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UNIVALENCE CRITERIA AND THE HYPERBOLIC METRIC

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1. Introduction. We shall consider restrictions on the derivative of a function $f \in H(\mathbf{B})$ (i.e., holomorphic in the unit disk $\mathbf{B}$) which imply that f is univalent. Perhaps the best known result of this type, due to Wolff [13], Warschawski [12] and Noshiro [11], involves only the argument of f'. It states that f is one-to-one if $f'(z) \neq 0$ and $\arg f'(z)$ lies in an interval of length π, $z \in \mathbf{B}$. If the length of the interval is larger than π, then f need not be univalent, and, in fact, the valence of f need not be bounded [6].

On the other hand, there is a criterion for univalence due to John [7] which involves only the modulus of f'. For non-constant $f \in H(\mathbf{B})$ let $M_f = \sup_{z\in\mathbf{B}}|f'(z)|$, $m_f = \inf_{z\in\mathbf{B}}|f'(z)|$ and $\mu_f = M_f/m_f$. The John constant γ is defined by $\gamma = \sup\{t: \mu_f \leqq t \text{ implies } f \text{ is univalent}\}$. If $\mu_f \leqq \gamma$, then f is univalent.

The condition $\mu_f < \infty$ is equivalent to $f'(\mathbf{B})$ lying in an annulus centered at zero. We may introduce symmetry relative to the unit circle by considering $g = f/\sqrt{m_f M_f}$. Then $M_g = \sqrt{M_f/m_f} = 1/m_g$, $\mu_g = \mu_f$, and, of course, f is univalent if and only if g is. It follows that

$$\frac{1}{2}\log\gamma = \sup\{M: e^{-M} \leqq |f'| \leqq e^{M} \Rightarrow f \text{ is univalent}\}.$$

The best known estimates for γ are $e^{\pi/2} \leqq \gamma \leqq e^{\pi}$; the lower and upper bounds being given by John [8] and Yamashita [14], respectively.

In the next section we consider the problem of determining which plane regions Ω have the property that $\log f'(\mathbf{B}) \subset \Omega$ implies f is univalent. The above two criteria correspond to the cases in which Ω is a horizontal or vertical strip, respectively. We obtain conditions on Ω, involving the hyperbolic metric on Ω, which insure that f is one-one. Our results rely on the following theorem due to Becker [3].

BECKER'S UNIVALENCE CRITERION. *If $f \in H(\mathbf{B})$, $f'(0) \neq 0$, and*

$$(1 - |z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leqq 1, \quad z \in \mathbf{B},$$

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then f is univalent.

In §4 we consider the case in which Ω is a rectangle with horizontal and vertical sides. This gives a chain of univalence criteria with the Wolff-Warshawski-Noshiro result and the John criterion as limiting cases.

In §3 we establish some properties of the hyperbolic metric which may be of independent interest.

2. The hyperbolic metric. Let $\Omega \subset \mathbf{C}$ be a hyperbolic region, i.e., $\mathbf{C}\backslash\Omega$ contains at least two points. Let ϕ be an analytic universal covering projection of $\mathbf{B}$ onto Ω. The hyperbolic metric, $\lambda_\Omega(z)|dz|$, is defined as follows: if $z \in \Omega$ and $w \in \phi^{-1}(z)$, then

$$\lambda_\Omega(z) = \frac{1}{|\phi'(w)|(1 - |w|^2)}.$$

The value of $\lambda_\Omega(z)$ is independent of both the choice of $w \in \phi^{-1}(z)$ and of the selection of the covering ϕ. The collection of analytic coverings of $\mathbf{B}$ onto Ω consists of the functions $\phi \circ T$, where T is a conformal self-mapping of $\mathbf{B}$. Thus, for fixed $z \in \Omega$, there is a unique analytic covering ϕ for which $\phi(0) = z$ and $\phi'(0) > 0$. In this case, $\lambda_\Omega(z) = 1/\phi'(0)$. If Ω is simply-connected, then ϕ is just a conformal mapping of $\mathbf{B}$ onto Ω. The function λ_Ω is real-analytic on Ω.

