ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 12, Number 2, Spring 1982

# INFINITE PERMUTABLE SUBGROUPS

FLETCHER GROSS

**1. Introduction.** Suppose $H$ is a core-free permutable subgroup of the group $G$. This means that $H$ contains no non-identity normal subgroup of $G$ and that $HK = KH$ for each subgroup $K$ in $G$. If $G$ is finite, then Itô and Szep [**6**] proved that $H$ must be nilpotent. This result was improved by Maier and Schmid [**7**] who showed that $H$ is contained in $Z_\infty(G)$, the hypercenter of $G$, when $G$ is finite. The motivation behind the present paper was to investigate what happens when $G$ is infinite.

It is known in general that $H$ must be residually a finite nilpotent group ([**1**] and [**8**]). This result seems less satisfying, however, when it is recalled that any free group is also residually a finite nilpotent group. Another approach to the structure of $H$ is to consider the subgroup of $H$ generated by all its elements of finite order. It follows from results in [**2**] that this subgroup, which I denote by $T(H)$, is both locally finite and locally nilpotent. It is natural then, to ask what can be said about $H/T(H)$. This question seemed even more pertinent when the author realized that in all the examples of core-free permutable subgroups previously known (to the author, at least), $H/T(H)$ is abelian. If it were true that $H/T(H)$ is locally nilpotent, then it would follow that $H$ is locally solvable.

It is shown in [**1**] and [**8**] how to construct examples in which $H$ is not nilpotent nor even solvable. These examples are constructed by taking the direct sum of groups of prime-power-order using infinitely many distinct primes. One consequence of the present paper is that even when $G$ is a $p$-group, $H$ need not be solvable. The major thrust of this paper, however, is to settle the question of whether $H$ need be locally nilpotent or locally solvable. We will do a little more than this by constructing an example in which $H/T(H)$ is not locally solvable.

As far as the result of Maier and Schmid is concerned, there are various natural ways to try to generalize this result to infinite groups. For example, one could work with ascending series and ask whether $H \leqq Z_\infty(G)$. Alternately, one could work with descending series and ask whether $[H, G; \infty]$ or $[G, H; \infty]$ is the identity. (This notation is explained in the next section.) The answers to all of these questions are no and the main result of this paper may be stated as follows.

Received by the editors on May 8, 1978.

THEOREM. *Let p be any prime. Then there exists a group G with a core-free permutable subgroup H such that*

(i) *$H/T(H)$ is not locally solvable. Indeed, there is a 2-generator subgroup of $H/T(H)$ which is not solvable.*
(ii) *$Z_\infty(G) = Z(G)$; $Z(G)$ is finite; and $H \cap Z(G) = 1$.*
(iii) *$[G, H; \infty]$ and $[H, G; \infty]$ are both infinite.*
(iv) *$T(G)$ is a locally finite p-group.*
(v) *$T(H)$ is a core-free permutable subgroup of $T(G)$.*
(vi) *$T(H)$ is not solvable.*
(vii) *H is residually a finite p-group.*

Most of this work was done while the author was a visitor at the University of Warwick. I wish to express my gratitude to T. Hawkes and everyone at Warwick for a most enjoyable stay. I am also grateful for a grant from NATO. Finally, I have had the benefit of some very helpful discussions with T.R. Berger, M. Newman, and S.E. Stonehewer.

**2. Notation and preliminary results.** The notation is mostly standard but a few symbols deserve explanation. If $A$ and $B$ are subgroups of a group $G$ and $\alpha$ is an ordinal, then $[A, B; \alpha]$ is defined as follows: $[A, B; 1] = [A, B]$. If $\alpha$ is not a limit ordinal, then $[A, B; \alpha] = [[A, B; \alpha - 1], B]$. If $\alpha$ is a limit ordinal, then $[A, B; \alpha] = \bigcap [A, B; \beta]$ where the intersection is over all $\beta < \alpha$. Eventually, $[A, B; \gamma] = [A, B; \gamma + 1]$ for some ordinal $\gamma$. Then we set $[A, B; \infty] = [A, B; \gamma]$.

$Z(G)$ is the center of $G$ and $Z_\alpha(G)$ is defined inductively by $Z_1(G) = Z(G)$, $Z_\alpha(G)/Z_{\alpha-1}(G) = Z(G/Z_{\alpha-1}(G))$ if $\alpha$ is not a limit ordinal, and $Z_\alpha(G) = \bigcup Z_\beta(G)$ where this union is over all $\beta < \alpha$ if $\alpha$ is a limit ordinal. The hypercenter, $Z_\infty(G)$, is defined to be $Z_\gamma(G)$ if $Z_{\gamma+1}(G) = Z_\gamma(G)$. The lower central series $\{L_n(G) | n = 1, 2, \ldots\}$ is defined by $L_1(G) = G$ and $L_{n+1}(G) = [L_n(G), G]$. If $G$ is nilpotent (solvable), then $c(G)$ ($d(G)$) denotes the class (derived length) of $G$.

