problem. For a Choquet simplex K, and for a Banach space E, does $A(K, E)$ have the Dunford-Pettis property whenever E does?

It turns out that the above problem is closely related to another still open problem of whether the space $C(\Omega, E)$ of all continuous E-valued functions defined on a compact Hausdorff space Ω has the Dunford-Pettis property whenever E does. In this paper, we will show that, for a Choquet simplex K, the space $A(K, E)$ has the Dunford-Pettis property whenever $C(K, E)$ has the Dunford-Pettis property.

We also observe that there are compact convex sets Z such that $A(Z, \mathbf{R})$ does not have the Dunford-Pettis property. For it can be shown that a *real* Banach space V has the Dunford-Pettis property if and only if the space $A(B(E^*), \mathbf{R})$ of all continuous and affine functions on the unit ball

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of E^*, has the Dunford-Pettis property. Finally, we exhibit a compact convex set C that is neither a Choquet simplex nor a dual unit ball of a Banach space with the Dunford-Pettis property such that $A(C, \mathbf{R})$ has the Dunford-Pettis property.

1. Definitions and preliminaries. If V is a real or complex Banach space, we shall denote by V^* the topological dual of V.

If Ω is a compact Hausdorff space, and if E is a Banach space, we shall denote by $C(\Omega, E)$ the Banach space of all continuous E-valued functions on Ω under the supremum norm.

If K is a compact convex subset of a locally convex Hausdorff space, and E is a Banach space, the symbol $A(K, E)$ will stand for the (closed) subspace of $C(K, E)$ consisting of affine functions.

In this paper we shall mostly consider Choquet simplexes, we shall mainly use the characterization given in **[8]**. For a detailed study of Choquet simplexes and other equivalent definitions we refer the reader to **[2]**, **[7]**, and **[8]**.

2. The Dunford-Pettis property for $A(K, E)$.

DEFINITION 2.1. A Banach space V has the *Dunford-Pettis prcperty* (DP) if whenever (x_n) and (x_n^*) are weakly null sequences in V and V^* respectively, then

$$\lim_n x_n^*(x_n) = 0.$$

The best-known spaces with the Dunford-Pettis property are $L^1(\mu)$ spaces, $C(\Omega)$ spaces, where Ω is a compact Hausdorff space, Banach spaces with the Schur property (i.e., weakly compact sets are norm compact) and complemented subspaces of Banach spaces with (DP).

The next theorem reduces the study of the Dunford-Pettis property of $A(K, E)$ to that of $C(K, E)$.

THEOREM 2.2 *Let K be a Choquet simplex, and let E be a Banach space, then $A(K, E)$ has the Dunford-Pettis property whenever $C(K, E)$ does.*

PROOF. Suppose that $C(K, E)$ has (DP). Let $(a_n)_{n\geq 1}$ and $(\ell_n)_{n\geq 1}$ be sequences in $A(K, E)$ and $A(K, E)^*$, respectively, such that $\lim_n a_n = 0$ weakly and $\lim_n \ell_n = 0$ weakly. Since K is a Choquet simplex it follows from **[8]** that there exists an isometric linear selection mapping S from $A(K, E)^*$ into $C(K, E)^*$, i.e., S: $A(K, E)^* \to C(K, E)^*$ is linear, $S(\ell) = \ell$ on $A(K, E)$ and $\|S(\ell)\| = \|\ell\|$ for each ℓ in $A(K, E)^*$. It follows that $\lim_n S(\ell_n) = 0$ weakly in $C(K, E)^*$. Also, since $\lim_n a_n = 0$ weakly in $A(K, E)$, $\lim_n a_n = 0$ weakly in $C(K, E)$. Hence, since $C(K, E)$ is supposed to have the Dunford-Pettis property, we have

$$\lim_n \ell_n(a_n) = \lim_n S(\ell_n)(a_n) = 0.$$

This completes the proof.

It is known [4] that if E has the Schur property or if $E = L^1$ [1], and Ω is a compact Hausdorff space, then $C(\Omega, E)$ has the Dunford-Pettis property. The following corollary is now immediate.

COROLLARY 2.3. *If K is a Choquet simplex and E is a Banach space that is either L^1 or has the Schur property, then $A(K, E)$ has the Dunford-Pettis property.*

When the Banach space E is a real or complex predual of an L^1-space one can say more about the space $A(K, E)$, namely we have the following theorem.