EXAMPLES. (i) $\lambda_{\mathbf{B}}(z) = 1/(1 - |z|^2)$.
(ii) If $H = \{z: \operatorname{Re} z > 0\}$, then $\lambda_H(z) = 1/(2 \operatorname{Re} z)$.
(iii) If $S(b) = \{z: |\operatorname{Re} z| < b\}$, then $\lambda_{S(b)}(z) = \pi/4b \cos(\pi \operatorname{Re} z/2b)$.

For a general discussion of the hyperbolic metric we refer the reader to **[1]**, **[5]**, and **[10]**. We shall need the following basic properties, which are stated without proof.

Assume Ω and Δ are hyperbolic plane regions.

CONFORMAL INVARIANCE. *If f is a conformal mapping of Ω onto Δ, then*

$$\lambda_\Delta(f(z))|f'(z)| = \lambda_\Omega(z).$$

PRINCIPLE OF HYPERBOLIC METRIC. *If $f \in H(\Omega)$ and $f(\Omega) \subset \Delta$, then*

$$\lambda_\Delta(f(z))|f'(z)| \leqq \lambda_\Omega(z).$$

Equality occurs at some point if and only if f is an anlaytic covering of Ω onto Δ.

MONOTONICITY. *If $\Omega \subset \Delta$, then for $z \in \Omega$, $\lambda_\Delta(z) \leqq \lambda_\Omega(z)$. If equality holds at a single point, then $\Omega = \Delta$.*

We may now prove the following distortion theorem, which, together with Becker's result, gives a criterion for univalence.

THEOREM 1. *Let Ω be a hyperbolic plane region and let $\Lambda(\Omega) = \inf\{\lambda_\Omega(z): z \in \Omega\}$. If $f \in H(\mathbf{B})$, $f'(z) \neq 0$, $z \in \mathbf{B}$, and $\log f'(\mathbf{B}) \subset \Omega$, then*

$$(1 - |z|^2)\left|\frac{f''(z)}{f'(z)}\right| \leqq \frac{1}{\Lambda(\Omega)}, \quad z \in \mathbf{B}.$$

If $\Lambda(\Omega) > 0$, then equality occurs at a single point if and only if $\log f'$ is a universal analytic covering of $\mathbf{B}$ onto Ω.

PROOF. Applying the Principle of Hyperbolic Metric to $\log f'$, we have

$$\lambda_\Omega(\log f'(z))\left|\frac{f''(z)}{f'(z)}\right| \leqq \lambda_{\mathbf{B}}(z) = \frac{1}{1 - |z|^2}, \quad z \in \mathbf{B},$$

and the result follows.

COROLLARY. *Under the hypotheses of Theorem 1, f is univalent if $\Lambda(\Omega) \geqq 1$.*

REMARK. From Example (iii) we see that $\Lambda(S(b)) = \pi/4b$. Thus, if $b \leqq \pi/4$ and $\log f'(\mathbf{B}) \subset S(b)$, then f is univalent. This gives $\gamma \geqq e^{\pi/2}$.

3. Evaluating $\Lambda(\Omega)$. Throughout this section Ω denotes a hyperbolic plane region.

PROPOSITION 1. *If a is a finite boundary point of Ω, then* $\lim_{z\to a} \lambda_\Omega(z) = \infty$.

PROOF. Let $b \in \mathbf{C}\backslash\Omega$, $b \neq a$, and let $\lambda_{a,b} = \lambda_{\mathbf{C}\backslash\{a,b\}}$. Since $\Omega \subset \mathbf{C}\backslash\{a, b\}$ and $w = (z - a)/(b - a)$ is a conformal mapping of $\mathbf{C}\backslash\{a, b\}$ onto $\mathbf{C}\backslash\{0, 1\}$, we see from Conformal Invariance and Monotonicity that

$$\lambda_\Omega(z) \geqq \lambda_{a,b}(z) = |b - a|^{-1} \lambda_{0,1}\left(\frac{z - a}{z - b}\right).$$