If $H$ is a subgroup of $G$, then $H_G$, the core of $H$ in $G$, is the intersection $\bigcap x^{-1}Hx$ as $x$ runs through all elements of $G$. If $H_G = 1$, then $H$ is said to be core-free. If $G$ is represented as a permutation group on the cosets of $H$, then those elements which move only finitely many cosets constitute a normal subgroup of $G$. The intersection of this subgroup with $H$ is denoted by $F(H, G)$. Alternately, $F(H, G)$ consists of those elements of $H$ which belong to all but a finite number of the groups $\{x^{-1}Hx | x \in G\}$. Clearly $H_G \trianglelefteq F(H, G) \trianglelefteq H$ and $F(H, G)/H_G$ is locally finite.

If $G$ is a $p$-group, then $\Omega_k(G)$ is the subgroup generated by all elements of order dividing $p^k$. For any group $G$, $T(G)$ is the subgroup generated by all elements of finite order in $G$. The set of all primes $p$ such that $G$ contains an element of order $p$ is denoted by $\pi(G)$. If $\{G_i | i \in I\}$ is a set of groups, then $\prod_{i \in I} G_i$ is the unrestricted direct product. Finally, $\mathbf{Z}$ and $\mathbf{Q}$ denote the additive groups of integers and rationals, respectively.

THEOREM 2.1. *Let $H$ be a core-free permutable subgroup, let $n$ be an integer, and let $K = \{x \in H | x^n = 1\}$. Then $K$ is a subgroup of $H$ and $K$ is nilpotent of class at most some function of $n$.*

PROOF. This follows immediately from Theorems 3.3 and 3.4 of [2].

COROLLARY 2.2. *Assume that $H$ is a core-free permutable subgroup. Then $T(H)$ is locally finite and locally nilpotent.*

PROOF. The theorem implies that a finitely generated subgroup of $T(H)$ is periodic and nilpotent. Since a periodic finitely generated, nilpotent group must be finite, the corollary follows.

COROLLARY 2.3. *Assume that $H$ is a core-free permutable subgroup. If $H/T(H)$ is locally nilpotent, then $H$ is locally solvable.*

PROOF. Let $K$ be a finitely generated subgroup of $H$. Since $H/T(H)$ is locally nilpotent, there must be an integer $n$ such that $L_n(K) \leqq T(H)$. Since $K$ is finitely generated, it follows from [5, Lemma 1.6] that there is a finite subset $S$ in $K$ such that $L_n(K)$ is generated by all conjugates in $K$ of the elements of $S$. We conclude from this that for some integer $m$, the set $\{x \in H | x^m = 1\}$ contains generators for $L_n(K)$. The theorem now implies that $L_n(K)$ is solvable and so $K$ is solvable.

I do not know whether or not the local nilpotence of $H/T(H)$ is sufficient to imply that $H$ is locally nilpotent. If $T(H)$ is replaced by $F(H, G)$, however, the result is true.

THEOREM 2.4. *Let $H$ be a core-free permutable subgroup of the group $G$. Then $H$ is locally nilpotent if and only if $H/F(H, G)$ is locally nilpotent.*

PROOF. We set $K = F(H, G)$ and assume that $H/K$ is locally nilpotent. Now if $H$ is abelian, the theorem is certainly true. Hence we assume that $H$ is not abelian. It follows from [3] that $|H\langle x\rangle : H|$ is finite for all $x \in G$.

Let $\Omega$ be the set of all right cosets of $H$ in $G$. Since $H$ is core-free, $G$ is faithfully represented as a transitive permutation group on $\Omega$. The orbits of $H$ on $\Omega$ have lengths $|H: H \cap x^{-1}Hx|$ for $x \in G$. Now $x^{-1}Hx \leqq H\langle x\rangle$ and so

$$|H: H \cap x^{-1}Hx| = |H(x^{-1}Hx): H| \leqq |H\langle x\rangle : H|.$$

Thus the orbits of $H$ all have finite length. It follows from this that if $x$ is an element of $K$, then $x$ has only finitely many conjugates in $H$.

Now suppose $M$ is a finitely generated subgroup of $H$. Then $MK/K$ is finitely generated and hence $MK/K$ is nilpotent. Then $L_n(M) \leqq K$ for some integer $n$. Lemma 1.6 of [5] implies that there is a finite subset $S$ in $M$ such that $L_n(M)$ is generated by $S$ and its conjugates in $M$. Since $S \subseteqq K$ and since $S$ is finite, there are only a finite number of conjugates of $S$ in $H$. Now $K$ is locally finite and so all the conjugates of $S$ in $H$ generate a finite normal subgroup of $H$. Thus there is a finite normal subgroup $N$ in $H$ such that $L_n(M) \leqq N \leqq K$.

