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ON SHELAH'S WHITEHEAD GROUPS AND CH

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ABSTRACT. Assuming that ZFC is consistent, it is consistent with GCH that there exists a non-free Whitehead group of cardinality ω_1. A proof of this result of Shelah is presented.

In **[10]** and **[11]** Shelah showed that the existence of a non-free Whitehead group (denoted W-group) is independent of ZFC. More precisely he showed that if the axiom of constructability ($V = L$) is added to ZFC then every W-group is free. However assuming Martin's axiom and not CH ($\neg$CH) (CH denotes the assertion that $2^\omega = \omega_1$), there is a non-free Whitehead group of cardinality ω_1. The reader should see **[6]** or **[8]** for a good account of these results. There remained the question of whether it is consistent with ZFC and GCH (GCH denotes the assertion that $2^\kappa = \kappa^+$ for all infinite cardinals κ) that there is a non-free W-group of cardinality ω_1. This was particularly interesting as it was known that CH gave information about W-groups. For instance, Chase **[2]** showed that if CH holds, then every W-group of cardinality ω_1 is strongly ω_1-free (i.e., every countable subgroup is contained in an ω_1-pure free subgroup). This last result can fail in the absence of CH (cf. **[10]** or **[8]**).

Shelah **[12]** established the consistency (relative to that of ZFC) of ZFC + GCH + "there exists a non-free W-group of cardinality ω_1". The general strategy was to find an axiom which approximates MA + $\neg$CH but which is consistent with GCH. One then uses this axiom to prove the group theoretic result. The proof in **[12]** that the set-theoretic axiom implies the existence of a non-free W-group is rather cryptic. The purpose of this paper is to elaborate this proof. I originally used a modification of the axiom in **[12]**. Following a suggestion of Shelah, the proof will be based on a more powerful axiom. The paper begins with set-theoretic preliminaries which culminate in the statement of AX(S). I will then prove the desired theorem.

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Set-theoretic preliminaries. If X is a set, $\mathscr{P}_{\omega_1}(X) = \{Y\colon Y \subseteqq X$ and $|Y| \leqq \omega\}$. ($|Y|$ denotes the cardinality of Y.) A set $\mathscr{C} \subseteqq \mathscr{P}_{\omega_1}(X)$ is *closed* if whenever $\{Y_n\colon n < \omega\} \subseteqq \mathscr{C}$ is such that $Y_{n+1} \supseteqq Y_n$ then $\bigcup_{n<\omega} Y_n \in \mathscr{C}$.

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It is *unbounded* if for all countable $Y \subseteqq X$ there is $Z \in \mathscr{C}$ such that $Z \supseteqq Y$. A set $\mathscr{C} \subseteqq \mathscr{P}_{\omega_1}(X)$ is a *cub*, if it is closed and unbounded. If $|X| = \omega_1$, an ω_1-*filtration* of X is a cub $\mathscr{C} = \{X_\alpha: \alpha < \omega_1\} \subseteqq \mathscr{P}_{\omega_1}(X)$ such that: if $\beta < \alpha$, then $X_\beta \subseteqq X_\alpha$; and if α is a limit ordinal, then $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$. Since an ordinal $\alpha = \{\beta: \beta < \alpha\}$, the usual notion of a cub for ω_1 is exactly that of an ω_1-filtration by ordinals of ω_1. Also if $|X| = \omega_1$ then every cub contains an ω_1-filtration. So for ω_1, the usual notion of a cub and that of a cub in $\mathscr{P}_{\omega_1}(\omega_1)$ are essentially the same. Recall that a set $S \subseteqq \mathscr{P}_{\omega_1}(X)$ is *stationary*, if for every cub $\mathscr{C}$, $\mathscr{C} \cap S \neq 0$.