It is known [**1**, p. 18], that

$$\log \lambda_{0,1}(z) = -\log|z| - \log\log\left(\frac{1}{|z|}\right) + O(1)$$

as $z \to 0$. Thus,

$$\lim_{z\to a} \lambda_\Omega(z) \geqq |b - a|^{-1} \lim_{z\to 0} \lambda_{0,1}(z) = \infty.$$

It is necessary that a be finite, as seen by Example (ii). Here, we have $\limsup_{z\to\infty} \lambda_H(z) = \infty$ and $\liminf_{z\to\infty} \lambda_H(z) = 0$. If Ω is bounded, then $\lim_{z\to a} \lambda_\Omega(z) = \infty$ for all $a \in \partial\Omega$, so λ_Ω necessarily has a minimum in Ω.

PROPOSITION 2. *If Ω is symmetric about the straight line L, then λ_Ω is symmetric about L.*

PROOF. There exist complex numbers a, b, $|a| = 1$, such that $f(z) = az + b$ maps L onto the real axis $\mathbf{R}$. If z and z^* are symmetric about L, then $f(z^*) = \overline{f(z)}$. By Conformal Invariance, $\lambda_\Omega(z) = \lambda_{f(\Omega)}(f(z))|f'(z)| = \lambda_{f(\Omega)}(f(z))$. Thus, it suffices to consider the case $L = \mathbf{R}$. Let $z \in \Omega$ and let ϕ be the analytic covering of $\mathbf{B}$ onto Ω with $\phi(0) = z$, $\phi'(0) > 0$. Since Ω is symmetric about $\mathbf{R}$, $\psi(\zeta) = \overline{\phi(\bar{\zeta})}$ is also an analytic covering of $\mathbf{B}$ onto Ω, and $\psi(0) = \bar{z}$, $\psi'(0) = \phi'(0)$. Thus $\lambda_\Omega(\bar{z}) = 1/\psi'(0) = 1/\psi'(0) = \lambda_\Omega(z)$.

THEOREM 2. *If Ω is convex, L is a straight line, and $\Omega \cap L \neq \emptyset$, then $1/\lambda_\Omega$ is concave on $\Omega \cap L$.*

PROOF. Consider distinct points $z_0, z_1 \in \Omega \cap L$ and let $z_t = (1-t)z_0 + tz_1$, $t \in [0, 1]$. Let ϕ_t be the conformal mapping of $\mathbf{B}$ onto Ω with $\phi_t(0) = z_t$, $\phi_t'(0) > 0$, and let $f_t = (1-t)\phi_0 + t\phi_1$. Then $f_t \in H(\mathbf{B})$, $f_t(0) = z_t$ and, since Ω is convex, $f_t(\mathbf{B}) \subset \Omega$. By the Principle of Hyperbolic Metric,

$$\lambda_\Omega(f_t(0))|f_t'(0)| \leqq \lambda_{\mathbf{B}}(0) = 1,$$

or equivalently,

$$\frac{1}{\lambda_\Omega(z_t)} \geqq (1-t)\,\phi_0'(0) + t\,\phi_1'(0) = \frac{1-t}{\lambda_\Omega(z_0)} + \frac{t}{\lambda_\Omega(z_1)}.$$

COROLLARY. *Suppose Ω is convex.*

(i) *If Ω is symmetric about a line L and L' is a line perpendicular to L, then the restriction of λ_Ω to $\Omega \cap L'$ attains a minimum value at $L \cap L'$.*

(ii) *If Ω is symmetric about two intersecting lines L and L', then λ_Ω attains a minimum at $L \cap L'$.*

PROOF. (i) By Proposition 2 and Theorem 2, the restriction of $1/\lambda_\Omega$ to $\Omega \cap L'$ is both concave and symmetric about $L \cap L'$, thus attaining a maximum at $L \cap L'$. Part (ii) follows from (i).

The proof of the following lemma is elementary and therefore omitted.