From the facts that $N$ is finite and normal in $H$, we conclude that there must be an integer $r$ such that $[N, H; r] = [N, H; r + 1]$. This implies that $[N, H; r] \leqq L_k(H)$ for all $k \geqq 1$. But $H$ is residually nilpotent ([1, Theorem 4.2] and [8, Theorem $C$]) and so $\bigcap L_k(H) = 1$ where the intersection is over all $k \geqq 1$. Therefore, $[N, H; r] = 1$. But then

$$L_{n+r}(M) \leqq [N, M; r] \leqq [N, H; r] = 1$$

and so $M$ is nilpotent.

The examples to be constructed later have the structure $G = HA$ where $H$ is a core-free permutable subgroup and $A$ is abelian. We now prove some facts about such groups starting with the special case when $G$ is a finite $p$-group.

LEMMA 2.5. *Let $G$ be a finite $p$-group and assume that $G = AH$ where $A$ is an abelian subgroup and $H$ is a core-free permutable subgroup of $G$. Assume that $x$ and $y$ are elements of $G$ and $x^{p^n} = y^{p^n} = 1$. Then $(xy)^{p^n} = 1$.*

PROOF. Since $A \cap H \trianglelefteq A$ and since $H_G = 1$, it must be true that $A \cap H = 1$. Suppose now that $z \in G$ and $z^p = 1$. Then $|H\langle z\rangle : H| \leqq p$ and $\langle z\rangle H = (\langle z\rangle H \cap A)H$. This implies that $|\langle z\rangle H \cap A| \leqq p$ and so $\langle z\rangle H \cap A \leqq \Omega_1(A)$. Then $z \in \Omega_1(A)H$ from which it follows that $\Omega_1(G) \leqq \Omega_1(A)H$. Since $|H\langle z\rangle : H| \leqq p$, $z$ must normalize $H$ and so $\Omega_1(G) \leqq N_G(H)$. Now $[G, \Omega_1(A)] \trianglelefteq G$ and

$$[G, \Omega_1(A)] = [AH, \Omega_1(A)] = [H, \Omega_1(A)] \leqq H.$$

Since $H_G = 1$, we conclude that $\Omega_1(A) \leqq Z(G)$. Then

$$\Omega_1(G) = \Omega_1(\Omega_1(A)H) = \Omega_1(A) \times \Omega_1(H).$$

But then the Frattini subgroup of $\Omega_1(G)$ is contained in $\Omega_1(H)$. Since $H$ is core-free, this implies that $\Omega_1(G)$ is elementary abelian. Hence, the theorem is proved if $n = 1$. We now assume that $n > 1$.

Let $M$ be the core of $H\Omega_1(G)$ in $G$. Then $M \geqq \Omega_1(G) = \Omega_1(A)\Omega_1(H)$ and $H\Omega_1(G) = H\Omega_1(A)$. Hence $M = \Omega_1(A) \times (M \cap H)$. Then the Frattini subgroup of $M$ is contained in $M \cap H$. Since $M \trianglelefteq G$ and since $H_G = 1$, we conclude that $M$ is elementary abelian. Then $M = \Omega_1(G)$ and so $H\Omega_1(G)/\Omega_1(G)$ is a core-free permutable subgroup in $G/\Omega_1(G)$.

Hence $G/\Omega_1(G)$ satisfies the hypothesis of the lemma and both $x^{p^{n-1}}$ and $y^{p^{n-1}}$ are contained in $\Omega_1(G)$. Induction then yields $(xy)^{p^{n-1}} \in \Omega_1(G)$. Since $\Omega_1(G)$ has exponent $p$, the theorem follows.

COROLLARY 2.6. *Let $G$ be a finite group, $H$ a core-free permutable subgroup of $G$, and $A$ an abelian subgroup of $G$. Assume that $G = HA$. Then $G$ is nilpotent, and if $x$ and $y$ are elements in $G$ such that $x^n = y^n = 1$, then $(xy)^n = 1$.*

PROOF. $H \leqq Z_\infty(G)$ by [7]. Then $G/Z_\infty(G)$ is abelian. This implies that $G$ is nilpotent. Then $G$ is the direct product of its Sylow subgroups and the corollary follows from the lemma.