Suppose $(P, <)$ is a partially ordered set (poset). Elements $p, q \in P$ are *compatible*, if there exist $r \in P$ such that $r \leqq p$ and $r \leqq q$. A set $D \subseteqq P$ is *dense* if for all $p \in P$ there is $q \in D$ such that $q \leqq p$. Let $\mathscr{D}(P) = \{D \subseteqq P$: D is dense$\}$. Where there can be no confusion we will let $\mathscr{D} = \mathscr{D}(P)$. Suppose $A \subseteqq P \cup \mathscr{D}$. An element $q \in P$ is *A-generic*, if for every $D \in A$ and $r \leqq q$ there is $p \in D \cap A$ such that p and r are compatible. A partial order P is *proper*, if there is a cub $\mathscr{C} \subseteqq \mathscr{P}_{\omega_1}(P \cup \mathscr{D})$ such that for all $A \in \mathscr{C}$ and $p \in A$ there is an A-generic $q \leqq p$.

The following theorem (which will not be used) summarizes some facts about proper posets.

THEOREM. (1) *If p satisfies the* c.c.c. (*i.e., all antichains—sets of pairwise incompatible elements—are countable*), *then P is proper.*

(2) *If P is ω_1-closed (i.e., whenever $p_1 \geqq p_2 \geqq \cdots$, there is $q \leqq p_n$ for all n), then P is proper.*

(3) ([13]). *P is proper if and only if for every κ forcing with P preserves stationary subsets of $P_{\omega_1}(\kappa)$.*

(4) ([15]). *If proper forcing is iterated with countable support, the resulting partial order is proper.*

PROOF. It is well known that (1) and (2) follow from (3). I will prove (1) and (2) to help reader become familiar with the definition of proper.

(1) Choose $\mathscr{C} \subseteqq \mathscr{P}_{\omega_1}(P \cup \mathscr{D})$ such that $C \in \mathscr{C}$, if and only if $D \in C \cap \mathscr{D}$ implies $D \cap C$ contains a maximal antichain. For $C \in \mathscr{C}$ and $p \in P$, p is C-generic.

(2) Take $\mathscr{C} \subseteqq \mathscr{P}_{\omega_1}(P \cup \mathscr{D})$ so that $C \in \mathscr{C}$ if and only if for all $D \in C \cap \mathscr{D}$, $D \cap C$ is dense in $C \cap P$. Suppose $C \in \mathscr{C}$ and $\{D_n: n < \omega\}$ is an enumeration of $\mathscr{D} \cap C$. Given $p \in C \cap P$, let $p_{-1} = p$. Inductively choose $p_{n+1} \leqq p_n$, so that $p_{n+1} \in D_{n+1} \cap C$. If $q \leqq p_n$ for all n, then for all $r \leqq q$ and $n < \omega$, r is compatible with (in fact $\leqq$) p_n and $p_n \in D_n \cap C$. Hence such a q is C-generic.

A poset P is *E-closed* ($E \subseteqq \omega_1$) if there exists a cub $\mathscr{C} \subseteqq \mathscr{P}_{\omega_1}(P \cup \mathscr{D} \cup \omega_1)$ such that for $A \in C$ the following implication is true. If $A \cap \omega_1 \in E$, $\{p_n: n < \omega\} \subseteqq P \cap A$, $p_{n+1} \leqq p_n$ and for all $D \in \mathscr{D} \cap A$ there is an n

such that $p_n \in D$, then there is $q \in P$ such that $q \leqq p_n$ for all n. It is now possible to define the needed axiom.

DEFINITION. Assume $S \subseteqq \omega_1$. Let AX(S) denote the following statement: if P is a poset of cardinality ω_1 which is proper and $(\omega_1 - S)$-closed, and $\{D_\alpha: \alpha < \omega_1\}$ is a collection of dense subsets of P, then there is a directed $G \subseteqq P$ such that for all $\alpha < \omega$, $G \cap D_\alpha \neq 0$.

NOTE. G is directed if for all $p, q \in G$ there is an $r \in G$ such that $r \leqq p$ and $r \leqq q$.

THEOREM 0. (SHELAH [13]). *If* ZFC *is consistent, then so is* ZFC + GCH + "*there exists a stationary set* $S \subseteqq \omega_1$ *such that* $(\omega_1 - S)$ *is stationary*" *and both* AX(S) *and* $\diamondsuit^*(\omega_1 - S)$ *hold.*

REMARK. I will not define $\diamondsuit^*(E)$ or $\diamondsuit(E)$. It suffices to know that $\diamondsuit^*(\omega_1 - S)$ implies $\diamondsuit(E)$ for all E such that E-S is stationary, (cf. [3]).