LEMMA. *Suppose f and g are real-analytic functions on an open set $U \subset \mathbf{C}$, and let ℓ be an open line segment contained in U. If f and g agree on a subset of ℓ which has a limit point in ℓ, then f and g agree on ℓ.*

THEOREM 3. *Assume Ω is convex, L is a line, and $\Omega \cap L \neq \phi$. If the restriction of λ_Ω to $\Omega \cap L$ attains a minimum at two distinct points, then Ω is either a strip or a half-plane.*

PROOF. Suppose the restriction of λ_Ω to $\Omega \cap L$ attains a minimum at distinct points z_1 and z_2. Since $1/\lambda_\Omega$ is concave on $\Omega \cap L$, λ_Ω is constant on the segment $[z_1, z_2]$. By the lemma, λ_Ω is necessarily constant on $\Omega \cap L$, say with value c. If L meets $\partial\Omega$ at a point $a \in \mathbf{C}$, then $\lambda_\Omega(z) \to e$ as $z \to a$ along L, contrary to Proposition 1. Thus Ω contains L. Being covex, Ω must be either a strip or a half-plane.

We have observed that λ_Ω does not have a minimum when Ω is a half-plane. In the case of the strip $S(b)$, the minimum exists and occurs at each point of the center line of the strip.

COROLLARY. *If Ω is convex and λ_Ω has a minimum, then either Ω is a strip or the minimum occurs at a unique point of Ω.*

4. The case of a rectangle. For $M, A \in (0, \infty]$, let $R(M, A) = \{z: |\text{Re } z| < M, |\text{Im } z| < A\}$ and let $\mathscr{F}(M, A) = \{f \in H(\mathbf{B}): \log f'(\mathbf{B}) \subset R(M, A)\}$. We wish to determine $\tau(A) = \sup \{M: f \in \mathscr{F}(M, A)$ implies f is univalent$\}$. Since $\mathscr{F}(M, A)$ increases with A, τ is a decreasing function on $(0, \infty]$. By the Wolff-Warschawski-Noshiro result, $\tau(A) = \infty$ for $0 < A \leqq \pi/2$. The John criterion gives $\tau(\infty) = (1/2) \log \gamma$.

Suppose $f \in \mathscr{F}(M, A)$ and $t > 0$. Let

$$f_t(z) = f(0) + \int_0^z [f'(\zeta)]^t \, d\zeta,$$

where the branch of the power function is determined by the choice of $\log f'$ satisfiyng $\log f'(\mathbf{B}) \subset R(M, A)$. Then $f_t \in \mathscr{F}(tM, tA)$ and $f_t \to f$ locally uniformly as $t \to 1$. If $f \in \mathscr{F}(\tau(A), A)$ and $0 < t < 1$, then $f_t \in \mathscr{F}(t\tau(A), tA) \subset \mathscr{F}(t\tau(A), A)$, implying f_t is univalent. Thus, f is one-to-one, and $\mathscr{F}(\tau(A), A)$ consists entirely of univalent functions.

We shall now apply our work in the preceding sections to obtain a lower bound for $\tau(A)$. If both M and A are finite, then $R(M, A)$ is convex, bounded and symmetric about both axes. Thus, $\lambda_{R(M, A)}$ has a minimum value, say $\Lambda(M, A)$, which is attained only at the origin. If exactly one of M and A is finite, then the minimum value, $\Lambda(M, A)$, is attained at each point of the center line of the strip and, in particular, at $z = 0$. $\Lambda(M, A)$ has the following properties.

PROPOSITION 3. *Assume at least one of M and A is finite.*

(i) $\Lambda(M, A) = \Lambda(A, M)$.

(ii) $\Lambda(tM, tA) = t^{-1}\Lambda(M, A)$, $0 < t < \infty$.

(iii) $\Lambda(M, A)$ *is strictly decreasing in each variable.*

(iv) $1/\Lambda(M, A)$ *is concave.*

PROOF. Parts (i) and (ii) follow from Conformal Invariance and the observation that $w = iz$ and $w = tz$ map $R(M, A)$ onto $R(A, M)$ and $R(tM, tA)$, respectively. If $M_1 < M_2$, then $R(M_1, A)$ is a proper subregion of (M_2, A). By Monotonicity, $\Lambda(M_1, A) = \lambda_{R(M_1, A)}(0) > \lambda_{R(M_2, A)}(0) =$