THEOREM 2.4. *If K is a Choquet simplex and E is a real (resp. complex) predual of an L^1-space, then $A(K, E)$ is a real (resp. complex) predual of an L^1-space. In particular $A(K, E)$ has the Dunford-Pettis property.*

PROOF. Let E be a real (resp. complex) Banach space such that E^* is isometrically isomorphic to an L^1-space. Without loss of generality one can assume that E^* is a real (resp. complex) L^1-space. It is known [9] that the dual of $C(K, E)$ is isometrically isomorphic to the Banach space $M(K, E^*)$ of all w^*-regular E^*-valued measures m defined on Σ the σ-field of Borel subsets of K, and that are of bounded variation with $\|m\| = |m|(K)$ where $|m|$ denotes the variation of m [3]. Since E^* is an L^1-space, the space $M(K, E^*)$ is easily checked to be an ordered linear space under the order defined as follows: for $m \in M(K, E^*)$, $m \geqq 0$ if $m(B) \geqq 0$ for all $B \in \Sigma$. Moreover, if $m \in M(K, E^*)$, one can define the absolute value of m as follows. Let $B \in \Sigma$ and let $\pi = (B_i)_{i \leqq n}$ denote a finite Borel partition of B. For each $B_i \in \pi$ let $|m(B_i)|$ denote the absolute value of $m(B_i)$ in E^*. Since E^* is an L^1-space, for each partition $\pi = (B_i)_{i \leqq n}$ of B we have

$$\Big\| \sum_{B_i \in \pi} |m(B_i)| \Big\| = \sum_{i=1}^{n} \Big\| m(B_i) \Big\| \leqq |m|(B).$$

Hence the family

$$\Big\{ \sum_{B_i \in \pi} |m(B_i)| \; : \; B = \bigcup B_i \Big\}_\pi$$

is a norm bounded family in E^* and is easily seen to be directed upward. Since E^* is an L^1-space, we can define the element $|m|_a(B)$ of E^* as follows:

$$|m|_a(B) = \sup_\pi \sum_{B_i \in \pi} |m(B_i)|$$

and

$$\left\| \, |m|_a(B) \right\| = \sup_{\pi} \left\| \sum_{B_i \in \pi} |m(B_i)| \right\|$$
$$= \sup_{\pi} \sum_{B_i \in \pi} \left\| m(B_i) \right\| = |m|(B).$$

Hence $|m|_a$ is a well defined set function on Borel subsets of K and takes its values in E^*, and if we compute its variation, we get $|\,|m|_a|(B) = |m|(B)$, for each $B \in \Sigma$. This shows that $|m|_a \in M(K, E^*)$ whenever $m \in M(K, E^*)$. Hence $M(K, E^*)$ is a Banach lattice. Moreover, it can easily be verified that the variation norm is additive on the positive elements. Thus $M(K, E^*)$ is itself linearly isometric to an L^1-space. Since K is a Choquet simplex, it follows from **[8]** that $A(K, E)^*$ is isometrically isomorphic to a closed linear subsapce of $C(K, E^*) \cong M(K, E^*)$ and that there exists a contractive linear projection from $M(X, E^*)$ onto $A(K, E)^*$. Thus, it follows from **[5]** that $A(K, E)^*$ is itself linearly isometric to an L^1-space. This completes the proof. The last assertion of Theorem 2.4 follows from the fact that if the dual V^* of a Banach space V has the Dunford-Pettis property then so does V.

3. More spaces with the Dunford-Pettis property. Theorem 2.4 shows that if K is a Choquet simplex then $A(K, \mathbf{R})$ has (DP). This raises the following question: Besides Choquet simplexes, for what compact convex set K, does $A(K, \mathbf{R})$ have (DP)?

Let us first observe that if V is a *real* Banach space, then a simple application of the Hahn-Banach theorem shows that the Banach space V is linearly isometric to $A_0(B(V^*))$, the Banach space of all affine and continuous real valued functions defined on the unit ball of V^*, and that are zero at the zero functional. Moreover, we have the following proposition.

PROPOSITION 3.1. *Let V be a real Banach space, then $A(B(V^*), \mathbf{R})$ is linearly isometric to $V \oplus_{\ell^1} \mathbf{R}$.*

Hence $A(B(V^*), \mathbf{R})$ has the Dunford-Pettis property if and only if V does. In particular, this shows that there are compact convex sets K such that $A(K, \mathbf{R})$ does not have the Dunford-Pettis property. These observations motivated the search for a compact convex set C that is neither a Choquet simplex nor the unit ball of a dual of a Banach space with the Dunford-Pettis property, such that $A(C, \mathbf{R})$ has (DP). The first step in this direction is given next.