A proof of Theorem 0 can be constructed by combining the iteration lemma ((4) of the theorem above) and the techniques of [12]. There are other axioms which deal with proper posets. In [1] there is an exposition of the proper forcing axiom and its consequences.

Group Theory. By "group" I shall mean "Abelian group". A group G is a W-group, if $\text{Ext}(G, Z) = 0$. This means that whenever

$$0 \to \mathbf{Z} \to H \overset{\sigma}{\to} G \to 0$$

is exact, there is a homomorphism $\theta: G \to H$ such that $\sigma\theta = \text{Id}$. The identity function (on any domain) will always be denoted "Id". In studying potential W-groups, it is enough to consider ω_1-free groups (i.e., groups where every countable subgroup is free), because of the following result of Stein.

THEOREM. *Every countable* W*-group is free.*

I will now review the structure of ω_1-free groups of cardinality ω_1. If G is ω_1-free and $H \subseteqq G$ is such that G/H is ω_1-free, then H is ω_1-*pure*. For $E, F \subseteqq \omega_1$, define $E \equiv F$ if there is a cub $C \subseteqq \omega_1$ such that $E \cap C = F \cap C$. Let $\tilde{E}$ denote the equivalence class of E. Suppose $\{G_\alpha: \alpha < \omega_1\}$ is an ω_1-filtration of an ω_1-free group G where $|G| = \omega_1$. Define $\Gamma(G) = \tilde{E}$, where $E = \{\alpha: G_\alpha \text{ is not an } \omega_1\text{-pure subgroup}\}$. (Note: $\{\alpha: G_\alpha \text{ is a subgroup}\}$ is a cub.) The function Γ was defined in [7] where it was shown that Γ does not depend on the filtration

THEOREM 1. *Let G be an* ω_1*-free group of cardinality* ω_1.

(1) (Eklof [5]). *G is free if and only if* $\Gamma(G) = \tilde{0}$.

(2) (Eklof [5]). *For any* $E \subseteqq \omega_1$, *there is an* ω_1*-free group* G' (*of cardinality* ω_1) *such that* $\Gamma(G') = \tilde{E}$.

(3) (Shelah [10]) *If E is stationary, $\diamondsuit(E)$ holds and $\Gamma(G) = \tilde{E}$, then G is not a W-group.*

The following theorem not only shows that the existence of non-free W-groups is consistent with GCH, but that it is consistent that the W-groups of cardinality ω_1 are characterized by the value of Γ.

THEOREM 2. 1. *If* ZFC *is consistent then so is* ZFC + GCH + "*there is a non-free W-group of cardinality ω_1*". 2. *If* ZFC *is consistent then so is* ZFC + GCH + "*there is a stationary set $S \subseteqq \omega_1$ such that: an ω_1-free group G of cardinality ω_1 is a W-group if and only if $\Gamma(G) \subseteqq \tilde{S}$*".

REMARK. It is known [4] (cf. [8]) that in 2.2 $\omega_1 - S$ is stationary. Also if ZFC is consistent then 2.2 is independent of 2.1 ([14]).

PROOF. Since 2.2 together with 1.2 imply 2.1, it suffices to prove 2.2. For the remainder of the proof assume $S \subseteqq \omega_1$ is a stationary set such that GCH + AX(S) + $\diamondsuit^*(\omega_1 - S)$ hold, and $(\omega_1 - S)$ is stationary.

Suppose G is an ω_1-free group of cardinality ω_1 and $\Gamma(G) = \tilde{E}$. If $\tilde{E} \nsubseteq \tilde{S}$, then $\diamondsuit(E)$ holds. So by 1.3, G is not a W-group.

It remains to show that if $\tilde{E} \subseteqq \tilde{S}$, then G is a W-group. Assume

$$0 \to \mathbf{Z} \to H \overset{\sigma}{\to} G \to 0$$

is exact. It must be shown that this sequence "splits". The basic idea is: define a poset P of partial splitting maps; show this poset is proper and (ω_1-S)-closed; and apply AX(S) to get the desired conclusion.