$\Lambda(M_2, A)$. Similiarly, $\Lambda(M, A)$ is strictly decreasing as a function of A. As for (iv), consider distinct finite points (M_0, A_0) and (M_1, A_1). For $t \in [0, 1]$, let $(M_t, A_t) = (1 - t)(M_0, A_0) + t(M_1, A_1)$, and let ϕ_t be the conformal mapping of $\mathbf{B}$ onto $R(M_t, A_t)$ with $\phi_t(0) = 0$, $\phi_t'(0) > 0$. Then

$$\frac{1}{\Lambda(M_t, A_t)} = \frac{1}{\lambda_{R(M_t, A_t)}(0)} = \phi_t'(0), \ 0 \leqq t \leqq 1.$$

Now, $\psi_t = (1 - t)\phi_0 + t\phi_1 \in H(\mathbf{B})$, $\psi_t(0) = 0$ and $\psi_t(\mathbf{B}) \subset R(M_t, A_t)$. By the Principle of Hyperbolic Metric,

$$\lambda_{R(M_t, A_t)}(\psi_t(0))\,|\psi_t'(0)| \leqq \lambda_{\mathbf{B}}(0) = 1,$$

or equivalently, $(1 - t)\phi_0'(0) + t\phi_1'(0) \leqq \phi_t'(0)$. This is the desired inequality.

When both M and A are finite, we can obtain an expression for $\Lambda(M, A)$ using the Jacobian elliptic functions sn, cn, and dn, relative to the parameter $\tau = iA/M$. We refer the reader to [**9**, Chapter VI, §3] for many of the results quoted below. Let $k = \sqrt{\lambda(t)}$, where here λ denotes the elliptic modular function. If

$$K = \int_0^1 \frac{dt}{[(1 - t^2)(1 - k^2t^2)]^{1/2}}$$

and $K' = K((1 - k^2)^{1/2})$, then

$$f(z) = \left(\frac{1 - \mathrm{cn}(z)}{1 + \mathrm{cn}(z)}\right)^{1/2}$$

maps the rectangle $R(K, K')$ conformally onto $\mathbf{B}$ with $f(0) = 0$ [**9**, pg. 297]. Moreover, $iA/M = \tau = iK'/K$, so $R(M, A)$ is similar to $R(K, K')$. By Proposition 3(ii), $\Lambda(M, A) = (K/M)\Lambda(K, K') = (K/M)|f'(0)|$. Various identities for the Jacobian ellitptic functions show that $f'(z) = \mathrm{dn}(z)/(1 + \mathrm{cn}(z))$. Furthermore, $\mathrm{dn}(0) = \mathrm{cn}(0) = 1$, so $f'(0) = 1/2$ and $\Lambda(M, A) = K/2M$. Although K is determined implicitly by M and A, it is possible to express $\Lambda(M, A)$ more explicitly in terms of M and A. If $q = e^{i\pi\tau} = e^{-2\pi A/M}$ then [**4**, pgs. 385, 410]

$$K = \frac{\pi}{2}\prod_{n=1}^{\infty}(1 - q^{2n})^2(1 + q^{2n-1})^4 = \frac{\pi}{2}(1 + \sum_{n=1}^{\infty} q^{n^2})^2,$$

and so

$$\Lambda(M, A) = \frac{\pi}{4M}(1 + 2\sum_{n=1}^{\infty} q^{n^2})^2.$$

Although it is not obvious that the preceding formula for $\Lambda(M, A)$ is symmetric in M and A, it can be established by means of identities for elliptic functions.

PROPOSITION 4. *Suppose* $M, A \in (0, \infty)$.

(i) $\lim_{M\to\infty} \Lambda(M, A) = \pi/4A = \Lambda(\infty, A)$.

(ii) $\lim_{A\to\infty} \Lambda(M, A) = \pi/4M = \Lambda(M, \infty)$.

(iii) $\lim_{M\to 0} \Lambda(M, A) = \infty = \lim_{A\to 0} \Lambda(M, A)$.

PROOF. As $A \to \infty, q \to 0$, so $\Lambda(M, A) \to \pi/4M$. That $\Lambda(M, \infty) = \pi/4M$ follows from Example (iii). Moreover, $M \to 0$ implies $q \to 0$ and, consequently, $\Lambda(M, A) \to \infty$. The other three conclusions follow from the symmetry of $\Lambda(M, A)$ (Proposition 3(i)).