THEOREM 2.7. *Let $H$ be a core-free permutable subgroup of the group $G$. Assume that $A$ is an abelian subgroup of $G$ such that $G = HA$. Then*
(i) $H \cap A = 1$
(ii) *If $x$ and $y$ are elements of $G$ and $x^n = y^n = 1$, then $(xy)^n = 1$.*
(iii) *$T(G)$ is locally finite and locally nilpotent.*
(iv) *$T(H)$ is a core-free permutable subgroup of $T(G)$.*
(v) $\pi(G) = \pi(A)$.
(vi) *$H$ is residually a finite nilpotent $\pi(A)$-group.*

PROOF. $(H \cap A)^G = (H \cap A)^{AH} = (H \cap A)^H \leqq H$. The fact that $H$ is core-free now yields $H \cap A = 1$. Now suppose $x$ and $y$ belong to $G$ and $x^n = y^n = 1$ but $(xy)^n \neq 1$. Then there is some $z \in A$ such that $(xy)^n$ does not belong to $z^{-1}Hz$. Since $|H\langle x\rangle : H|$ must divide $n$ and since $H\langle x\rangle = H(H\langle x\rangle \cap A)$, we see that $H\langle x\rangle \cap A$ and $H\langle y\rangle \cap A$ have orders dividing $n$. Since $A$ is abelian, this implies that $A$ contains a finite subgroup $B$ such that $t^n = 1$ for all $t \in B$ and $HB$ contains both $x$ and $y$. Next, let $M$ be the core of $H$ in $HB\langle z\rangle$. Since $(xy)^n$ is not contained in $z^{-1}Hz$, $HB\langle z\rangle/M$ is a counter-example to part (ii) of the theorem. Thus in proving (ii), we may assume that $G = HB\langle z\rangle$ and $M = 1$. By Corollary 2.6, we may assume that $G$ is infinite. Since $H_G = 1$, this implies that $|G: H|$ is infinite. Therefore, $\langle z\rangle$ is infinite. But then $z$ normalizes $H$ by [**1**, Theorem 4.1] or by [**8**, Lemma 2.1], and then

$$1 = H_G = \bigcap_{t \in B} t^{-1}Ht.$$

This implies that $H$ is a core-free permutable subgroup of $HB$. Since $|HB: H| = |B|$ is finite, $HB$ must be finite. Then $(xy)^n = 1$ by Corollary 2.6. Thus (ii) is proved.

An immediate consequence of (ii) is that $T(G)$ must be periodic. Let $L$ be a finitely generated subgroup of $T(G)$. If $x \in T(G)$, then $H\langle x\rangle \cap A$ must be finite. Then there is a finite subgroup $L_1$ in $A$ such that $L \leqq L_1H$. This implies that $|L: L \cap H|$ is finite and so $L \cap H$ is finitely generated.

But $L \cap H \leqq T(G) \cap H = T(H)$ and $T(H)$ is locally finite by Corollary 2.2. It follows from this that $L$ is finite. Finally, since $L$ satisfies (ii) (i.e., if $x$ and $y$ belong to $L$ and $x^n = y^n = 1$, then $(xy)^n = 1$), $L$ must be nilpotent. This proves (iii).

Since $T(H) = H \cap T(G)$, we find that $T(H)$ is a permutable subgroup of $T(G)$. If $x$ is an element of infinite order in $A$, then $x$ normalizes $H$([1] or [8]). Hence

$$1 = H_G = \bigcap y^{-1}Hy$$

where the intersection is over all $y \in T(A)$. This implies that $T(H)$ is core-free in $T(G)$. Hence, (iv) is proved.

Suppose next that $p$ is a prime contained in $\pi(G)$ but not in $\pi(A)$. Then, if $x$ is an element of order $p$ in $G$, we find that $|H\langle x\rangle \cap A| \neq p$. But $H\langle x\rangle = H(H\langle x\rangle \cap A)$ and $|H\langle x\rangle : H|$ divides $p$. Hence $x$ must be contained in $H$. Then $H$ contains all elements of order $p$ in $G$ which contradicts $H_G = 1$. Thus, $\pi(G) = \pi(A)$.

Now let $y$ be any element in $G$ and let $H_y$ be the core of $H$ in $H\langle y\rangle$. Then $H$ is a subdirect product of the groups $\{H/H_y | y \in G\}$. If $|H\langle y\rangle : H|$ is infinite, then $|H/H_y| = 1$ by [1] or [8]. If $|H\langle y\rangle : H|$ is finite, then by (v) applied to $(H\langle y\rangle/H_y)$, we see that $H/H_y$ is a $\pi(\langle y\rangle H_y/H_y)$-group. Since $|H\langle y\rangle : H| = |H\langle y\rangle \cap A|$, we see that $H/H_y$ is a $\pi(A)$-group for all $y \in G$. This proves (vi).

**3. Construction of the examples.** Before proceeding to the infinite groups, we need to review the groups constructed in [4]. Throughout this section we fix some notation. For the benefit of the reader, a glossary is included at the end.