Before defining P, it is convenient to choose a "nice" ω_1-filtration of G. I wish to choose $\{G_\alpha : \alpha < \omega_1\}$ an ω_1-filtration of G (by subgroups) such that: if $E = \{\alpha : G_\alpha \text{ is not } \omega_1\text{-pure}\}$ then $E \subseteqq S$ and every element of E is a limit ordinal. Let $\{G'_\alpha : \alpha < \omega_1\}$ be an ω_1-filtration of G by subgroups and let $E' = \{\alpha : G'_\alpha \text{ is not } \omega_1\text{-pure}\}$. Let $C_1 \subseteqq \omega_1$ be a cub such that $C_1 \cap E' \subseteqq S$. Let C be the closure (under countable unions) of $C_1 \cap (\omega_1 - E')$. Finally let $G_\alpha = G'_\beta$ where β is the least element of $C \geqq \alpha$. If $\alpha \in C$, then $G_\alpha = G'_\alpha$. If $\alpha \notin C$, then $G_\alpha = G'_\beta$ for some $\beta \in (\omega_1 - E')$. So if G_α is not ω_1-pure, then $\alpha \in C \cap E' \subseteqq S$.

Let $P = \{\theta : \operatorname{dom}(\theta) = G_{\alpha+1} \text{ and } \sigma\theta = \mathrm{Id}\}$. (All maps between groups will be assumed to be homomorphisms.) For $\theta, \psi \in P$ let $\theta \leqq \psi$ if $\psi \subseteqq \theta$. Since $2^\omega = \omega_1$, $|P| = \omega_1$. Assume for the moment that P is proper and $(\omega_1 - S)$-closed.

CLAIM 1. *For each $\alpha < \omega_1$, $D_\alpha = \{\theta \in P : \operatorname{dom}(\theta) \supseteqq G_\alpha\}$ is dense.*

PROOF OF CLAIM 1. Assume $\psi \in P$ and $\operatorname{dom}(\psi) = G_{\beta+1}$. If $\alpha \leqq \beta + 1$, there is nothing to prove. Suppose $\alpha > \beta + 1$. Since $G_{\beta+1}$ is ω_1-pure, $G_{\beta+1}$ is a direct summand of $G_{\alpha+1}$. So there exists $\theta \in P$ such that $\operatorname{dom}(\theta) = G_{\alpha+1}$ and $\theta \supseteqq \psi$ (cf. [9], Theorem 5, p. 15).

By AX(S) there exists a directed subset $\Phi \subseteqq P$ such that $\Phi \cap D_\alpha \neq 0$ for all $\alpha < \omega_1$. Let $\psi = \bigcup_{\theta \in \Phi} \theta$. The ψ is the desired splitting.

It remains to show P is proper and $(\omega_1 - S)$-closed.

CLAIM 2. *The poset P is $(\omega_1 - S)$-closed.*

PROOF OF CLAIM 2. Let $\mathscr{C} \subseteqq \mathscr{P}_{\omega_1}(P \cup \mathscr{D} \cup \omega_1)$ be such that for all $C \in \mathscr{C}$: if $\alpha \in C$, then $D_\alpha \in C$; if $\psi \in C \cap P$ and $G_\alpha \subseteqq \operatorname{dom} \psi$, then $\alpha \in C$; and if $\alpha \in C$, then $(\alpha + 1) \in C$. Suppose $C \in \mathscr{C}$, $C \cap \omega_1 = \beta \in (\omega_1 - S)$ and $\{\theta_n : n < \omega\}$ are as in the definition of $(\omega_1 - S)$-closed. Let $\psi = \bigcup_{n<\omega} \theta_n$. By the choice of $\mathscr{C}$, $\operatorname{dom}(\psi) = G_\beta$ and $\sigma\psi = \mathrm{Id}$. Since G_β is ω_1-pure there exists $\theta \in P$ such that $\operatorname{dom}(\theta) = G_{\beta+1}$ and $\theta \supseteqq \psi$.