We have shown that for fixed $A \in (0, \infty)$, $\Lambda(M, A)$ is a strictly decreasing function of M on $(0, \infty)$ with limits ∞ and $\pi/4A$ at the left and right end points, respectively. If $A \leqq \pi/4$, then $\Lambda(M, A) > 1$ for all $M \in (0, \infty)$. For $A > \pi/4$ there is a unique value of M, say $\Upsilon(A)$, such that $\Lambda(\Upsilon(A), A) = 1$. If $0 < A \leqq \pi/4$, we set $\Upsilon(A) = \infty$.

PROPOSITION 5. *The function Υ has the following properties.*

(i) $\Upsilon(A) \leqq \tau(A)$, $0 < A \leqq \infty$,

(ii) *Υ is a strictly decreasing, convex function on* $(\pi/4, \infty)$,

(iii) *The graph of $M = \Upsilon(A)$ is symmetric about $M = A$,*

(iv) *$\Upsilon(A) \to \infty$ as A decreases to $\pi/4$.*

PROOF. (i) follows directly from the Corollary to Theorem 1. For (ii), consider $\pi/4 < A_0 < A_1 < \infty$ and let $(M_t, A_t) = t(\Upsilon(A_1), A_1) + (1 - t)(\Upsilon(A_0), A_0)$, $0 \leqq t \leqq 1$. By Proposition 3 (iii), $1 = \Lambda(\Upsilon(A_0), A_0) > \Lambda(\Upsilon(A_0), A_1)$, so $\Upsilon(A_1) < \Upsilon(A_0)$. Furthermore, by Proposition 3 (iv),

$$\frac{1}{\Lambda(M_t, A_t)} \geqq \frac{t}{\Lambda(\Upsilon(A_1), A_1)} + \frac{1-t}{\Lambda(\Upsilon(A_0), A_0)} = 1.$$

Thus, $\Lambda(M_t, A_t) \leqq 1 = \Lambda(\Upsilon(A_t), A_t)$, which gives $\Upsilon(A_t) \leqq M_t = t\,\Upsilon(A_1) + (1 - t)\,\Upsilon(A_0)$, $0 \leqq t \leqq 1$. Part (ii) follows from Proposition 3(i). As to (iv), if $\Upsilon(A)$ had a finite limit, L, as A decreases to $\pi/4$, then $1 = \Lambda(\Upsilon(A), A) \to \Lambda(L, \pi/4)$, contrary to the remarks preceding this proposition.

The following two examples provide an upper estimate for $\tau(A)$.

EXAMPLE 1. Suppose $\pi/2 < A < \pi$, $0 < r < 1$, and consider the function f such that

$$f'(z) = (1 + rz)^{A/\arcsin r}, f'(0) = 1.$$

Then

$$\frac{A \log(1-r)}{\text{arscin } r} < \text{Re}\,\{\log f'(z)\} < \frac{A \log(1+r)}{\arcsin r}$$

$|\arg f'(z)| < A$, and $f \in \mathscr{F}(-A \log(1 - r)/\arcsin r, A)$. Now the univalence of f is determined by that of $\exp\{(1 + A/\arcsin r) \log(1 + rz)\}$.

The image of **B** under $w = \log(1 + rz)$ is a convex region D, which lies in the strip $\{w\colon |\mathrm{Im}\ w| < \arcsin r\}$. The points $\log(1 - r^2)^{1/2} \pm i \arcsin r$ lie on the boundary of D. Thus, by the periodicity of $\exp(z)$, f will fail to be univalent if and only if $A + \arcsin r > \pi$. For $r > \sin A$, we have $\arcsin r > \pi - A$, so f is not univalent, and $\tau(A) \leqq A \log(1 - r)/\arcsin r$. Letting r decrease to $\sin A$, we obtain

$$\tau(A) \leqq \frac{-A \log(1 - \sin A)}{\pi - A}, \qquad \pi/2 < A < \pi.$$

The quantity on the right side has limit π as $A \to \pi$.