Let X_1 and X_2 be two compact convex sets. Denote by $BA(X_1 \times X_2)$ (or simply BA) the Banach space of all continuous functions that are affine for each variable (separately). It is well known that BA separates the points of the compact convex set $X_1 \times X_2$, and contains the constant functions.

LEMMA 2.9. *If K is a compact convex set and E is a Banach space, then $A(K, E)$ embeds isometrically into $BA(K \times B(E^*))$. Moreover if E is a real Banach space, the space $A(K, E)$ considered as a subspace of $BA(K \times B(E^*))$ is complemented in $BA(K \times B(E^*))$ with complement $A(K, \mathbf{R})$ and $BA(K \times B(E^*))$ is linearly isomorphic to $A(K, E) \oplus_{\ell^1} A(K, \mathbf{R})$.*

PROOF. Let I: $A(K, E) \to BA(K \times B(E^*))$ be defined as follows: for a in $A(K, E)$,

$$I(a)(x, x^*) = x^*(a(x)) \text{ for each } (x, x^*) \text{ in } K \times B(E^*).$$

It is easily checked that I defines an embedding of $A(K, E)$ into $BA(K \times B(E^*))$. Moreover, if E is a real Banach space, we can define the following map P: $BA(K \times B(E^*)) \to A(K, E)$, where for each b in $BA(K \times B(E^*))$ we let $Pb(x)(x^*) = (b(x, x^*) - b(x, -x^*)/2$, for each x in K and x^* in $B(E^*)$. Note that for each x in K and b in $BA(K \times B(E^*))$, $Pb(x)$ is in $A_0(B(E^*))$, and hence it is in E, since E is identified with $A_0(B(E^*))$. This shows that P is well defined and takes its values in $A(K, E)$. Moreover it is easy to check that P is a linear projection and that $\|P\| = 1$. An element b in BA is in the complement of $A(K, E)$ if $Pb = 0$; that is, for each x^* in $B(E^*)$ and for each x in K, $(b(x, x^*) - b(x, -x^*))/2 = 0$. It follows that $b(x, x^*) = (b(x, x^*) + b(x, -x^*))/2 = b(x, 0)$. This shows that if b is in the complement of $A(K, E)$, then b can be identified with the element $b(\cdot, 0)$ of $A(K, \mathbf{R})$. Conversely, each element of $A(K, \mathbf{R})$ defines an element of $BA(K \times B(E^*))$ as follows: for each x in K and x^* in $B(E^*)$, $\bar{a}(x, x^*) = a(x)$. It follows that $P\bar{a} = 0$. Also it can easily be shown that for each b in $BA(K \times B(E^*))$,

$$\|b\| \leqq \|Pb\| + \|b - P(b)\| \leqq 2\|b\|.$$

Hence $BA(K \times B(E^*))$ is linearly isomorphic to $A(K, E) \oplus_{\ell^1} A(K, \mathbf{R})$. The following proposition is now immediate.

PROPOSITION 3.2. *If K is a Choquet simplex and if, E is a real Banach space with Schur property, then $BA(K \times B(E^*))$ has the Dunford-Pettis property.*

The next example ends the search.

EXAMPLE 3.3. A compact convex set C that is neither a Choquet simplex nor the unit ball of the dual of a Banach space with the Dunford-Pettis property such that $A(C, \mathbf{R})$ has the Dunford-Pettis property.

Let K be a Choquet simplex and let E be a real Banach space with the Schur property. Consider $BA = BA(K \times B(E^*))$, and let C denote the state space of BA; that is $C = \{\ell \in BA^* \ \ell(1) = \|\ell\| = 1\}$. We know that C is a weak*-compact convex subset of BA^* and it can be shown that BA

is linearly isometric to $A(C, \mathbf{R})$. Hence $A(C, \mathbf{R})$ has the Dunford-Pettis property by Proposition 3.2. Moreover C is neither a unit ball in a dual space nor a Choquet simplex because it follows from [**6**, 2.10] that the state space of BA is a Choquet simplex if and only if K and $B(E^*)$ are Choquet simplexes. But $B(E^*)$ is never a Choquet simplex, for any measure μ of the form $(\varepsilon_{x^*} + \varepsilon_{-x^*})/2$, where x^* is an extreme point of $B(E^*)$, is a maximal probability measure whose barycenter is 0 [**7**] or [**2**].

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NOTE ADDED IN PROOF. Recently, M. Talagrand has constructed a separable Banach space E, such that E^* has the Schur property but such that $C(\Delta, E)$ fails the Dunford-Pettis property, here $\Delta = \{0, 1\}^{\mathbf{N}}$ is the Cantor group. This answers in the negative the problem mentioned in this paper.