CLAIM 3. *P is proper.*

PROOF OF CLAIM 3. For $C \in \mathscr{P}_{\omega_1}(P \cup \mathscr{D})$, let $\beta(C) = \{\alpha$: there exists $\theta \in C \cap P$ such that $\operatorname{dom}(\theta) \supseteqq G_\alpha\}$. Let $C' \subseteqq P_{\omega_1}(P \cup \mathscr{D})$ be the cub such that $C \in \mathscr{C}'$ if:

(i) $\beta(C)$ is a limit ordinal;

(ii) if $\theta \in C \cap P$ and $\theta \leqq \psi$, then $\psi \in C$;

(iii) $\alpha < \beta(C)$ if and only if $D_\alpha \in C$;

(iv) $D_\alpha \cap C$ is dense in $C \cap P$, for all $\alpha < \beta(C)$;

(v) if $\theta \in C \cap P$ and $\theta \subseteqq \psi$, then for all finite $X \subseteqq G_{\beta(C)}$ there is $\psi' \in C \cap P$ such that $\theta \subseteqq \psi'$ and for all $x \in X$, $\psi'(x) = \psi(x)$.

Let $\mathscr{C}'' = \{C \in \mathscr{C}' : \beta(C) \notin E\}$. Note: $\mathscr{C}''$ is not a cub but it is unbounded (as $\omega_1 - S$ is stationary). Let $\mathscr{C}$ be the closure (under the union of countable chains) of $\mathscr{C}''$.

Suppose $C \in \mathscr{C}$. If $\beta(C) \notin E$, the verification that P is proper is similar to the proof of claim 2. Assume $\beta(C) = \beta \in E$. Suppose $\theta \in P \cap C$, and $\{D^n | n < \omega\}$ is an enumeration of $\mathscr{D} \cap C$. Choose $\{C_n : n < \omega\} \subseteqq \mathscr{C}''$ so that for each $n < \omega$: $C_{n+1} \supseteqq C_n$; $\beta_{n+1} > \beta_n$ (where $\beta_n = \beta(C_n)$); $\beta_n \notin E$; $C = \bigcup_{n<\omega} C_n$; $D^n \in C_n$; and $\theta \in C_0$. Now choose for each $n < \omega$ groups A_n and B_n such that:

$$G_{\beta_n} \oplus A_n = G_{\beta+1};$$
$$G_{\beta_n} \oplus B_n = G_{\beta_{n+1}};$$
$$A_{n+1} \oplus B_n = A_n.$$

To do this it suffices: to choose A_0 a complementary summand (in $G_{\beta+1}$) of G_{β_0}; to let $B_0 = A_0 \cap G_{\beta_1}$; to choose A_1 a complementary summand (in A_0) of B_0; and so on.

Let $C_{\beta+1} = \langle G_\beta \cup \{g_i : i < \omega\}\rangle$. ($\langle X \rangle$ denotes the group generated by X.) Assume $g_n \in A_n$ for all n. Let π_0 be the projection of $G_{\beta+1}$ on G_{β_0} relative to A_0. Let π_{n+1} be the projection of $G_{\beta+1}$ on B_n relative to $G_{\beta_n} \oplus A_{n+1}$. Notice that $\pi_n(g_m) = 0$, if $m \geqq n$.

Choose $\psi \in P$ so that $\mathrm{dom}(\psi) = G_{\beta+1}$ and $\theta \subseteq \psi$. Let $\theta_{-1} = \theta$. Define the sequence $\theta_n \in C_n$ inductively as follows:

(a) choose $\theta' \supseteq \theta_{n-1}$, such that $\theta' \in C_n$ and $\theta'(\pi_n(g_m)) = \psi(\pi_n(g_m))$ for all $m < \omega$;

(b) choose $\theta_n \supseteq \theta'$, such that $\theta_n \in C_n \cap D^n$.

It must be shown that the choices above are possible. I will first deal with (a). If $n = 0$, take $\theta' = \theta$. Assume $n > 0$. Choose $\theta'' \supseteq \theta_{n-1}$ such that $\mathrm{dom}(\theta'') = G_{\beta_{n-1}}$ and $\sigma\theta'' = \mathrm{Id}$. Let ψ' be the unique homorphism whose domain is $G_{\beta+1}$ such that $\psi' \supseteq \theta''$ and $\psi'|A_{n-1} = \psi|A_{n-1}$ (where $\psi|A_{n-1}$ denotes the restriction of ψ to A_{n-1}). Such a ψ' exists since $G_{\beta+1} = G_{\beta_{n-1}} \oplus A_{n-1}$. Note that $\psi' \in P$. Since $\{\pi_n(g_m): m < \omega\}$ is finite, (v) of the definition at $\mathscr{C}'$ guarantees the existence of $\theta' \in C_n \cap P$ such that $\theta_{n-1} \subseteq \theta'$ and for all m $\theta'(\pi_n(g_m)) = \psi'(\pi_n(g_m)) = \psi(\pi_n(g_m))$. The choice in (*b*) is possible, since $D^n \cap C_n$ is dense in $C_n \cap P$.