Let $p$ be a fixed prime, $e = (3 + (-1)^p)/2$, and $r = p^e - p^{e-1}$. Thus $e = 1$ and $r = p - 1$ if $p$ is odd, while $e = 2$ and $r = 2$ if $p = 2$. Let $n$ be a positive integer, ler $\Gamma_n$ be the additive group $\mathbf{Z}/p^n\mathbf{Z}$, and let $\Delta_{n,k}$ be the set of elements of order $p^k$ in $\Gamma_n$. Let $x_n$ be the permutation of $\Gamma_n$ given by

$$(p^n\mathbf{Z} + a)x_n = p^n\mathbf{Z} + a + 1.$$

If $0 \leqq m \leqq n$, then set $x_{n,m} = x_n^{p^{n-m}}$. If $0 \leqq m \leqq n - e$ and $1 \leqq i \leqq r$, then let $\theta_{n,m,i}$ be the orbit under $\langle x_{n,m}\rangle$ of

$$p^n\mathbf{Z} + ip^{n-m-1} - (e - 1)p^{n-m-e}.$$

Then $\Delta_{n,m+e}$ is the disjoint union of the sets $\{\theta_{n,m,i} | 1 \leqq i \leqq r\}$. Next let $\pi_{n,m,i}$ be the permutation on $\theta_{n,m,i}$ induced by $x_{n,m}$ and let

$$A_{n,m} = \left\{\prod_{i=1}^{r} \pi_{n,m,i}^{c_i} \,\middle|\, \sum_{i=1}^{r} c_i = 0\right\}.$$

Then $A_{n,m}$ is an abelian group which is the direct product of $r-1$ copies of a cyclic group of order $p^m$.

Now let $G_n$ be the group generated by $x_n$ and $\{A_{n,m} | 0 \leqq m \leqq n-e\}$. Let $H_n$ be the stabilizer in $G_n$ of the zero element of $\Gamma_n$. Then $G_n = \langle x_n \rangle H_n$ and $H_n$ is a core-free permutable subgroup of $G_n$ [4]. Next let $R_n$ be the subgroup of $H_n$ consisting of those elements which fix every element in $\Omega_{n-1}(\Gamma_n)$. Then $R_n$ is faithfully represented as a permutation group on $\Delta_{n,n}$. If $0 \leqq k \leqq n$, then

$$p^k\mathbf{Z} + a \to p^n\mathbf{Z} + ap^{n-k}$$

determines a one-to-one correspondence between $\Delta_{k,k}$ and $\Delta_{n,k}$. In this way, we may consider $R_k$ as a permutation group on $\Delta_{n,k}$. Since $\Gamma_n$ is the disjoint union of the sets $\{\Delta_{n,k} | 0 \leqq k \leqq n\}$, we may consider the direct product $\prod_{k=1}^{n} R_k$ as a permutation group on $\Gamma_n$. It is shown in [4] that this group is in fact $H_n$.

Shortly, we will construct a group in which $\prod_{k=1}^{\infty} R_k$ is a core-free permutable subgroup. Before doing this, however, we consider a particular 2-generator subgroup. One consequence will be that $\prod_{k=1}^{\infty} R_k$ is not locally solvable.

To start, we set

$$y_n = \prod_k \prod_{i=1}^{r} (\pi_{n,\, n-e-2k+2,\, i})^{(-1)^i p^{2-e}}$$

where $k$ runs over all positive integers such that $2k \leqq n-e+2$. Since

$$\sum_{i=1}^{r} (-1)^i p^{2-e} = 0,$$

$y_n$ belongs to $G_n$. We now describe specifically how $y_n$ acts on $\Gamma_n$. Since $y_n$ fixes $p^n\mathbf{Z}$, we look at $(p^n\mathbf{Z} + a)y_n$ where $a$ is a positive integer.

Suppose first that $p > 2$ and that $p^k$ is the largest power of $p$ dividing $a$. Then $a \equiv i\,p^k \pmod{p^{k+1}}$ where $1 \leqq i \leqq p-1$. Then

$$(p^n\mathbf{Z} + a)y_n = \begin{cases} p^n\mathbf{Z} + a & \text{if } k \text{ is odd,} \\ p^n\mathbf{Z} + a + (-1)^i p^{k+2} & \text{if } k \text{ is even.} \end{cases}$$

Now suppose that $p = 2$ and that $4^k$ is the largest power of 4 dividing $a$. Then $a \equiv i4^k \pmod{4^{k+1}}$ where $1 \leqq i \leqq 3$. Then

$$(2^n\mathbf{Z} + a)y_n = \begin{cases} 2^n\mathbf{Z} + a - 4^{k+1} & \text{if } i = 1, \\ 2^n\mathbf{Z} + a & \text{if } i = 2, \\ 2^n\mathbf{Z} + a + 4^{k+1} & \text{if } i = 3. \end{cases}$$

Next (again assuming that $p$ is any prime), let

$$u_n = x_n[y_n, x_n^{p+1}]x_n^{-1}$$

and

$$v_n = [x_n^p, y_n].$$

Then $u_n$ and $v_n$ are contained in $\langle x_n, y_n\rangle'$. A straightforward but tedious calculation shows that $u_n$ and $v_n$ fix the set

$$\{p^n\mathbf{Z} + p^2a | a \in \mathbf{Z}\}.$$

The mapping $p^n\mathbf{Z} + p^2a \rightarrow p^{n-2}\mathbf{Z} + a$ induces a homomorphism of $\langle u_n, v_n\rangle$ onto a permutation group on $\Gamma_{n-2}$. The calculation referred to above shows that this homomorphism maps $u_n$ onto $x_{n-2}$ and $v_n$ onto $y_{n-2}$. Here, if $n \leqq 2$, we set $x_{n-2} = y_{n-2} = 1$.