EXAMPLE 2. Suppose $\pi < A$, $0 < r < 1$, and consider the function f determined by

$$f'(z) = (1 + rz)^{-iA/\log(1-r)}, f'(0) = 1.$$

Estimates on the real and imaginary parts of $\log f'$ show that $f \in \mathscr{F}(-A \arcsin r/\log(1 - r), A)$. For the univalence of f we consider $\exp\{(1 - iA/\log(1 - r)) \log(1 + rz)\}$. The region D in Example 1 is symmetric about both the real axis and the line $\mathrm{Re}\ w = \log(1 - r^2)^{1/2}$. It can be shown that D contains the disk of radius $\arcsin r$ centered at $\log(1 - r^2)^{1/2}$, but we omit the details. Then, the image of **B** under $(1 - iA/\log(1 - r)) \log(1 + rz)$ is a convex region containing a disk of radius $\rho(r) = |1 - iA/\log(1 - r)| \arcsin r$, and by periodicity of $\exp(z)$, f fails to be univalent when $\rho(r) > \pi$. Now, $\rho(0) = A > \pi$ and $\rho(1) = \pi/2$, so there is a smallest positive root, $r_0(A)$, of $\rho(r) = \pi$. This gives the implicit estimate

$$\tau(A) \leqq \frac{-A \arcsin(r_0(A))}{\log(1 - r_0(A))}.$$

This estimate cannot be sharp, since we lost some ground by considering the largest disk contained in D.

2. Comments. It is not known if the constant 1 in Becker's Theorem is sharp. It would be of considerable interest to determine the supremum, say c, of constants k such that $(1 - |z|^2)\ |f''(z)/f'(z)| \leqq k$ implies f is one-to-one. Of course $c \geqq 1$. An example of Becker **[2]** shows that $c \leqq 4/e$. From Proposition 5 we obtain $\tau(A) \geqq \Upsilon(A) = \infty$, $0 < A \leqq \pi/4$, whereas the Wolff-Warschawski-Noshiro Theorem gives $\tau(A) = \infty$ for $0 < A \leqq \pi/2$. To obtain the latter conclusion from our method would require c to be 2. On the other hand, if one could demonstrate that $\gamma = e^{\pi/2}$, then it would follow that $c = 1$.

After the completion of this research, S. Yamashita brought to our attention the work of Avhadiev and Aksent'ev [*Sufficient conditions for univalence of analytic functions*, Soviet Math. Dokl, **12** (1971), 859–863]. Their paper overlaps with the application, in §4, of our main results.

We would like to thank J. Becker for several useful communications. In particular, he has shown that $\tau(A) \leqq 2\Upsilon(A/2)$. His proof, which he has permitted us to give here, goes as follows. Let $\phi_{M,A}$ denote a univalent mapping of $\mathbf{B}$ onto $R(M, A)$ sending zero to zero. For $A > \pi/4$, $\Upsilon(A)$ is the unique value of M for which $\Lambda(M, A) = 1$, or equivalently for which $|\phi_{M,A}(0)| = 1$. Now, suppose $M > 2\Upsilon(A/2)$ and let $a = \phi_{M,A}(0)$. Then, for $A > \pi/2$,

$$|a| > |\phi'_{2\Upsilon(A/2),A}(0)| = 2|\phi'_{\Upsilon(A/2),A/2}(0)| = 2.$$

If

$$f(z) = \int_0^z \exp\{\phi_{M,A}(\zeta^n)\}\, d\zeta = z + \frac{a}{n+1} z^{n+1} + \dots,$$

then $f \in \mathscr{F}(M, A)$, but since $|a| > 2$, f is not univalent when n is sufficiently large [5, page 494]. Thus $\tau(A) < M$ for each $M > 2\Upsilon(A/2)$. We note that $2\Upsilon(A/2) \to \pi/2$ as $A \to +\infty$.

Finally, Prokhorov, Szynal and Waniurski [*New upper estimate of the John constant*, Abstracts Amer. Math. Soc 1 (1980), 380] have announced the estimate $\Upsilon \leqq 19.93$.

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