Let $\rho = \bigcup_{n<\omega}\theta_n$. By the choice of $\{\theta_n: n < \omega\}$, if $\rho \subseteq \rho'$ and $\rho' \in P$ then ρ' is C-generic. (For all $\rho'' \leq \rho'$ and all n, ρ'' is compatible with $\theta_n \in D^n \cap C$.)

Note that $\rho: G_\beta \to H$ and $\sigma\rho = \mathrm{Id}$. Further for all $m, n < \omega$, $\rho(\pi_n(g_m)) = \psi(\pi_n(g_m))$. The following claim is used to complete the proof.

CLAIM 4. *If* $\sum_{i=0}^n a_i g_i = g$, $g \in G_\beta$, *and* $a_i \in \mathbf{Z}$ *for* $i \leq n$; *then* $\sum_{i=0}^n a_i\psi(g_i) = \rho(g)$.

Assume Claim 4. So there exists a unique homorphism $\rho': G_{\beta+1} \to H$ such that $\rho \subseteq \rho'$ and $\rho'(g_n) = \psi(g_n)$ for all n.

PROOF OF CLAIM 4. Choose m such that $g \in G_{\beta_m}$. So

$$\sum_{i=0}^n a_i g_i = g = \sum_{k=0}^m \sum_{i=0}^n a_i \pi_k(g_i).$$

Hence

$$\begin{aligned}\rho(g) &= \sum_{k=0}^m \sum_{i=0}^n a_i \rho(\pi_k(g_i)) \\ &= \sum_{k=0}^m \sum_{i=0}^n a_i \psi(\pi_k(g_i)) \\ &= \sum_{i=0}^n a_i \psi(g_i).\end{aligned}$$

Appendix. For those readers who demand that theorems follow only from results whose proofs are accessible, I will present a modified version of the axiom in **[12]**. From this new axiom theorems 2.1 and 2.2 follow as above. The proof that the new axiom is consistent is similar to the proofs in **[12]**. Since this appendix is intended as a supplement to **[12]**, several definitions will be omitted.

In the following S will always be a set of limit ordinals contained in ω_1.

DEFINITION. A tree $(T, <)$ is defined to be S-fair if

(1) the height of T is ω_1,

(2) every node of T has successors of arbitrarily large height $< \omega_1$,

(3) $|T| = \omega_1$,

(4) for each $\delta \in \omega_1 - S$, every δ-branch in T has a successor in T_δ, and

(5) if $\langle X_\alpha : \alpha < \omega_1 \rangle$ is an ω_1-filtration of T, then there exists a cub $C \subseteqq \omega_1$ such that for all $\delta \in C \cap S$ and $x \in T|\delta$ there is a $T^* \subseteqq T|\delta$ such that

(i) $x \in T^*$,

(ii) every element of T^* has successors in T^* of arbitrarily large height $< \delta$,

(iii) if $\bar{\delta} \in (C \cap \delta - S)$ and $a \in (T^*|\bar{\delta} \cap X_{\bar{\delta}})$, then there exists $b \in (T^*|\bar{\delta} \cap X_{\bar{\delta}})$ such that $a < b$ and $\{c | c \in T|\bar{\delta}$ and $b < c\} \subset T^*$, and

(iv) every δ-branch in T^* has an extension in T_δ.

Let SAM be the statement: There is a stationary set $S \leqq \omega_1$ such that: every S-fair tree has an ω_1-branch; $(\omega_1 - S)$ is stationary and $\diamondsuit^*(\omega_1 - S)$ holds.

THEOREM. *If* ZFC *is consistent, then so is* ZFC + GCH + SAM.

The theorem above is a special case of Theorem 0.

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