Lemma 3.1. There are elements $x$ and $y$ in $\prod_{n=1}^{\infty} G_n$ such that for all positive integers $m$, $\langle x, y\rangle^{(m)}$ contains an element of infinite order.

Proof. Let $x$ and $y$ be the elements of $\prod_{n=1}^{\infty} G_n$ whose $n$-th components are $x_n$ and $y_n$, respectively. Let $u = x[y, x^{p+1}]x^{-1}$ and $v = [x^p, y]$. It follows from the previous discussion that there is a homomorphism of $\langle u, v\rangle$ onto $\langle x, y\rangle$. Since $\langle u, v\rangle \leqq \langle x, y\rangle'$, we see that $\langle x, y\rangle$ is a homomorphic image of a subgroup of $\langle x, y\rangle^{(m)}$. Since $\langle x\rangle$ is infinite, the lemma is proved.

Corollary 3.2. There are elements $x$ and $y$ in $\prod_{n=1}^{\infty} R_n$ such that for all positive integers $m$, $\langle x, y\rangle^{(m)}$ contains an element of infinite order.

Proof. If $n > e$, then $G_{n-e}$ is a homomorphic image of $R_n$ [**4**, Lemma 3.19]. Then $\prod_{n=1}^{\infty} G_n$ is a homomorphic image of $\prod_{n=1}^{\infty} R_n$ and the corollary follows immediately.

Now set $R = \prod_{n=1}^{\infty} R_n$ and let $\Lambda$ be the Sylow $p$-subgroup of $\mathbf{Q}/\mathbf{Z}$. $R$ operates on $\Lambda$ as follows: $R$ fixes the zero of $\Lambda$. If $f \in R$, if $a$ is an integer not divisible by $p$, and if $n$ is a positive integer, then

$$p^n\mathbf{Z} + a \in \Delta_{n,n}$$

and so $(p^n\mathbf{Z} + a)f(n) = p^n\mathbf{Z} + b$ for some integer $b$. Then set

$$(\mathbf{Z} + a/p^n)f = \mathbf{Z} + b/p^n.$$

It is easily verified that this is well-defined and that in this way $R$ is faithfully represented as a permutation group on $\Lambda$.

Next, if $x \in \Lambda$, let $T_x$ be the permutation of $\Lambda$ given by $yT_x = y + x$. Set $X = \{T_x | x \in \Lambda\}$. Then $X$ is a group isomorphic to $\Lambda$ and $X$ acts as a regular permutation group on $\Lambda$.

We now let $G$ be the permutation group generated by $R$ and $X$. Set $H$ equal to the stabilizer of the zero element of $\Lambda$. Clearly $H$ contains $R$. The following theorem is the principal result of this paper.

THEOREM 3.3. *$H$ is a core-free permutable subgroup of $G = HX$. Furthermore, $H = R$ and the following are true:*

(i) *There is a 2-generator subgroup of $H/T(H)$ which is not solvable.*
(ii) *$Z_\infty(G) = Z(G)$; $Z(G)$ is finite; and $H \cap Z(G) = 1$.*
(iii) *$[G, H; \infty]$ and $[H, G; \infty]$ both contain $X$ and hence are infinite.*
(iv) *$T(G)$ is a locally finite $p$-group.*
(v) *$T(H)$ is a core-free permutable subgroup of $T(G)$.*
(vi) *$T(H)$ is not solvable.*
(vii) *$H$ is residually a finite $p$-group.*

PROOF. Since $X$ is transitive, we conclude that $G = HX$ and that $H$ is core-free in $G$. Suppose now that $n$ is a positive integer and let $\Lambda_n = \Omega_n(\Lambda)$. Set $X_n = \{T_x | x \in \Lambda_n\}$ and let $P_n$ be the subgroup of $G$ generated by $R$ and $X_n$. Then $P_n$ fixes the set $\Lambda_n$. If $K_n = \{g \in P_n | xg = x \text{ for all } x \in \Lambda_n\}$, then $K_n$ is a normal subgroup of $P_n$ and $P_n/K_n$ is a permutation group acting on $\Lambda_n$.

The mapping

$$p^n\mathbf{Z} + a \to \mathbf{Z} + a/p^n$$

establishes a one-to-one correspondence between $\Gamma_n$ and $\Lambda_n$. If we make this identification, then the image of $R$ in $P_n/K_n$ is permutation isomorphic to $\prod_{k=1}^n R_k = H_n$. The image of $X_n$ in $P_n/K_n$ is permutation isomorphic to $\langle x_n \rangle$. Thus $P_n/K_n$ is isomorphic to $G_n = \langle x_n \rangle H_n$ where $R$ is mapped onto $H_n$ and $X_n$ onto $\langle x_n \rangle$. $H_n$ is the stabilizer in $G_n$ of the zero of $\Gamma_n$. This implies that $RK_n = H \cap P_n$. Since $H_n$ is a permutable subgroup of $G_n$, we see that $RK_n$ is a permutable subgroup of $P_n$.

It is immediate that $G$ is the ascending union $\bigcup_{n \geqq 1} P_n$. This implies that $H$ is the ascending union $\bigcup_{n \geqq 1} RK_n$. Now $R$ fixes $\Lambda_n$ for all $n$. If $h \in H$ and $n$ is any integer, then we can find an integer $m$ such that $h \in RK_m$ and $\Lambda_n \subseteqq \Lambda_m$. Then $h$ must fix $\Lambda_n$ and the permutation on $\Lambda_n$ induced by $h$ is also induced by some element of $R$.

In the correspondence between $\Gamma_n$ and $\Lambda_n$, $\Delta_{n,n}$ corresponds with the set-theoretic difference $\Lambda_n$-$\Lambda_{n-1}$. If $h \in H$, then $h$ will induce a permutation $h_n$ on $\Lambda_n$-$\Lambda_{n-1}$, and if $\Lambda_n$-$\Lambda_{n-1}$ is identified with $\Delta_{n,n}$, $h_n$ is an element of $R_n$. But then $h$ is the same permutation on $\Lambda$ as $f \in R$ where $f(n) = h_n$ for all $n$. Thus $H = R$.

Then $RK_n = H$ and so $H$ is a permutable subgroup of $P_n$ for all $n$. If $g$ is any element of $G$, then $g \in P_n$ for some $n$. But then $\langle g \rangle H = H \langle g \rangle$. This implies that $H$ is a permutable subgroup of $G$.

Since $X$ is an abelian $p$-group, Theorem 2.7 implies that $T(G)$ is a locally finite $p$-group, $T(H)$ is a core-free permutable subgroup of $T(G)$, and $H$ is residually a finite $p$-group. Corollary 3.2 implies that there is a 2-generator subgroup of $H/T(H)$ which is not solvable. $T(H)$ contains a copy of

$R_n$ and $d(R_n) \to \infty$ as $n \to \infty$ [4]. It now follows that $T(H)$ is not solvable.

From [4, Lemma 4.4], we find that there are elements $t_n$ and $t$ in $H$ such that $xt = (p^e + 1)x$ for all $x \in \Lambda$ and

$$xt_n = \begin{cases} (p^e + 1)x & \text{if } x \in \Lambda_n, \\ x & \text{if } x \notin \Lambda_n. \end{cases}$$

We now easily conclude that $[T_x, t] = T_{p^e x}$ for all $x \in \Lambda$. Then $[X, \langle t \rangle] = X$ which implies that $X$ is contained in both $[G, H; \infty]$ and $[H, G; \infty]$. Also $[T_x, t] = 1$ if, and only if $x \in \Lambda_e$. Hence $C_X(t) = X_e$.

Since $X$ is an abelian regular permutation group, we have $Z(G) \leqq C_G(X) = X$. This implies that $Z(G) \leqq X_e$. However, it is an easy matter to verify that $X_e \leqq Z(G)$ (this follows, for example, from [4, Lemma 3.2(4)]). Thus $Z(G) = X_e$. Since $X_e$ is cyclic of order $p^e$ and $X \cap H = 1$, we will be done once we show that $Z_2(G) = Z(G)$.

Suppose now that $g \in Z_2(G)$. Then $[g, G] \leqq Z(G) = X_e$. This implies that $[g, h^{p^e}] = [g, h]^{p^e} = 1$ for all $h \in G$. Since $\{(T_x)^{p^e} | x \in \Lambda\} = X$, we see that $g \in C_G(X) = X$. Hence $g = T_x$ for some $x \in \Lambda$. Choose $n$ such that $x \in \Lambda_n$. Then

$$y[T_x, t_n] = \begin{cases} y + p^e x & \text{if } y \in \Lambda_n, \\ y & \text{if } y \notin \Lambda_n. \end{cases}$$

Hence $[T_x, t_n]$ cannot be a non-identity element of $X$ since $X$ acts regularly on $\Lambda$. Since $[T_x, G] \leqq Z(G) < X$, we conclude that $[T_x, t_n] = 1$. Hence $p^e x = 0$ and so $x \in \Lambda_e$. But then $T_x \in Z(G)$. Therefore, $Z(G) = Z_2(G)$ and the theorem is proved.

## Glossary

| | |
|---|---|
| $p$ | a prime |
| $e$ | $e = 1$ if $p > 2$, $e = 2$ if $p = 2$ |
| $r$ | $r = p - 1$ if $p > 2$, $r = 2$ if $p = 2$ |
| $\Gamma_n$ | $\mathbf{Z}/p^n\mathbf{Z}$ |
| $\Delta_{n,k}$ | set of elements of order $p^k$ in $\Gamma_n$ |
| $x_n$ | permutation $p^n\mathbf{Z} + a \to p^n\mathbf{Z} + a + 1$ |
| $x_{n,m}$ | $x_n^{p^{n-m}}$ if $0 \leqq m \leqq n$ |
| $\theta_{n,m,i}$ | orbit under $\langle x_{n,m} \rangle$ of $p^n\mathbf{Z} + ip^{n-m-1} - (e-1)p^{n-m-e}$ if $1 \leqq i \leqq r$ and $0 \leqq m \leqq n - e$ |
| $\pi_{n,m,i}$ | permutation on $\theta_{n,m,i}$ induced by $x_{n,m}$ |
| $A_{n,m}$ | $\left\{\prod_{i=1}^{r} \pi_{n,m,i}^{c_i} \middle\vert \sum_{i=1}^{r} c_i = 0\right\}$ |
| $G_n$ | $\langle x_n, A_{n,m} \vert 0 \leqq m \leqq n - e \rangle$ |
| $H_n$ | $\{g \in G_n \vert (p^n\mathbf{Z})g = p^n\mathbf{Z}\}$ |
| $R_n$ | $\{g \in H_n \vert \alpha g = \alpha$ for all $\alpha \in \Omega_{n-1}(\Gamma_n).\}$ |

| | |
|---|---|
| $y_n$ | $\prod_k \prod_{i=1}^{r} (\pi_{n,\, n-e-2k+2,\, i})^{(-1)^i p^{2-e}}$ where $1 \leqq k \leqq [(n-e+2)/2]$ |
| $u_n$ | $x_n[y_n, x^{p+1}]x_n^{-1}$ |
| $v_n$ | $[x_n^p, y_n]$ |
| $R$ | $\prod_{n=1}^{\infty} R_n$ |
| $\Lambda$ | Sylow $p$-subgroup of $\mathbf{Q}/\mathbf{Z}$ |
| $T_x$ | If $x, y \in \Lambda$, then $yT_x = y + x$ |
| $X$ | $\{T_x \mid x \in \Lambda\}$ |
| $G$ | $\langle R, X\rangle$ |
| $H$ | $\{g \in G \mid (\mathbf{Z})g = \mathbf{Z}\}$ |
| $\Lambda_n$ | $\Omega_n(\Lambda)$ |
| $X_n$ | $\{T_x \mid x \in \Lambda_n\}$ |
| $P_n$ | $\langle R, X_n\rangle$ |
| $K_n$ | $\{g \in P_n \mid xg = x$ for all $x \in \Lambda_n\}$ |
| $t$ | $xt = (p^e + 1)x$ for all $x \in \Lambda$ |
| $t_n$ | $xt = (p^e + 1)x$ if $x \in \Lambda_n$, $xt = x$ otherwise |

## References


**1.** R. Bradway, F. Gross, and W.R. Scott, *The nilpotence class of core-free quasinormal subgroups*, Rocky Mt. J. Math. **1** (1971), 375–382.

**2.** F. Gross, *p-subgroups of core-free quasinormal subgroups*, Rocky Mt. J. Math. **1** (1971), 541–550.

**3.** ———, *Subnormal, core-free quasinormal subgroups are solvable*, Bull. London Math. Soc. **7** (1975), 93–95.

**4.** F. Gross and T.R. Berger, *A universal example of core-free permutable subgroup*, Rocky Mtn. J. of Math., **12** (1982), 345–365.

**5.** P. Hall, *The Edmonton notes on nilpotent groups*, Queen Mary College, 1969.

**6.** N. Itô and J. Szep, *Über die Quasinormalteiler von endlichen Gruppen*, Acta Sci. Math. (Szeged) **23** (1962), 168–170.

**7.** R. Maier and P. Schmid, *The embedding of quasinormal subgroups in finite groups*, Math. Z. **131** (1973), 269–272.

**8.** S.E. Stonehewer, *permutable subgroups of infinite groups*, Math. Z. **125** (1972), 1–16.

**9.** ———, *Permutable subgroups of some finite permutation groups*, Proc. London Math. Soc. **(3) 28** (1974), 222–236.


MATHEMATICS DEPARTMENT, UNIVERSITY OF UTAH, SALT LAKE CITY, UT 84112